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ASYMPTOTIC ALMOST PERIODICITY OF C-SEMIGROUPS

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Let $\{T(t)\}_{t\geq 0}$ be a C-semigroup on a Banach space X with generator A. We will investigate the asymptotic almost periodicity of $\{T(t)\}$ via the Hille-Yosida space of its generator.

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1. Introduction. Motivated by the abstract Cauchy problem

$$\frac{d}{dt}u(t) = Au(t) \quad (t \geq 0),\ u(0) = x, \tag{1.1}$$

a generalization of strongly continuous semigroups, C-semigroups, has recently received much attention (see [6, 7, 17]). The operator A generating a C-semigroup leads to (1.1) having a unique solution, whenever $x = Cy$ for some y in the domain of A. It is well known that the class of operators that generate C-semigroups is much larger than the class of operators that generate strongly continuous semigroups.

On the other hand, the asymptotic almost periodicity of C_0-semigroup has been studied systematically in [2, 4, 8, 11, 12, 14, 15, 16]. It was shown that when A generates an asymptotically almost periodic C_0-semigroup on a Banach space X, then X can be decomposed into the direct sum of two subspaces X_a and X_s, and the mild solutions with initial values taken from X_a are almost periodic and thus can be extended to the whole line while the mild solutions from X_s are vanishing at infinity.

In this paper, we will discuss the asymptotic almost periodicity of C-semigroups. We show that if A generates an asymptotically almost periodic C-semigroup, then the range of C has an analogous decomposition. The technique we use here is the Hille-Yosida space for A, which is a maximal imbedded subspace of X such that the part of A on this subspace generates a C_0-semigroup. The crucial facts are that a mild solution of the abstract Cauchy problem is asymptotically almost periodic in the Hille-Yosida space if and only if it is asymptotically almost periodic in X, and the mild solution is vanishing at infinity in the Hille-Yosida space if and only if it is vanishing in X.

Under suitable spectral conditions, we obtain a theorem for asymptotic almost periodicity of C-semigroups that is more easily testified (Theorem 3.7).

At last, we give a theorem for asymptotically almost periodic integrated semigroups (Theorem 3.9).

Although there are papers devoted to asymptotic properties of individual solution [1, 2, 3, 4], our main concern is some kind of global property.

Throughout the paper, all operators are linear. We write $D(A)$ for the domain of an operator A, $R(A)$ for the range, and $\rho(A)$ for the resolvent set; X will always be a Banach space, and the space of all bounded linear operators on X will be denoted by $B(X)$ while C will always be a bounded, injective operator on X. Finally, $\mathbb{R}^+$ will be the half-line $[0,+\infty)$, and $\mathbb{C}^+$ will be the half-plane $\{z \mid z=\tau+iw,\ \tau,w\in\mathbb{R} \text{ and } \tau>0\}$.

2. Preliminaries.

We start with the definitions and properties of C-semigroups.

Definition 2.1. A family $\{T(t)\}_{t\geq 0}\subset B(X)$ is a *C-semigroup* if

(a) $T(0)=C$,

(b) the map $t\mapsto T(t)x$, from $[0,+\infty)$ into X, is continuous for all $x\in X$,

(c) $CT(t+s)=T(t)T(s)$.

The *generator* of $\{T(t)\}_{t\geq 0}$, A is defined by

$$Ax = C^{-1}\left[\lim_{t\downarrow 0}\frac{1}{t}\big(T(t)x-Cx\big)\right] \tag{2.1}$$

with

$$D(A)=\left\{x\in X \mid \lim_{t\downarrow 0}\frac{1}{t}\big(T(t)x-Cx\big) \text{ exists and is in } R(C)\right\}. \tag{2.2}$$

The complex number λ is in $\rho_C(A)$, the *C-resolvent set* of A, if $(\lambda-A)$ is injective and $R(C)\subseteq R(\lambda-A)$. And we denote by $\sigma_C(A)$ the set of all points in complex plane which are not in the C-resolvent of A.

For the basic properties of C-semigroups and their generators, we refer to [7]. Next, we introduce the Hille-Yosida space for an operator.

