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EXAMPLES OF FRIEDRICHS MODEL OPERATORS WITH A CLUSTER POINT OF EIGENVALUES

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A family of selfadjoint operators of the Friedrichs model is considered. These symmetric type operators have one singular point, zero of order m. For every $m > 3/2$, we construct a rank 1 perturbation from the class Lip 1 such that the corresponding operator has a sequence of eigenvalues converging to zero. Thus, near the singular point, there is no singular spectrum finiteness condition in terms of a modulus of continuity of a perturbation for these operators in case of $m > 3/2$.

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1. Statement of the problem and main result. Consider selfadjoint operators S_m, $m > 0$, given by

$$S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot, \varphi)\varphi \tag{1.1}$$

on the domain of functions $u(t) \in L_2(\mathbb{R})$ such that $|t|^m u(t) \in L_2(\mathbb{R})$. Here $\varphi \in L_2(\mathbb{R})$ and t is the independent variable. The action of the operator S_m can be written as follows:

$$(S_m u)(t) = \operatorname{sgn} t \cdot |t|^m u(t) + \varphi(t) \int_{\mathbb{R}} u(x)\overline{\varphi(x)}\,dx. \tag{1.2}$$

The function φ is assumed to satisfy the smoothness condition

$$\left|\varphi(t+h) - \varphi(t)\right| \le \omega(|h|), \quad |h| \le 1, \tag{1.3}$$

where the function $\omega(t)$ (the modulus of continuity of the function φ) is monotone and satisfies a Dini condition

$$\omega(t) \downarrow 0 \quad \text{as } t \downarrow 0, \qquad \int_0^1 \frac{\omega(t)}{t}\,dt < \infty. \tag{1.4}$$

For the operators S_m, the absolutely continuous spectrum fills the real axis $\mathbb{R}$. The behavior of the singular spectrum of the operators S_m is of interest. Note that we define the singular spectrum as the union of the point spectrum and the singular continuous one. The structure of the spectrum $\sigma_{\text{sing}}(S_1)$ (the singular spectrum of the operator $S_1 = t \cdot + (\cdot, \varphi)\varphi$) has been studied in detail in [1, 2, 3, 4, 5, 6, 7, 8, 9]. It is shown in [1, 7] that for this operator, there exists an exact condition of the singular spectrum finiteness. Namely, if $\omega(t) = O(\sqrt{t})$ as $t \to 0^+$, $\sigma_{\text{sing}}(S_1)$ consists of at most a finite number of eigenvalues of finite multiplicity (the singular continuous spectrum is missing). But if $\liminf \omega(t)/\sqrt{t} = +\infty$ as $t \to 0^+$, then we construct examples showing that a nontrivial singular spectrum appears, in particular, the operator S_1 has accumulation points of eigenvalues. Note that the real appearance of a nontrivial singular spectrum in the Friedrichs model was, for the first time, shown by Pavlov and Petras [8].

Using the simple change of variables $\operatorname{sgn} t \cdot |t|^m = x$, we can show that outside any neighborhood of the origin, the structure of the spectrum $\sigma_{\text{sing}}(S_m)$ is identical with the one of the operator S_1. This is due to the smoothness of this change of variables outside any neighborhood of the origin, and also to the local character of the main results of [1, 2, 3, 4, 5, 6, 7, 8, 9]. Namely, suppose that conditions (1.3), (1.4), and also some additional conditions on the function φ are fulfilled only in a certain interval $(c,d) \subset \mathbb{R}$, then, the main results of [1, 2, 3, 4, 5, 6, 7, 8, 9], concerning the structure of $\sigma_{\text{sing}}(S_1)$, will remain true in any closed subinterval $\Delta \subset (c,d)$. At the same time, as shown in this paper, for the operator S_m, $m > 3/2$, the behavior of the singular spectrum has quite different character in a neighborhood of the origin. In this case, it turns out that the singular spectrum can appear for any modulus of continuity $\omega(t)$. Hence, near zero, there is no condition of the singular spectrum finiteness in terms of the modulus of continuity of $\varphi(t)$ like for the operator S_1. Here, we can also use the pointed change of variables but, since, for instance, $(\operatorname{sgn} t \cdot |t|^m)'_{|t=0} = 0$ for $m > 1$, it is not smooth (that is, not a diffeomorphism) near zero. In this sense, zero is a singular point of the operators S_m, $m \neq 1$, and needs a special attention.

We start with a formulation of the main theorem of this paper on the construction of a function $\varphi(t)$ such that the corresponding operator S_m, $m > 3/2$, has a nontrivial singular spectrum near the singular point zero and, in particular, a sequence of eigenvalues converging to the origin. The proof of this theorem will be obtained as a combination of a sequence of lemmas. Observe that the actual modulus of continuity $\widetilde{\omega}(h) := \sup\{|\varphi(t_1) - \varphi(t_2)| : |t_1 - t_2| < h\}$ of the function φ always satisfies the additional constraint of semiadditivity $\widetilde{\omega}(t_1 + t_2) \leq \widetilde{\omega}(t_1) + \widetilde{\omega}(t_2)$ for all $t_1, t_2 \geq 0$.

