A NOTE ON RIESZ ELEMENTS IN C*-ALGEBRAS

DAVID LEGG

Department of Mathematics Indiana University-Purdue University Fort Wayne, Indiana

(Received December 5, 1977)

<u>ABSTRACT</u>. It is known that every Riesz operator R on a Hilbert space can be written R = Q + C, where C is compact and both Q and CQ - QC are quasinilpotent. This result is extended to a general C^* -algebra setting.

1. INTRODUCTION.

In [3], Smyth develops a Riesz theory for elements in a Banach algebra with respect to an ideal of algebraic elements. In [1], Chui, Smith and Ward show that every Riesz operator on a Hilbert space is decomposible into R=Q+C, where C is compact and both Q and CQ-QC are quasinilpotent. In this paper we use Smyth's work to show that the analogous result holds in an arbitrary C^* -algebra.

2. DEFINITIONS AND NOTATION.

Let A be a C*-algebra, and let F be a two-sided ideal of algebraic elements

94 D. LEGG

of A. An element T ϵ A is a Riesz element if its coset T + \overline{F} in A/ \overline{F} has spectral radius 0. A point λ ϵ $\sigma(T)$ is a finite pole of T if it is isolated in $\sigma(T)$ and the corresponding spectral projection lies in F. Let $E\sigma(T) = \{\lambda \in \sigma(T) : \lambda \in T\}$ is not a finite pole of T}. Smyth has shown that T is a Riesz element if and only if $E\sigma(T) \subseteq \{0\}$, [3, Thm. 5.3]. Smyth also showed that if T is a Riesz element, then T = Q + U, where Q is quasinilpotent and U ϵ \overline{F} . [3, Thm. 6.9]. This is a generalization of West's result [4, Thm. 7.5]. We now extend the result of Chui, Smith and Ward [1, Thm. 1] by showing that UQ - QU is quasinilpotent, where T = Q + U is the Smyth decomposition.

3. OUTLINE OF SMYTH'S CONSTRUCTION.

Let T be a Riesz element, and label the elements of $\sigma(T) \setminus E\sigma(T)$ by λ_n , $n=1,2,\ldots$, in such a way that $|\lambda_n|^2 |\lambda_{n+1}|$, $\lambda_n \to 0$ as $n \to \infty$. Each λ_n is a finite pole, so each spectral projection P_n is in F. Let $S_n = P_1 + \ldots + P_n$, then find a self-adjoint projection Q_n satisfying $S_n Q_n = Q_n$ and $Q_n S_n = S_n$. Let $V_n = Q_n - Q_{n-1}$, and define $U = \sum_i \lambda_k V_i$. U is clearly in \overline{F} and $Q_i = T_i - U$ is shown to be quasinilpotent.

4. THEOREM 1 UQ - QU is quasinilpotent.

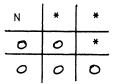
PROOF. For any S ϵ A, let $\overset{\boldsymbol{\circ}}{S}$ denote the left regular representation of S. Then by Lemma 6.6 in Smyth [3], we have that Q_n A is an invariant subspace of $\overset{\boldsymbol{\circ}}{Q}$. Since $Q_n = Q_n Q_n$, we have $Q_n \epsilon Q_n A$. Hence $\overset{\boldsymbol{\circ}}{Q}(Q_n) \epsilon Q_n A$, say $\overset{\boldsymbol{\circ}}{Q}(Q_n) = Q_n S$ for some S ϵ A. That is, $QQ_n = Q_n S$. Now let v ϵ range Q_n , say v = $Q_n x$. Then $Qv = QQ_n x = Q_n Sx$ belongs to range Q_n . Hence we see that range Q_n is an invariant subspace of Q. It follows that Q has an operator matrix representation of the form

where $A_{ij} = V_i QV_j$. With respect to this blocking, we have

Hence
$$O(\lambda_{1}-\lambda_{2})A_{12} - - - (\lambda_{1}-\lambda_{n})A_{1n} - - - \\ OO \\ (\lambda_{1}-\lambda_{2})A_{12} - - - (\lambda_{1}-\lambda_{n})A_{1n} - - - \\ (\lambda_{1}-\lambda_{n})A_{1n}$$

96 D. LEGG

Now let P be the orthogonal projection onto \bigcup_n range \mathbb{Q}_n , and let $A_n = (P - \mathbb{Q}_n)(U\mathbb{Q} - \mathbb{Q}U)(P - \mathbb{Q}_n)$. It is easy to see that $||A_n|| \le |\lambda_n|||\mathbb{Q} - \mathbb{Q}U$ as $n \to \infty$. Hence $U\mathbb{Q} - \mathbb{Q}U - A_n$ converges in the uniform norm to $U\mathbb{Q} - \mathbb{Q}U$ as $n \to \infty$. But $U\mathbb{Q} - \mathbb{Q}U - A_n$ has the form



where N is nilpotent. It follows that UQ - QU - A_n has no non-zero eigenvalues. Thm. 3.1, p. 14 of [2] can now be easily modified to show that UQ - QU has no non-zero eigenvalues. Since UQ - QU belongs to \overline{F} , this means $\sigma(UQ - QU) \subseteq \{0\}$, i.e., UQ - QU is quasinilpotent.

REFERENCES

- Chui, C. K., Smith, P. W., and Ward, J. D., A note on Riesz operators, <u>Proc. Amer. Math. Soc.</u>, 60, (1976), 92-94.
- Gohberg, I. C., and Krein, M. G., <u>Introduction to the theory of linear non-selfadjoint operators</u>, "Nauka," Moscow, 1965; English transl., Transl. Math Monographs, vol. 18, Amer. Math. Soc., Providence, R. I. 1969.
- Smyth, M. R. F., Riesz theory in Banach algebras, <u>Math Z.</u>, 145, (1975), 145-155.
- West, T. T., The decomposition of Riesz operators, <u>Proc. London Math. Soc.</u>, <u>III</u>, Ser. 16, (1966), 737-752.

AMS (MOS) Subject Classification numbers 47 B 05, 47 C 10

KEY WORDS AND PHRASES. C* algebra, quasinilpotent operators, Riesz elements.