Definition 2.2. Suppose that A has no eigenvalues in $(0,\infty)$ and is a closed linear operator. The *Hille-Yosida space for* A, Z_0 is defined by $Z_0=\{x\in X \mid$ The Cauchy problem (1.1) has a bounded uniformly continuous mild solution $u(\cdot,x)\}$ with

$$\|x\|_{Z_0}=\sup\{\|u(t,x)\|:\ t\geq 0\} \quad \text{for } x\in Z_0. \tag{2.3}$$

Lemma 2.3 (see [7]). *Suppose that A generates a strongly uniformly continuous bounded C-semigroup $\{T(t)\}_{t\geq 0}$ on X. Then, the Hille-Yosida space for*

A, Z_0 is a Banach space and

$$Z_0 = \{x \mid t \longrightarrow C^{-1}T(t)x \text{ is uniformly continuous and bounded}\} \tag{2.4}$$

with

$$\|x\|_{Z_0} = \sup\{\|C^{-1}T(t)x\|:\ t \geq 0\}. \tag{2.5}$$

Moreover, $Z_0 \hookrightarrow X$, that is, Z_0 can be continuously imbedded in X, $A|_{Z_0}$ generates a contraction C_0-semigroup $S(t) = C^{-1}T(t)$ on Z_0, and $T(t)x = S(t)Cx$ for all $x \in X$.

Let $J = \mathbb{R}^+$ or $\mathbb{R}$. The spaces of all bounded continuous functions from J into X will be denoted by $C_b(J,X)$, $C_0(\mathbb{R}^+,X)$ will designate the set of those $\varphi \in C_b(\mathbb{R}^+,X)$ that vanish at infinity on $\mathbb{R}^+$, and we will hereafter assume that each of these spaces is equipped with the supremum norm. Moreover, for $f \in C_b(J,X)$, $w \in J$, we put $f_w(t) = f(t+w)$ $(t \in J)$ and let $H(f) = \{f_w : w \in J\}$ denote the set of all translates of f.

Definition 2.4. (a) A function $f \in C(J,X)$ is said to be *almost periodic* if for every $\varepsilon > 0$, there exists $l > 0$ such that every subinternal of J of length l contains, at least, one τ satisfying $\|f(t+\tau) - f(t)\| \leq \varepsilon$ for $t \in J$. The space of all almost periodic functions will be denoted by $\mathrm{AP}(J,X)$.

(b) A function $f \in C_b(\mathbb{R}^+,X)$ is said to be *asymptotically almost periodic* if for every $\varepsilon > 0$, there exist $l > 0$ and $M > 0$ such that every subinterval of $\mathbb{R}^+$ of length l contains, at least, one τ satisfying $\|f(t+\tau) - f(t)\| \leq \varepsilon$ for all $t \geq M$. The space of all asymptotically almost periodic functions will be denoted by $\mathrm{AAP}(\mathbb{R}^+,X)$.

Lemma 2.5 (see [10, 15, 16]). *For a function $f \in C(\mathbb{R}^+,X)$, the following statements are equivalent:*

(a) $f \in \mathrm{AAP}(\mathbb{R}^+,X)$*;*

(b) *there exist uniquely determined functions $g \in \mathrm{AP}(\mathbb{R},X)$ and $\varphi \in C_0(\mathbb{R}^+,X)$ such that $f = g|_{\mathbb{R}^+} + \varphi$;*

(c) *$H(f)$ is relatively compact in $C_b(\mathbb{R}^+,X)$.*

Definition 2.6. Let $\{F(t)\}_{t\in J} \subseteq B(X)$ be a strongly continuous operator family.

(a) $F(t)$ $(t \in J)$ is *almost periodic* if $F(\cdot)x$ is almost periodic for every $x \in X$.

(b) $F(t)$ $(t \in \mathbb{R}^+)$ is *asymptotically almost periodic* if $F(\cdot)x$ is asymptotically almost periodic for every $x \in X$.

3. Main results. In order to characterize the asymptotic almost periodicity of C-semigroups, we need the following result.