Theorem 1.1 (main result). *Let a nonnegative, monotone function $\omega(t)$, $t \geq 0$, be semiadditive such that $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ for all $t_1, t_2 \geq 0$. Then for any $m > 3/2$, a compactly supported function φ satisfying the smoothness condition $|\varphi(t+h) - \varphi(t)| \leq \omega(|h|)$, $h \in \mathbb{R}$, is constructed and such that the corresponding operator $S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot, \varphi)\varphi$ has a sequence of eigenvalues converging to zero.*

Note that the result of Theorem 1.1 can be formulated in terms of real zeros of some analytic functions. Define in the upper half plane an analytic function

$M_m(z)$ in the following way:

$$M_m(z) = 1 + \int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{\operatorname{sgn} t \cdot |t|^m - z} dt, \quad \operatorname{Im} z > 0. \tag{1.5}$$

It is easily shown that under conditions (1.3) and (1.4), the function $M_m(z)$ is continuously extended up to the real axis on the intervals $(-\infty;0)$ and $(0;+\infty)$. We define, for $\lambda \in \mathbb{R} \setminus \{0\}$, the value $M_m(\lambda) := M_m(\lambda + i0)$ and the roots set $N := \{\lambda \in \mathbb{R} \setminus \{0\} : M_m(\lambda) = 0\}$. Then, we have the following inclusion $\sigma_{\text{sing}}(S_m) \subseteq N \cup \{0\}$. (See, e.g., [10] where similar assertions are proved for the function $1 + \int_{-\infty}^{+\infty} (|\varphi^2(t)|/(t^2 - z)) dt$ or [8] for the function $1 + \int_{-\infty}^{+\infty} (|\varphi^2(t)|/(t - z)) dt$.) Further, the exact condition $\omega(t) = O(\sqrt{t})$, as $t \to 0^+$, appears to guarantee that outside any neighborhood of the origin, there is at most a finite number of zeros of the function $M_m(\lambda)$. At the same time, Theorem 1.1 means that, for $m > 3/2$, the function $M_m(\lambda)$ can have a sequence of zeros converging to the origin for any monotone, nonnegative, and semiadditive function ω satisfying condition (1.4).

2. Construction of the function φ

Lemma 2.1. *Let the function φ belong to $L_2(\mathbb{R})$. Then a point $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot, \varphi)\varphi$ if and only if the following conditions hold:*

$$\frac{\varphi(t)}{\operatorname{sgn} t \cdot |t|^m - \lambda} \in L_2(\mathbb{R}), \tag{2.1}$$

$$1 + \int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{\operatorname{sgn} t \cdot |t|^m - \lambda} dt = 0. \tag{2.2}$$

Proof. Solving the eigenvalue problem $S_m v = \lambda v$, we find that $v(t) = C\varphi(t)/(\operatorname{sgn} t \cdot |t|^m - \lambda)$. (Remark that from now on, we denote by C various constants which may be different even in a single chain of inequalities.) This expression will be an eigenfunction if it is from the space $L_2(\mathbb{R})$ and satisfies the equation $S_m v - \lambda v = 0$, that is,

$$(\operatorname{sgn} t \cdot |t|^m - \lambda) \frac{C\varphi(t)}{\operatorname{sgn} t \cdot |t|^m - \lambda} + \varphi(t) \int_{-\infty}^{+\infty} \frac{C|\varphi^2(x)|}{\operatorname{sgn} x \cdot |x|^m - \lambda} dx = 0, \tag{2.3}$$

or

$$\left(1 + \int_{-\infty}^{+\infty} \frac{|\varphi^2(x)|}{\operatorname{sgn} x \cdot |x|^m - \lambda} dx\right) \varphi(t) = 0. \tag{2.4}$$

The proof is complete. □

Lemma 2.2. *Suppose that a nonnegative, monotone function* $\omega(t)$ *is semiadditive* $\omega(t_1+t_2) \le \omega(t_1)+\omega(t_2)$ *for all* $t_1, t_2 \ge 0$. *If* $\liminf_{t\to 0^+} \omega(t)/t = 0$, *then* $\omega(t) = 0$ *for all* $t \ge 0$.

