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ON HILL'S EQUATION WITH A DISCONTINUOUS COEFFICIENT

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We research the asymptotic formula for the lengths of the instability intervals of the Hill's equation with coefficients $q(x)$ and $r(x)$, where $q(x)$ is piecewise continuous and $r(x)$ has a piecewise continuous second derivative in open intervals $(0,b)$ and (b,a) $(0<b<a)$.

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1. Introduction. We consider the second-order equation

$$y''(x)+\{\lambda r(x)-q(x)\}y(x)=0,\quad -\infty<x<\infty, \tag{1.1}$$

where $q(x)$ and $r(x)$ are real valued and all have period a. Also $q(x)$ is piecewise continuous, $r''(x)$ is piecewise continuous in $(0,b)$ and (b,a) where $0<b<a$ and $r(x)\ge r_0$ (>0).

Those values of complex parameter λ, for which periodic or antiperiodic problem associated with (1.1) and the x-interval $(0,a)$ has a nontrivial solution $y(x,\lambda)$, are called eigenvalues. An equation giving the eigenvalues of the periodic or the antiperiodic problem can be found. For the purpose, after we use the Liouville transformation

$$t=\int_0^x r^{1/2}(u)\,du,\qquad z(t)=r^{1/4}(x)y(x), \tag{1.2}$$

the periodic boundary value problem becomes the boundary value problem

$$\begin{gathered}\ddot z+\{\lambda-Q(t)\}z(t)=0,\quad 0\le t\le A,\\ z(A)=\sigma z(0),\qquad \sigma\dot z(A)+\rho z(A)=\dot z(0)+\tau z(0)\end{gathered} \tag{1.3}$$

and the antiperiodic boundary value problem becomes the boundary value problem

$$\begin{gathered}\ddot z+\{\lambda-Q(t)\}z(t)=0,\quad 0\le t\le A,\\ z(A)=-\sigma z(0),\qquad \sigma\dot z(A)+\rho z(A)=-\dot z(0)-\tau z(0),\end{gathered} \tag{1.4}$$

where

$$r(x)=\begin{cases} r_1(x), & \text{if } 0\le x<b,\\ r_2(x), & \text{if } b<x\le a,\end{cases}$$

$$Q(t)=\begin{cases} \dfrac{q(x)}{r_1(x)}-r_1^{-3/4}(x)\{r_1^{-1/4}(x)\}'', & \text{if } 0\le t<B,\\ \dfrac{q(x)}{r_2(x)}-r_2^{-3/4}(x)\{r_2^{-1/4}(x)\}'', & \text{if } B<t\le A,\end{cases} \tag{1.5}$$

$$A=\int_0^a r^{1/2}(u)du,\qquad B=\int_0^b r^{1/2}(u)du,\qquad \sigma=\left\{\frac{r(a)}{r(0)}\right\}^{1/4},$$

$$\rho=r^{-1/4}(0)\{r^{-1/4}(x)\}'_{x=a},\qquad \tau=r^{-1/4}(0)\{r^{-1/4}(x)\}'_{x=0}.$$

Let $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ denote the solutions of problem (1.3), satisfying the initial conditions

$$\theta(0,\lambda)=1,\qquad \frac{\partial}{\partial t}\theta(t,\lambda)\Big|_{t=0}=0;$$
$$\varphi(0,\lambda)=0,\qquad \frac{\partial}{\partial t}\varphi(t,\lambda)\Big|_{t=0}=1. \tag{1.6}$$

Then, we can define the Hill discriminant (see [1]) of (1.1) by the function

$$F(\lambda)=\theta(A,\lambda)+\sigma^2\frac{\partial}{\partial t}\varphi(t,\lambda)\Big|_{t=A}+(\rho\sigma-\tau)\varphi(A,\lambda). \tag{1.7}$$

Thus, the eigenvalues of the periodic boundary value problem coincide with the roots λ of

$$F(\lambda)=2\sigma \tag{1.8}$$

and also the eigenvalues of the antiperiodic boundary value problem coincide with the roots of

$$F(\lambda)=-2\sigma. \tag{1.9}$$

Each of the periodic and antiperiodic boundary value problems has a countable infinity real eigenvalues with the points accumulate at $+\infty$. Denote by

$$\mu_0<\mu_2^-\le\mu_2^+<\cdots<\mu_{2n}^-\le\mu_{2n}^+\cdots \tag{1.10}$$

the eigenvalues of the periodic boundary value problem, and by

$$\mu_1^-\le\mu_1^+<\cdots<\mu_{2n+1}^-\le\mu_{2n+1}^+\cdots \tag{1.11}$$

the eigenvalues of the antiperiodic boundary value problem (the equality holds in the case of double eigenvalue). These values occur in the order

$$\mu_0 < \mu_1^- \le \mu_1^+ < \mu_2^- \le \mu_2^+ < \cdots < \mu_n^- \le \mu_n^+ < \cdots. \tag{1.12}$$