Lemma 3.1. *Assume that* $\{T(t)\}_{t\geq 0}$ *is a* C*-semigroup on* X *generated by* A *and* A *has no eigenvalues in* $(0,\infty)$. *Suppose that* $T(\cdot)x : \mathbb{R}^+ \mapsto X$ *is asymptotically almost periodic for some* $x \in X$. *Then,*

(a) *there exist* $y \in X$ *and* $\varphi \in C_0(\mathbb{R}^+, X)$ *such that, for all* $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in \mathrm{AP}(\mathbb{R}^+, X)$, *and*

$$T(t)x = C^{-1}T(t)y + \varphi(t); \tag{3.1}$$

(b) *if* $\{T(t)\}$ *is strongly uniformly continuous and bounded, then there exist* $y, z \in Z_0$ *such that* $S(\cdot)y \in \mathrm{AP}(\mathbb{R}^+, Z_0)$, $S(\cdot)z \in C_0(\mathbb{R}^+, Z_0)$, *and*

$$T(t)x = S(t)y + S(t)z \quad \forall t \geq 0, \tag{3.2}$$

where Z_0 *is the Hille-Yosida space for* A *and* $\{S(t)\}$ *is the* C_0*-semigroup generated by* $A|_{Z_0}$.

Proof. (a) By Lemma 2.5, there exist uniquely determined functions $h \in \mathrm{AP}(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that $T(\cdot)x = h|_{\mathbb{R}^+} + \varphi$, and, from the proof of Lemma 2.5 (see [14]), we know that there exists $0 < t_n \to \infty$ such that

$$h(t) = \lim_{n\to\infty} T(t + t_n)x \quad \forall t \geq 0. \tag{3.3}$$

Let $y = h(0)$, then we have $\lim_{n\to\infty} T(t_n)x = y$. Therefore, for each $t \geq 0$,

$$Ch(t) = \lim_{n\to\infty} CT(t+t_n)x = T(t)\lim_{n\to\infty} T(t_n)x = T(t)y, \tag{3.4}$$

which implies $T(t)y \in R(C)$ for all $t \geq 0$ and $C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \mathrm{AP}(\mathbb{R}^+, X)$, so that

$$T(t)x = C^{-1}T(t)y + \varphi(t) \quad \forall t \geq 0. \tag{3.5}$$

(b) Let y and $\varphi(t)$ be the same as in (a). Since $C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \mathrm{AP}(\mathbb{R}^+, X)$, we have that $C^{-1}T(t)y$ is uniformly continuous and bounded in $[0, +\infty)$, and so it follows that $y \in Z_0$ by Lemma 2.3. Setting $z = Cx - y$, then obviously, $z \in Z_0$. Hence, $S(t)y = C^{-1}T(t)y$ and

$$S(t)z = C^{-1}T(t)z = C^{-1}T(t)Cx - C^{-1}T(t)y = T(t)x - C^{-1}T(t)y = \varphi(t) \tag{3.6}$$

so that

$$T(t)x = S(t)y + S(t)z \quad \forall t \geq 0. \tag{3.7}$$

Next, we show that $S(\cdot)y \in \mathrm{AP}(\mathbb{R}^+, Z_0)$ and $S(\cdot)z \in C_0(\mathbb{R}^+, Z_0)$. By the definition of almost periodicity, we know, for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of $\mathbb{R}^+$ of length l contains, at least, one τ satisfying

$$\sup_{t\geq 0} ||S(t+\tau)y - S(t)y|| < \varepsilon, \tag{3.8}$$

hence,

$$\begin{aligned}
&\sup_{t\geq 0}\|S(t+\tau)y-S(t)y\|_{Z_0}\\
&\qquad= \sup_{t\geq 0,\, s\geq 0}\|C^{-1}T(s)S(t+\tau)y-C^{-1}T(s)S(t)y\|\\
&\qquad= \sup_{t\geq 0,\, s\geq 0}\|C^{-1}T(s)C^{-1}T(t+\tau)y-C^{-1}T(s)C^{-1}T(t)y\|\\
&\qquad= \sup_{t\geq 0,\, s\geq 0}\|C^{-1}T(t+s+\tau)y-C^{-1}T(t+s)y\|\\
&\qquad= \sup_{t\geq 0,\, s\geq 0}\|S(t+s+\tau)y-S(t+s)y\|\\
&\qquad= \sup_{t\geq 0}\|S(t+\tau)y-S(t)y\|<\varepsilon,
\end{aligned}\tag{3.9}$$