Proof. According to the semiadditivity, $\omega(nt) \le n\omega(t)$, $n \in \mathbb{N}$. Therefore, using the notation $[t]$ for the integer part of t, by the monotonicity of $\omega(t)$, we have for all $t, x > 0$,

$$\begin{aligned} \omega(t) &= \omega\left(\left(\frac{t}{x}\right)x\right) \le \omega\left(\left(\left[\frac{t}{x}\right]+1\right)x\right) \le \left(\left[\frac{t}{x}\right]+1\right)\omega(x) \\ &\le \left(\frac{t}{x}+1\right)\omega(x) = (t+x)\frac{\omega(x)}{x}. \end{aligned} \tag{2.5}$$

Thus,

$$\begin{aligned} \omega(t) &\le \liminf_{x\to 0^+}(t+x)\frac{\omega(x)}{x} = \lim_{\delta\to 0^+} \inf_{0<x<\delta}(t+x)\frac{\omega(x)}{x} \\ &\le \lim_{\delta\to 0^+}(t+\delta)\inf_{0<x<\delta}\frac{\omega(x)}{x} = t\cdot\liminf_{x\to 0^+}\frac{\omega(x)}{x} = 0. \end{aligned} \tag{2.6}$$

The lemma is proved. □

Corollary 2.3. *If additionally* $\omega(t) \to 0^+$ *as* $t \to 0^+$, *then for any* $t_0 > 0$, *there exists a constant* $C > 0$ *such that*

$$Ct \le \omega(t), \quad t \in [0; t_0]. \tag{2.7}$$

Proof. It suffices to show that there exists a positive ε less than t_0 such that the inequality $Ct \le \omega(t)$ is fulfilled with a certain constant $C > 0$ for all $t \in [0;\varepsilon]$. If this is the case, put $\tilde{C} := \min\{C, \omega(\varepsilon)/t_0\}$. Then, the inequality $\tilde{C}t \le \omega(t)$ already holds for all $t \in [0; t_0]$. Indeed, using the monotonicity of ω, we have for $t \in [\varepsilon; t_0]$

$$\tilde{C}t \le \frac{\omega(\varepsilon)}{t_0}\cdot t \le \inf_{x\in[\varepsilon;t_0]}\left(\frac{\omega(x)}{x}\right)\cdot t \le \frac{\omega(t)}{t}\cdot t = \omega(t). \tag{2.8}$$

We prove that there are such constants ε and C. By assuming the converse, we construct two positive sequences C_k and t_k tending to 0 such that $\omega(t_k) \le C_k t_k$. Hence, $\lim_{k\to+\infty} \omega(t_k)/t_k = 0$. Since

$$\liminf_{t\to 0^+}\frac{\omega(t)}{t} \le \liminf_{k\to+\infty}\frac{\omega(t_k)}{t_k} = 0, \tag{2.9}$$

by Lemma 2.2, $\omega = 0$. This completes the proof. □

Remark that the nonnegativity of ω is a consequence of its semiadditivity and monotonicity as $\omega(t+0) \le \omega(t)+\omega(0)$, and hence $0 \le \omega(0) \le \omega(t)$.

Corollary 2.3 shows that it is sufficient to prove Theorem 1.1 for $\omega(t) = C_\omega t$ with an arbitrary constant $C_\omega > 0$.

Let $\{u_n\}_{n=0}^{+\infty}$ and $\{\varepsilon_n\}_{n=1}^{+\infty}$ be two sequences from the interval $(0;10^{-1})$ satisfying the condition

$$u_n < \varepsilon_n < \frac{u_{n-1}}{8}, \quad n = 1,2,\ldots. \tag{2.10}$$

On the real axis, we define a sequence of functions φ_n as follows:

$$\varphi_n(t) := \begin{cases} \omega\left(t - \dfrac{u_n}{2}\right), & t \in \left[\dfrac{1}{2}u_n; \dfrac{3}{4}u_n\right], \\ \omega(u_n - t), & t \in \left[\dfrac{3}{4}u_n; u_n\right], \\ 0, & t \notin \left[\dfrac{1}{2}u_n; u_n\right], \end{cases} \tag{2.11}$$

where $\omega(t) = C_\omega t$.

We will show that for any real-valued Lipschitz function $\gamma(t)$ compactly supported in the interval $(-\infty;-1)$, it is possible to select the sequences u_n and ε_n, and a bounded sequence of nonnegative numbers $\{c^n\}_{n=1}^{+\infty}$ such that the points $\lambda_n := (u_n + \varepsilon_n)^m$ will be eigenvalues of the operator $S_m = \operatorname{sgn} t \cdot |t|^m \cdot +(\cdot,\varphi)\varphi$ with the function

$$\varphi(t) := K \cdot \sum_{k=1}^{+\infty} (c^k)^{1/2} \varphi_k(t) + \gamma(t). \tag{2.12}$$

Here $K > 0$ is a parameter. It will be shown that for all K large enough, the function φ satisfies the required smoothness condition $|\varphi(t+h) - \varphi(t)| \le \omega(|h|)$, $t,h \in \mathbb{R}$.