If λ lies in any of the open intervals $(-\infty,\mu_0)$ and (μ_n^-,μ_n^+) $(n=1,2,\dots)$, then all nontrivial solutions of (1.1) are unbounded in $(-\infty,\infty)$. These kind of intervals are called the instability intervals of (1.1). Apart from $(-\infty,\mu_0)$, some or all of the instability intervals vanish for the case of double eigenvalues. If λ lies in any of the complementary open intervals (μ_{n-1}^+,μ_n^-) $(n=1,2,\dots)$, then all solutions of (1.1) are bounded in $(-\infty,\infty)$, and these intervals are called the stability intervals of (1.1). We are interested in the lengths I_n of the instability intervals

$$I_{2n} = \mu_{2n}^+ - \mu_{2n}^-, \qquad I_{2n+1} = \mu_{2n+1}^+ - \mu_{2n+1}^-. \tag{1.13}$$

Eastham [2] has studied (1.1) where the second derivative of $r(x)$ is piecewise continuous on interval $(0,a)$. But we studied (1.1) where the second derivative of $r(x)$ is piecewise continuous on intervals $(0,b)$ and (b,a) $(0<b<a)$. On the other hands, our method is different from his method and is based on using Rouche's theorem about roots of analytic functions.

2. An asymptotic formula of the hill discriminant

Proposition 2.1. *Let $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ be the solutions of problem (1.3) such that $\theta(0,\lambda)=1$, $\dot\theta(0,\lambda)=0$, $\varphi(0,\lambda)=0$, and $\dot\varphi(0,\lambda)=1$. For $\lambda=s^2$, these solutions verify the following integral equations:*

$$\theta(t,\lambda)=\begin{cases}\cos st+\displaystyle\int_0^t\frac{\sin s(t-\xi)}{s}Q_1(\xi)\theta(\xi,\lambda)d\xi, & 0\le t<B,\\ k\cos sB\cos s(t-B)-\dfrac{1}{k}\sin sB\sin s(t-B)\\ \quad +m\dfrac{\cos sB\sin s(t-B)}{s}\\ \quad +\displaystyle\int_0^B\Big\{k\frac{\cos s(t-B)\sin s(B-\xi)}{s}\\ \qquad +\dfrac{1}{k}\dfrac{\sin s(t-B)\cos s(B-\xi)}{s}\\ \qquad +m\dfrac{\sin s(t-B)\sin s(B-\xi)}{s^2}\Big\}Q_1(\xi)\theta(\xi,\lambda)d\xi\\ \quad +\displaystyle\int_B^t\frac{\sin s(t-\xi)}{s}Q_2(\xi)\theta(\xi,\lambda)d\xi, & B<t\le A,\end{cases} \tag{2.1}$$

$$\varphi(t,\lambda)=\begin{cases}\dfrac{\sin st}{s}+\displaystyle\int_0^t\dfrac{\sin s(t-\xi)}{s}Q_1(\xi)\varphi(\xi,\lambda)d\xi, & 0\le t<B,\\ k\dfrac{\cos s(t-B)\sin sB}{s}+\dfrac{1}{k}\dfrac{\sin s(t-B)\cos sB}{s}\\ \quad +m\dfrac{\sin sB\sin s(t-B)}{s^2}\\ \quad +\displaystyle\int_0^B\Big\{k\dfrac{\cos s(t-B)\sin s(B-\xi)}{s}\\ \qquad +\dfrac{1}{k}\dfrac{\sin s(t-B)\cos s(B-\xi)}{s}\\ \qquad +m\dfrac{\sin s(t-B)\sin s(B-\xi)}{s^2}\Big\}Q_1(\xi)\varphi(\xi,\lambda)d\xi\\ \quad +\displaystyle\int_B^t\dfrac{\sin s(t-\xi)}{s}Q_2(\xi)\varphi(\xi,\lambda)d\xi, & B<t\le A,\end{cases}\tag{2.2}$$

where $Q_1(t)=q(x)/r_1(x)-r_1^{-3/4}(x)\{r_1^{-1/4}(x)\}''$, $Q_2(t)=q(x)/r_2(x)-r_2^{-3/4}(x)\{r_2^{-1/4}(x)\}''$, $k=r^{1/4}(b+0)/r^{1/4}(b-0)$, $m=\{r^{-1/4}(b-0)\}'/r^{1/4}(b+0)-\{r^{-1/4}(b+0)\}'/r^{1/4}(b-0)$.