which yields $S(t)y\in \mathrm{AP}(\mathbb{R}^+,Z_0)$. Also,

$$\|S(t)z\|_{Z_0}=\sup_{s\geq 0}\|C^{-1}T(s)S(t)z\|=\sup_{s\geq 0}\|S(t+s)z\|=\sup_{s\geq t}\|S(s)z\| \tag{3.10}$$

converges to 0 as $t\to+\infty$ since $S(t)z\in C_0(\mathbb{R}^+,X)$, and this yields $S(t)z\in C_0(\mathbb{R}^+,Z_0)$. □

Now, we can prove the following theorem which characterizes the asymptotic almost periodicity of C-semigroups.

Theorem 3.2. *Let $\{T(t)\}$ be a C-semigroup generated by A on X. Then, $\{T(t)\}$ is asymptotically almost periodic if and only if $R(C)\subseteq X_{0a}+X_{0s}$, where $X_{0a}=\{x\mid x\in Z_0,\ S(t)x\in \mathrm{AP}(\mathbb{R}^+,Z_0)\}$ and $X_{0s}=\{x\mid x\in Z_0,\ S(t)x\in C_0(\mathbb{R}^+,Z_0)\}$, where Z_0 is the Hille-Yosida space for A, $\{S(t)\}$ is the C_0-semigroup generated by $A|_{Z_0}$.*

Proof. Necessity. First, it follows from [17, Lemma 1.6.(a)] and the uniform boundedness theorem that $\{T(t)\}$ is bounded and strongly uniformly continuous. By Lemma 3.1(b), for every $x\in X$, there exist $y,z\in Z_0$ such that $S(\cdot)y\in \mathrm{AP}(\mathbb{R}^+,Z_0)$, $S(\cdot)z\in C_0(\mathbb{R}^+,Z_0)$, and $T(t)x=S(t)y+S(t)z$ for all $t\geq 0$. Choosing $t=0$, we obtain $Cx=y+z$, that is, $R(C)\subseteq X_{0a}+X_{0s}$.

Sufficiency. Since $R(C)\subseteq X_{0a}+X_{0s}$, for any $x\in X$, there exist $y\in X_{0a}$ and $z\in X_{0s}$ such that $Cx=y+z$. Hence, by Lemma 2.3, we have $T(t)x=S(t)Cx=S(t)y+S(t)z$ while $S(t)y\in \mathrm{AP}(\mathbb{R}^+,Z_0)$ and $S(t)z\in C_0(\mathbb{R}^+,Z_0)$. Since Z_0 is continuously imbedded in X, it follows that $T(t)x$ is asymptotically almost periodic by Lemma 2.5, that is, $\{T(t)\}$ is an asymptotically almost periodic C-semigroup. □

From the proof of Lemma 3.1(b), we know that for $y,z\in Z_0$, $S(t)y\in \mathrm{AP}(\mathbb{R}^+,X)$ and $S(t)z\in C_0(\mathbb{R}^+,X)$ if and only if $S(t)y\in \mathrm{AP}(\mathbb{R}^+,Z_0)$ and $S(t)z\in C_0(\mathbb{R}^+,Z_0)$, respectively. Hence, we have the following corollary.

Corollary 3.3. *Let $\{T(t)\}_{t\geq 0}$ be a C-semigroup generated by A on X. Then, $\{T(t)\}$ is asymptotically almost periodic if and only if for any $x\in X$, there* *exist $y, z \in Z_0$ such that $Cx = y + z$, $C^{-1}T(t)y \in \mathrm{AP}(\mathbb{R}^+, X)$, and $C^{-1}T(t)z \in C_0(\mathbb{R}^+, X)$, where Z_0 is the Hille-Yosida spaces for A.*

In the case of $\overline{R(C)} = X$, we have the following theorem.