Lemma 2.4. *For the points λ_n to be eigenvalues of the operator S_m defined by (1.1), it is necessary and sufficient that*

$$\int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{\operatorname{sgn} t \cdot |t|^m - \lambda_1} dt = -1, \tag{2.13}$$

$$\int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{(\operatorname{sgn} t \cdot |t|^m - \lambda_n)(\operatorname{sgn} t \cdot |t|^m - \lambda_{n+1})} dt = 0, \quad n = 1,2,\ldots. \tag{2.14}$$

Proof. We need to verify conditions (2.1) and (2.2) for $\lambda = \lambda_n$, $n = 1,2,\ldots$. As $u_n < u_{n-1}/8$, the supports of the functions φ_n are disjoint. Thus, the function φ is bounded and compactly supported. According to (2.10), $\lambda_n = (u_n + \varepsilon_n)^m < t^m$ for $t \ge u_{n-1}/2$. Therefore, φ vanishes identically in a neighborhood of λ_n. Hence, condition (2.1) is fulfilled for any $\lambda = \lambda_n$, $n = 1,2,\ldots$.

Now putting in (2.2) $\lambda = \lambda_n$, $n = 1,2,\dots$ we then pass from the obtained system of equations to an equivalent one by subtracting the nth equation from the $(n+1)$st for $n = 1,2,\dots$. As a result, we get system (2.13) and (2.14). The lemma is proved. □

Since $\varphi(t) = 0$ for $t > \lambda_1$, it follows that

$$\alpha_m := \int_{-\infty}^{+\infty} \frac{\varphi^2(t)}{\operatorname{sgn} t \cdot |t|^m - \lambda_1} dt < 0. \tag{2.15}$$

Therefore, after solving the homogeneous system (2.14), the first equality (2.13) will be satisfied by replacing the function φ by $\varphi/\sqrt{|\alpha_m|}$.

Substituting expression (2.12) for $\varphi(t)$ in (2.14), we obtain a system of linear equations for the unknowns c^n,

$$\sum_{k=1}^{n-1} (-d_{nk}c^k) + d_{nn}c^n + \sum_{k=n+1}^{+\infty} (-d_{nk}c^k) = \gamma_n, \quad n = 1,2,\dots, \tag{2.16}$$

with the coefficients

$$d_{nk} := K^2 \int_{u_k/2}^{u_k} \frac{\varphi_k^2(t)}{|(t^m - \lambda_n)(t^m - \lambda_{n+1})|} dt, \tag{2.17}$$

$$\gamma_n := \int_{-\infty}^{-1} \frac{\gamma^2(t)}{(|t|^m + \lambda_n)(|t|^m + \lambda_{n+1})} dt. \tag{2.18}$$

In the next section we show that the linear system (2.16) has a nonnegative solution in the space l_∞ of bounded sequences.

3. Solution of the linear system

Lemma 3.1. *The coefficients d_{nk} of the linear system (2.16) satisfy the following inequalities:*

$$d_{nn} \geq K^2 \frac{C_1}{\varepsilon_n^m u_n^{m-3}}, \tag{3.1}$$

$$\sum_{k=1}^{n-1} d_{nk} \leq K^2 \frac{C_2}{u_{n-1}^{2m}}, \tag{3.2}$$

$$\sum_{k=n+1}^{+\infty} d_{nk} \leq K^2 \frac{C_3 u_{n+1}^3}{\varepsilon_n^m \varepsilon_{n+1}^m}, \tag{3.3}$$

with some positive constants C_1, C_2, and C_3.

Proof. For arbitrary $a,b > 0$ and $m \geq 1$, the following inequality obviously holds:

$$|a-b| \cdot \max\{a^{m-1}, b^{m-1}\} \leq |a^m - b^m| \leq |a-b| \cdot \max\{a^{m-1}, b^{m-1}\} m. \tag{3.4}$$

Thus,

$$\frac{1}{|a-b|\cdot\max\{a^{m-1},b^{m-1}\}m} \le \frac{1}{|a^m-b^m|} \le \frac{1}{|a-b|\cdot\max\{a^{m-1},b^{m-1}\}}. \tag{3.5}$$