Proof. It is clear that $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ verify the integral equations for $0\le t<B$.

Let $B<t\le A$. Since $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ are solutions of problem (1.3), we have

$$\begin{aligned}\ddot\theta(t,\lambda)+\{\lambda-Q(t)\}\theta(t,\lambda)&=0,\\ \ddot\varphi(t,\lambda)+\{\lambda-Q(t)\}\varphi(t,\lambda)&=0.\end{aligned}\tag{2.3}$$

First, we multiply the equalities above by $\sin s(t-\xi)/s$ and take integral of the obtained equalities from B to t. Then, we get

$$\begin{aligned}\theta(t,\lambda)&=\cos s(t-B)\theta(B+0,\lambda)+\frac{\sin s(t-B)}{s}\dot\theta(B+0,\lambda)\\ &\quad+\int_B^t\frac{\sin s(t-\xi)}{s}Q_2(\xi)\theta(\xi,\lambda)d\xi,\\ \varphi(t,\lambda)&=\cos s(t-B)\varphi(B+0,\lambda)+\frac{\sin s(t-B)}{s}\dot\varphi(B+0,\lambda)\\ &\quad+\int_B^t\frac{\sin s(t-\xi)}{s}Q_2(\xi)\varphi(\xi,\lambda)d\xi.\end{aligned}\tag{2.4}$$

Using the equalities $y(b-0)=y(b+0)$ and $y'(b-0)=y'(b+0)$, we obtain

$$\begin{gathered}\theta(B+0,\lambda)=k\theta(B-0,\lambda),\qquad \varphi(B+0,\lambda)=k\varphi(B-0,\lambda),\\ \dot{\theta}(B+0,\lambda)=m\theta(B-0,\lambda)+\frac{1}{k}\dot{\theta}(B-0,\lambda),\\ \dot{\varphi}(B+0,\lambda)=m\varphi(B-0,\lambda)+\frac{1}{k}\dot{\varphi}(B-0,\lambda).\end{gathered} \tag{2.5}$$

When these equalities are written in (2.4), the proof is completed. □

Proposition 2.2. *Let $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ be the solutions of problem (1.3) such that $\theta(0,\lambda)=1$, $\dot{\theta}(0,\lambda)=0$, $\varphi(0,\lambda)=0$, and $\dot{\varphi}(0,\lambda)=1$. Then*

$$\begin{aligned}F(\lambda)=&\,k\{\cos s(A-B)\cos sB-\sigma^2\sin s(A-B)\sin sB\}\\&+\frac{1}{k}\{\sigma^2\cos s(A-B)\cos sB-\sin s(A-B)\sin sB\}+O\left(\frac{e^{|\operatorname{Im}s|A}}{|s|}\right),\end{aligned} \tag{2.6}$$

where $|\lambda|\to\infty$, that is, $|s|\to\infty$.

Proof. Using integral equations (2.1) and (2.2), we have

$$\theta(t,\lambda)=\begin{cases}\cos st+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|}\right), & 0\le t<B,\\ k\cos sB\cos s(t-B)-\dfrac{1}{k}\sin sB\sin s(t-B)+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|}\right), & B<t\le A,\end{cases} \tag{2.7}$$

$$\varphi(t,\lambda)=\begin{cases}\dfrac{\sin st}{s}+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|^2}\right), & 0\le t<B,\\ k\dfrac{\cos s(t-B)\sin sB}{s}+\dfrac{1}{k}\dfrac{\sin s(t-B)\cos sB}{s}+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|^2}\right), & B<t\le A,\end{cases} \tag{2.8}$$

$$\dot{\varphi}(t,\lambda)=\begin{cases}\cos st+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|}\right), & 0\le t<B,\\ -k\sin sB\sin s(t-B)+\dfrac{1}{k}\cos sB\cos s(t-B)+O\left(\dfrac{e^{|\operatorname{Im}s|t}}{|s|}\right), & B<t\le A,\end{cases} \tag{2.9}$$

where $|s|$ goes to positive infinity. When we put $\theta(A,\lambda)$, $\varphi(A,\lambda)$, and $\dot{\varphi}(A,\lambda)$ in (1.7), we get the result. □

3. The asymptotic formulas for the lengths of the instability intervals. First, we research the eigenvalues of the periodic boundary value problem, using Rouche's theorem which is stated as follows.