Theorem 3.4. *Assume that $\{T(t)\}_{t\geq 0}$ is a C-semigroup on X and $\overline{R(C)} = X$. Then, $\{T(t)\}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_a + X_s$, where $X_a = \{x \mid x \in X,\ T(t)x \in \mathrm{AP}(\mathbb{R}^+, X)\}$ and $X_s = \{x \mid x \in X,\ T(t)x \in C_0(\mathbb{R}^+, X)\}$.*

Proof. Necessity. By Lemma 3.1(a), for every $x \in X$, there exist $y \in X$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that for all $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in \mathrm{AP}(\mathbb{R}^+, X)$, and

$$T(t)x = C^{-1}T(t)y + \varphi(t). \tag{3.11}$$

It follows that $T(t)Cx = CT(t)x = T(t)y + C\varphi(t)$. Setting $z = Cx - y$, then $T(t)z = T(t)Cx - T(t)y = C\varphi(t) \in C_0(\mathbb{R}^+, X)$; on the other hand, $T(\cdot)y \in \mathrm{AP}(\mathbb{R}^+, X)$ since C is bounded. So, we have $R(C) \subseteq X_a + X_s$.

Sufficiency follows from the fact that $R(C)$ is dense in X and $\mathrm{AAP}(\mathbb{R}^+, X)$ is closed in the space of all bounded uniformaly continuous functions from $\mathbb{R}^+$ to X. □

The following result clarifies the relations between the generator of an asymptotically almost periodic C-semigroup and of an asymptotically almost periodic C_0-semigroup.

Theorem 3.5. *Suppose that A is closed and has no eigenvalues in $(0, \infty)$, and assume that $C^{-1}AC = A$. Then, A generates an asymptotically almost periodic C-semigroup if and only if $R(C) \subseteq Z_{\mathrm{aap}} \equiv (Z_0)_a + (Z_0)_s$, where $(Z_0)_a = \{x \mid x \in X$, the Cauchy problem (1.1) has an almost periodic mild solution $u(\cdot, x)\}$ and $(Z_0)_s = \{x \mid x \in X$, the Cauchy problem (1.1) has a mild solution $u(\cdot, x) \in C_0(\mathbb{R}^+, X)\}$. And Z_{aap} is the maximal continuously imbedded subspace on which A generates an asymptotically almost periodic C_0-semigroup.*

Proof. Necessity holds by Lemma 3.1 and the relations between C-semigroup and solutions of the corresponding Cauchy problem [7, Theorem 3.13].

Sufficiency. By Definition 2.2, we know that both $(Z_0)_a$ and $(Z_0)_s$ are contained in Z_0; so $R(C) \subseteq Z_0$. Thus, by [7, Theorem 5.17] and [9, Corollary 3.14], A generates a bounded C-semigroup; since all mild solutions with initial data taken from $R(C)$ are asymptotically almost periodic, so is the C-semigroup.

Now, suppose that $Y \hookrightarrow X$ and $A|_Y$ generates a contraction asymptotically almost periodic C_0-semigroup, then $Y \hookrightarrow Z_0$ since Z_0 is maximal (cf. [7, Theorem 5.5]). So, $Y \hookrightarrow Z_{aap}$ follows from the fact that the asymptotic almost periodicity of the mild solution of the abstract Cauchy problem in Z_0 is equivalent to the same property in X, and the mild solution $u(t)$ converges to 0 as $t \to \infty$ in X is equivalent to $u(t) \to 0$ in Z_0. □

Remark 3.6. (a) $(Z_0)_a$ is the maximal subspace on which A generates an almost periodic C_0-semigroup.

(b) From the proof of Lemma 3.1 and Theorems 3.2 and 3.5, it is clear that $(Z_0)_a = X_{0a}$, $(Z_0)_s = X_{0s}$.

When $\sigma_C(A) \cap i\mathbb{R}$ is countable, we obtain a result which is more easily testified for asymptotic almost periodicity of C-semigroups.