Using inequalities (3.5) and (2.10) with elementary estimates and a simple change of variables $x := u_n - t$, we get

$$\begin{aligned}
d_{nn} &\ge K^2\int_{3u_n/4}^{u_n}\frac{\omega^2(u_n-t)dt}{((u_n+\varepsilon_n)^m-t^m)\cdot(t^m-(u_{n+1}+\varepsilon_{n+1})^m)}\\
&\ge K^2\int_{3u_n/4}^{u_n}\omega^2(u_n-t)dt\\
&\quad\cdot\frac{1}{((u_n+\varepsilon_n)^m-(3u_n/4)^m)\cdot(u_n^m-(u_{n+1}+\varepsilon_{n+1})^m)}\\
&\ge K^2\int_0^{u_n/4}\omega^2(x)dx\\
&\quad\cdot\frac{1}{(m(u_n+\varepsilon_n-(3/4)u_n)(u_n+\varepsilon_n)^{m-1})\cdot(m(u_n-u_{n+1}-\varepsilon_{n+1})u_n^{m-1})}\\
&\ge K^2\frac{C_\omega^2}{3}\left(\frac{u_n}{4}\right)^3\cdot\frac{1}{(m(u_n/4+\varepsilon_n)(u_n+\varepsilon_n)^{m-1})\cdot(mu_n^m)}\\
&\ge K^2C\frac{u_n^3}{(u_n+\varepsilon_n)^m\cdot u_n^m}\ge K^2\frac{C_1}{\varepsilon_n^m\cdot u_n^{m-3}}.
\end{aligned} \tag{3.6}$$

Inequality (3.2) is established analogously,

$$\begin{aligned}
\sum_{k=1}^{n-1}d_{nk} &= \sum_{k=1}^{n-1}K^2\int_{u_k/2}^{u_k}\frac{\varphi_k^2(t)dt}{(t^m-(u_n+\varepsilon_n)^m)\cdot(t^m-(u_{n+1}+\varepsilon_{n+1})^m)}\\
&\le K^2\sum_{k=1}^{n-1}\int_{u_k/2}^{u_k}\frac{\varphi_k^2(t)dt}{((u_{n-1}/2)^m-(u_n+\varepsilon_n)^m)\cdot((u_{n-1}/2)^m-(u_{n+1}+\varepsilon_{n+1})^m)}\\
&\le K^2\int_0^{u_1}\left(\sum_{k=1}^{+\infty}\varphi_k^2(t)\right)dt\frac{1}{((u_{n-1}/2-u_n-\varepsilon_n)(u_{n-1}/2)^{m-1})}\\
&\quad\times\frac{1}{((u_{n-1}/2-u_{n+1}-\varepsilon_{n+1})(u_{n-1}/2)^{m-1})}\\
&\le K^2C\frac{1}{(u_{n-1}/2-u_{n-1}/8-u_{n-1}/8)^2\cdot(u_{n-1}/2)^{2(m-1)}}\le K^2\frac{C_2}{u_{n-1}^{2m}}.
\end{aligned} \tag{3.7}$$

We took into account that the functions $\varphi_k(t)$ are uniformly bounded and have disjoint supports, therefore $\int_0^{u_1}(\sum_{k=1}^{+\infty}\varphi_k^2(t))dt \le C$. Finally, we have

$$\begin{aligned}
&\sum_{k=n+1}^{+\infty} d_{nk}\\
&\quad= K^2\sum_{k=n+1}^{+\infty}\int_{u_k/2}^{u_k}\frac{\varphi_k^2(t)dt}{((u_n+\varepsilon_n)^m-t^m)\cdot((u_{n+1}+\varepsilon_{n+1})^m-t^m)}\\
&\quad\le K^2\sum_{k=n+1}^{+\infty}\int_{u_k/2}^{u_k}\frac{\varphi_k^2(t)dt}{((u_n+\varepsilon_n)^m-u_{n+1}^m)\cdot((u_{n+1}+\varepsilon_{n+1})^m-u_{n+1}^m)}\\
&\quad\le K^2\sum_{k=n+1}^{+\infty}\int_{u_k/2}^{u_k}\frac{\omega^2(u_k-t)dt}{\big((u_n+\varepsilon_n-u_{n+1})(u_n+\varepsilon_n)^{m-1}\big)\cdot\big(\varepsilon_{n+1}(u_{n+1}+\varepsilon_{n+1})^{m-1}\big)}\\
&\quad\le K^2\sum_{k=n+1}^{+\infty}\int_{u_k/2}^{u_k}\frac{\omega^2(u_{n+1}-t)dt}{\big((u_n+\varepsilon_n-u_n/8)(u_n+\varepsilon_n)^{m-1}\big)\cdot\varepsilon_{n+1}(u_{n+1}+\varepsilon_{n+1})^{m-1}}\\
&\quad\le K^2\int_0^{u_{n+1}}\omega^2(u_{n+1}-t)dt\frac{C}{(u_n+\varepsilon_n)^m\cdot\varepsilon_{n+1}(u_{n+1}+\varepsilon_{n+1})^{m-1}}\\
&\quad\le K^2u_{n+1}^3\frac{C_3}{\varepsilon_n^m\cdot\varepsilon_{n+1}^m}.
\end{aligned}\tag{3.8}$$

The lemma is proved. □

We rewrite system (2.16) in matrix form

$$(I+A)\vec{c}=f,\tag{3.9}$$

where the column vectors $\vec{c}=(c^1,c^2,\dots)^T$, $f=(\gamma_1/d_{11},\gamma_2/d_{22},\dots)^T$, and the infinite matrix A has the entries $(A)_{nk}=(\delta_{nk}-1)\cdot d_{nk}/d_{nn}$. Equation (3.9) will be considered in the Banach space l_∞.

In the sequel, we consider the sequences u_n and ε_n defined as follows:

$$u_n=u_{n-1}^{\alpha},\quad \varepsilon_n=u_{n-1}^{\beta},\quad n=1,2,\dots,\tag{3.10}$$

with some $u_0\in(0;10^{-1})$ and $\alpha>\beta>2$. It is evident that inequality (2.10) is fulfilled.

Lemma 3.2. *For every $m>3/2$, the numbers α and β can be found satisfying the inequality $\alpha>\beta>2$ such that the following estimate:*

$$d_{nn}\ge K^2C_1\tag{3.11}$$

holds for all $n=1,2,\dots$.

Proof. Substituting (3.10) in (3.1), we get

$$d_{nn} \geq K^2 \frac{C_1}{u_{n-1}^{\beta m+\alpha(m-3)}}. \tag{3.12}$$

As $u_{n-1} \in (0;1)$ for $m \geq 3$, the assertion of the lemma is true with any $\alpha > \beta > 2$. Now let $m \in (3/2;3)$. Then, the inequality $\beta m + \alpha(m-3) > 0$ is fulfilled if $\alpha < \beta m/(3-m)$. It must be $\alpha > \beta > 2$, therefore we require that $\beta m/(3-m) > \beta$. As the last inequality holds for any $\beta > 0$, we can take for $m \in (3/2;3)$ any

$$\beta > 2, \qquad \alpha \in \left(\beta; \frac{\beta m}{(3-m)}\right). \tag{3.13}$$

This completes the proof. □

By (2.18), $\gamma_n > 0$, and

$$\gamma_n = \int_{-\infty}^{-1} \frac{|\gamma(t)|^2 dt}{(|t|^m+\lambda_n)(|t|^m+\lambda_{n+1})} \leq \int_{-\infty}^{-1} |\gamma(t)|^2 dt \equiv \|\gamma\|_2^2. \tag{3.14}$$

Hence, for the l_∞-norm of the vector f equal to $\|f\|_\infty = \sup_n(\gamma_n/d_{nn})$, we obtain the following estimate:

$$\|f\|_\infty \leq \frac{\|\gamma\|_2^2}{K^2 C_1}. \tag{3.15}$$

Lemma 3.3. *For every $m > 3/2$, the numbers α and β can be found satisfying the inequality $\alpha > \beta > 2$ such that $\|A\| < 1$ for all u_0 small enough.*

Proof. By the definition,

$$\begin{aligned} \|A\| &= \sup_{\|s\|_\infty=1} \|As\|_\infty \leq \sup_{\|s\|_\infty=1} \sup_n \sum_{k=1}^{+\infty} |A_{nk}s_k| \\ &\leq \sup_n \sum_k |A_{nk}| = \sup_n \sum_{k\neq n} \frac{d_{nk}}{d_{nn}} \\ &= \sup_n \left(\sum_{k=1}^{n-1} \frac{d_{nk}}{d_{nn}} + \sum_{k=n+1}^{+\infty} \frac{d_{nk}}{d_{nn}} \right). \end{aligned} \tag{3.16}$$

Hence, it suffices to require that

$$S_1 := \frac{\left(\sum_{k=1}^{n-1} d_{nk}\right)}{d_{nn}} < \frac{1}{4}, \qquad S_2 := \frac{\left(\sum_{k=n+1}^{+\infty} d_{nk}\right)}{d_{nn}} < \frac{1}{4}. \tag{3.17}$$

Using (3.1), (3.2), (3.3), and (3.10), we get

$$S_1 \le \frac{C_2}{C_1} \cdot \frac{\varepsilon_n^m u_n^{m-3}}{u_{n-1}^{2m}} = \frac{C_2}{C_1} \cdot u_{n-1}^{\alpha(m-3)+(\beta-2)m},$$
$$S_2 \le \frac{C_3}{C_1} \cdot \frac{u_n^{m-3} u_{n+1}^3}{\varepsilon_{n+1}^m} = \frac{C_3}{C_1} \cdot u_n^{3\alpha-m\beta+m-3}. \tag{3.18}$$

Thus, if there exist $\alpha > \beta > 2$ such that

$$\alpha(m-3)+(\beta-2)m > 0, \qquad 3\alpha - m\beta + m - 3 > 0, \tag{3.19}$$

then

$$S_1 \le \left(\frac{C_2}{C_1}\right) \cdot u_0^{\alpha(m-3)+(\beta-2)m}, \qquad S_2 \le \left(\frac{C_3}{C_1}\right) \cdot u_0^{3\alpha-m\beta+m-3}. \tag{3.20}$$

Therefore, (3.17) will be true for all u_0 small enough.