Let $f : \mathbb{R}^+ \to X$ be strongly measurable, and let $\widetilde{f}$ be the Laplace transform of f,

$$\widetilde{f}(z) = \int_0^{+\infty} e^{-tz} f(t)\,dt. \tag{3.12}$$

We assume that $\widetilde{f}(z)$ exists for all z in $\mathbb{C}^+$, so $\widetilde{f}$ is holomorphic in $\mathbb{C}^+$ (usually, f will be bounded). A point $\lambda = i\eta$ in $i\mathbb{R}$ is said to be a regular point for $\widetilde{f}$ if there is an open neighborhood U of λ in $\mathbb{C}$ and a holomorphic function $g : U \to X$ such that $g(z) = \widetilde{f}(z)$ whenever $z \in U \cap \mathbb{C}^+$. The singular set E of $\widetilde{f}$ is the set of all points of $i\mathbb{R}$ which are not regular points.

Theorem 3.7. *Let $\{T(t)\}_{t\ge 0}$ be a C-semigroup generated by A in X, and let $\sigma_C(A) \cap i\mathbb{R}$ be countable. Then, the following assertions are equivalent:*

(a) *$\{T(t)\}$ is asymptotically almost periodic;*

(b) *$\{T(t)\}$ is bounded, strongly uniformly continuous and, for every $r \in \sigma_C(A) \cap i\mathbb{R}$, $x \in X$, $\lim_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x\,dt$ exists uniformly for $s \ge 0$.*

Proof. (a)⇒(b). It follows from the properties of asymptotically almost periodic functions (cf. [4]).

(b)⇒(a). Given $x \in X$, let $f(t) = T(t)x$, and then we have that $f(t)$ is bounded, uniformly continuous and $\widetilde{f}(\lambda) = (\lambda - A)^{-1}Cx\,(\operatorname{Re}\lambda > 0)$. Let E be the singular set of $\widetilde{f}$ in $i\mathbb{R}$, then $E \subseteq \sigma_C(A) \cap i\mathbf{R}$, and then it follows that E is countable by the assumption. Moreover, for each $ir \in \sigma_C(A) \cap i\mathbb{R}$,

$$\lim_{\lambda\to 0} \lambda \widetilde{f}_s(\lambda + ir) = \lim_{\lambda\to 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x\,dt \tag{3.13}$$

exists uniformly for $s \ge 0$, where $f_s(t) = f(s+t)$. Therefore, $f(t) = T(t)x$ is asymptotically almost periodic by [5, Theorem 4.1]; so, $\{T(t)\}$ is asymptotically almost periodic. □

Remark 3.8. The result of Theorem 3.7 can be deduced directly from [8, Theorem 4] with the assumption on $\sigma_C(A)$ replaced by that on $\sigma(A)$; while with the aid of [5, Theorem 4.1], the result can be improved.

We end this paper with a theorem for integrated semigroups, see [13] for the definitions and basic properties of integrated semigroups.

Theorem 3.9. *Suppose that A generates a bounded n-times integrated semigroup $\{T(t)\}_{t\geq 0}$ and $\sigma(A)\cap i\mathbb{R}$ is at most countable. Then, the following assertions are equivalent:*

(a) *$\{T(t)\}$ is asymptotically almost periodic;*

(b) *for every $r\in\sigma(A)\cap i\mathbb{R}$, $x\in X$, the limit*

$$\lim_{\lambda\to 0}\lambda\int_0^{+\infty} e^{-(\lambda+ir)t}T(t+s)x\,dt \tag{3.14}$$

exists uniformly for $s\geq 0$.

Proof. We only need to show (b)⇒(a).

We first recall that for bounded integrated semigroup $\{T(t)\}$, we have

$$(\lambda-A)^{-1}x=\lambda^n\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt \tag{3.15}$$

for $\operatorname{Re}\lambda>0$, that is,

$$\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt=\frac{1}{\lambda^n}(\lambda-A)^{-1}x \tag{3.16}$$

for $\operatorname{Re}\lambda>0$, so that $\tilde{T}(\lambda):=\int_0^{+\infty}e^{-\lambda t}T(t)x\,dt$ can be extended holomorphically to a connected open neighborhood V of $(i\mathbb{R}\backslash\sigma(A))\backslash\{0\}$, hence the singular set of $\tilde{T}(\lambda)$ in $i\mathbb{R}$ is contained in $(\sigma(A)\cap i\mathbb{R})\bigcup\{0\}$. By our assumption and [5, Theorem 4.1], we derive (a) from (b). □

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