Let $m \ge 3$. The first inequality in (3.19) holds for all $\alpha > \beta > 2$. The second one is true for $\alpha > (\beta m + 3 - m)/3$. It follows that we can take any $\beta > 2$ and $\alpha > \max\{\beta, (\beta m + 3 - m)/3\} = (\beta m + 3 - m)/3$. Note that this choice of α and β is consistent with the choice made for $m \ge 3$ in Lemma 3.2.

Now let $m \in (3/2; 3)$. Then, (3.19) is equivalent to

$$\alpha < \frac{(\beta m - 2m)}{(3-m)}, \qquad \alpha > \frac{(\beta m + 3 - m)}{3}. \tag{3.21}$$

As it must be $\alpha > \beta$, find β such that

$$\frac{(\beta m + 3 - m)}{3} < \beta < \frac{(\beta m - 2m)}{(3-m)}. \tag{3.22}$$

The inequality $(\beta m + 3 - m)/3 < \beta$ is fulfilled for all $\beta > 1$. The inequality $\beta < (\beta m - 2m)/(3-m)$ is equivalent to $\beta > 2m/(2m-3)$. Hence, for $m \in (3/2; 3)$, we can take any $\beta > \max\{2; 2m/(2m-3)\} = 2m/(2m-3)$ and $\alpha \in (\beta; (\beta m - 2m)/(3-m))$. We also see that α and β satisfy (3.13). This completes the proof. □

By virtue of the inequality $\|A\| < 1$, equation (3.9) has a unique solution in l_∞,

$$\vec{c} = (I + A)^{-1} f. \tag{3.23}$$

It is easily seen that all the components c^n of the vector $\vec{c} = (c^1, c^2, \dots)^T$ are nonnegative. Indeed, $f_n = \gamma_n / d_{nn} > 0$ since, by (2.17) and (2.18), $\gamma_n, d_{nn} > 0$. The Neumann series $\vec{c} = (I + (-A) + (-A)^2 + \cdots) f$ includes powers of a matrix with nonnegative entries such that $(-A)_{nk} = (1 - \delta_{nk}) \cdot d_{nk}/d_{nn} \geq 0$ since $d_{nk} \geq 0$. Consequently, the vector $\vec{c}$ is obtained as a result of the action of a matrix with nonnegative entries on a vector whose components are also nonnegative.

4. Smoothness of the function φ. Introducing the convenient notations

$$u_n^1 := u_n, \qquad u_n^2 := \frac{u_n}{2}, \qquad \Delta_n := [u_n^2; u_n^1],$$
$$\Delta_n^2 := \left[u_n^2; \frac{(u_n^1 + u_n^2)}{2}\right], \qquad \Delta_n^1 := \left[\frac{(u_n^1 + u_n^2)}{2}; u_n^1\right], \tag{4.1}$$

we can write (2.11) as follows:

$$\varphi_n(t) := \begin{cases} \omega(t - u_n^2), & t \in \Delta_n^2, \\ \omega(u_n^1 - t), & t \in \Delta_n^1, \\ 0, & t \notin \Delta_n. \end{cases} \tag{4.2}$$

Lemma 4.1. *Suppose that ω is monotone and semiadditive (and thus nonnegative). Then, the function*

$$\psi(t) := \sum_{k=1}^{+\infty} (c^k)^{1/2} \varphi_k(t) \tag{4.3}$$

satisfies the following smoothness condition:

$$|\psi(t+h) - \psi(t)| \leq \sup_k (c^k)^{1/2} \omega(|h|), \quad t, h \in \mathbb{R}. \tag{4.4}$$

Inequality (4.4) remains true if $\{\Delta_n\}_{n=1}^{+\infty}$ is an arbitrary sequence of disjoint finite intervals.

Proof. If $t_1 \in \Delta_n$ and $t_2 \notin \Delta_n$, then, as the intervals Δ_k are disjoint for different k, for some $i \in \{1, 2\}$, we have

$$\psi(t_1) = (c^n)^{1/2} \cdot \omega(|t_1 - u_n^i|) \leq (c^n)^{1/2} \cdot \omega(|t_1 - t_2|). \tag{4.5}$$

Thus,

$$|\psi(t_2) - \psi(t_1)| \leq \max\{\psi(t_2), \psi(t_1)\} \leq \sup_k (c^k)^{1/2} \cdot \omega(|t_1 - t_2|). \tag{4.6}$$

For $t,\delta > 0$, $\omega(t+\delta) - \omega(t) \le \omega(\delta)$ according to the semiadditivity of ω. Therefore, for $t_1, t_2 \in \Delta_n^i$, we obtain

$$\begin{aligned} |\psi(t_1) - \psi(t_2)| &= (c^n)^{1/2} \cdot |\omega(|t_1 - u_n^i|) - \omega(|t_2 - u_n^i|)| \\ &\le (c^n)^{1/2} \cdot \omega(|t_1 - t_2|). \end{aligned} \tag{4.7}$$

Now, let $t_1 \in \Delta_n^i$ and $t_2 \in \Delta_n^j$, $i \ne j$. Assume that $|t_1 - u_n^i| \ge |t_2 - u_n^j|$, then

$$\begin{aligned} |\psi(t_1) - \psi(t_2)| &= (c^n)^{1/2} \cdot (\omega(|t_1 - u_n^i|) - \omega(|t_2 - u_n^j|)) \\ &\le (c^n)^{1/2} \cdot (\omega(|t_1 - u_n^j|) - \omega(|t_2 - u_n^j|)) \\ &\le (c^n)^{1/2} \cdot \omega(|t_1 - t_2|). \end{aligned} \tag{4.8}$$

The lemma is proved. □

By (2.12), the function $\varphi(t) = K\psi(t) + \gamma(t)$ where the function γ is assumed to satisfy a Lipschitz condition $|\gamma(t+h) - \gamma(t)| \le C_\gamma |h|$, $h \in \mathbb{R}$. Since the functions ψ and γ have disjoint supports and $\omega(t) = C_\omega t$, we see that the following inequality:

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| &\le K \sup_k (c^k)^{1/2} \cdot \omega(|h|) + C_\gamma |h| \\ &\le \left(K \sup_k (c^k)^{1/2} + \frac{C_\gamma}{C_\omega} \right) \omega(|h|) \end{aligned} \tag{4.9}$$

holds for all $t, h \in \mathbb{R}$.

After finding the solution of system (2.14), we satisfy (2.13) replacing the function φ by $\varphi/\sqrt{|\alpha_m|}$. This replacement corresponds to a passage from the functions ψ and γ to $\psi/\sqrt{|\alpha_m|}$ and $\gamma/\sqrt{|\alpha_m|}$, respectively. Therefore, for the new function φ, the following smoothness condition will be fulfilled:

$$|\varphi(t+h) - \varphi(t)| \le \left(K \sup_k (c^k)^{1/2} |\alpha_m|^{-1/2} + \frac{C_\gamma |\alpha_m|^{-1/2}}{C_\omega} \right) \omega(|h|), \quad h \in \mathbb{R}. \tag{4.10}$$

Lemma 4.2. *The constant*

$$M := K \sup_k (c^k)^{1/2} |\alpha_m|^{-1/2} + \frac{C_\gamma |\alpha_m|^{-1/2}}{C_\omega}, \tag{4.11}$$

in the smoothness condition (4.10), satisfies the inequality $M \le C/K$, and hence, it is less than one for all K large enough.

Proof. Since

$$
\begin{aligned}
|\alpha_m| &= \int_{-\infty}^{+\infty} \frac{\varphi^2(t)}{|\operatorname{sgn} t \cdot |t|^m - \lambda_1|} dt \\
&\geq K^2 \int_0^1 \frac{\psi^2(t)}{\lambda_1 - t^m} dt \geq K^2 \int_0^1 \psi^2(t) dt \equiv K^2 \|\psi\|_2^2,
\end{aligned}
\tag{4.12}
$$

it follows that $|\alpha_m|^{-1/2} \leq 1/(K\|\psi\|_2)$. From (3.23), we obtain the inequality

$$
\|\vec{c}\|_\infty \leq (1 - \|A\|)^{-1} \|f\|_\infty. \tag{4.13}
$$

In view of relation (3.15), then

$$
\|\vec{c}\|_\infty \leq (1 - \|A\|)^{-1} \frac{\|\gamma\|_2^2}{K^2 C_1}. \tag{4.14}
$$

Therefore, for the first summand in (4.11), we get

$$
K \sup_k (c^k)^{1/2} |\alpha_m|^{-1/2} \leq K(1 - \|A\|)^{-1/2} \frac{\|\gamma\|_2}{K C_1^{1/2}} \cdot \frac{1}{K\|\psi\|_2} \leq \frac{C}{K}. \tag{4.15}
$$

Likewise, we find that

$$
\frac{C_\gamma |\alpha_m|^{-1/2}}{C_\omega} \leq \frac{C_\gamma}{C_\omega} \cdot \frac{1}{K\|\psi\|_2} \leq \frac{C}{K}. \tag{4.16}
$$

The lemma is proved. □

As a result, the function φ satisfies the smoothness condition

$$
|\varphi(t+h) - \varphi(t)| \leq \omega(|h|), \quad h \in \mathbb{R}, \tag{4.17}
$$

for all K large enough. Theorem 1.1 is, thus, completely proved.

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