Theorem 3.1 (Rouche's theorem). *If $f(w)$ and $g(w)$ are analytic functions inside and on a simple closed contour Γ and $|g(w)| < |f(w)|$ at each point on Γ, then $f(w)$ and $f(w)+g(w)$ have the same number of zeros, counting multiplicities, inside Γ.*

Let $\sigma = \{r(a)/r(0)\}^{1/4} = 1$. Then, define

$$\Phi^+(s) = F(\lambda) - 2 = \frac{k^2+1}{k}\cos sA - 2 + O\left(\frac{e^{|\operatorname{Im} s|A}}{|s|}\right). \tag{3.1}$$

Let $f(s) = ((k^2+1)/k)\cos sA - 2$, $g(s) = O(e^{|\operatorname{Im} s|A}/|s|)$ and

$$\begin{aligned}\Gamma_{2n+1/2} = \Bigg\{ s \in \mathbb{C} : |\operatorname{Re} s| &= \frac{1}{A}\left[\left(2n+\frac{1}{2}\right)\pi + \arccos\frac{2k}{k^2+1}\right], \\ |\operatorname{Im} s| &= \frac{1}{A}\left[\left(2n+\frac{1}{2}\right)\pi + \arccos\frac{2k}{k^2+1}\right]\Bigg\}.\end{aligned} \tag{3.2}$$

In order to apply Rouche's theorem to our case, we need the following lemma.

Lemma 3.2. *There is a positive number C such that*

$$|f(s)| \geq C e^{A|\operatorname{Im} s|}, \quad s \in \Gamma_{2n+1/2}, \tag{3.3}$$

where C does not depend on s and n.

Proof. Let $s = u + iv$. Then

$$\begin{aligned}|f(s)|^2 &= \left|\frac{k^2+1}{k}\cos sA - 2\right|^2 \\ &= 4 + \frac{1}{4}\left(\frac{k^2+1}{k}\right)^2\left[e^{2vA} + e^{-2vA} + 2\cos 2uA\right] \\ &\quad - 2\frac{k^2+1}{k}\left(e^{vA} + e^{-vA}\right)\cos uA.\end{aligned} \tag{3.4}$$

On the vertical edge of square contour $\Gamma_{2n+1/2}$, take

$$\begin{aligned}u &= -\left(2n+\frac{1}{2}\right)\frac{\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1}, \\ u &= \left(2n+\frac{1}{2}\right)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1}.\end{aligned} \tag{3.5}$$

When the value of u is written in (3.4), we have

$$|f(s)|^2 \geq \frac{1}{4}\left(\frac{k^2+1}{k}\right)^2 e^{2|v|A} \tag{3.6}$$

and hence

$$|f(s)| \geq \frac{1}{2}\left(\frac{k^2+1}{k}\right)e^{|v|A}. \tag{3.7}$$

On the other hand, we take that

$$\begin{aligned} v &= -\left(2n+\frac{1}{2}\right)\frac{\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1}, \\ v &= \left(2n+\frac{1}{2}\right)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} \end{aligned} \tag{3.8}$$

on the horizontal edge of square contour $\Gamma_{2n+1/2}$. Since the function $|f(s)|^2$ has minimum values at the points $u = p\pi/A$ where p is even, we have

$$|f(s)|^2 \geq \frac{1}{16}\left(\frac{k^2+1}{k}\right)^2 e^{2|v|A}. \tag{3.9}$$

This completes the proof. □

From Lemma 3.2, we have a positive number C' such that

$$\left|\frac{g(s)}{f(s)}\right| \leq \frac{C'}{|s|}. \tag{3.10}$$

For all $s \in \Gamma_{2n+1/2}$,

$$|s| \geq \left(2n+\frac{1}{2}\right)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} \tag{3.11}$$

and there exits a natural number n_0 such that, for all $n \geq n_0$,

$$C' < \left(2n+\frac{1}{2}\right)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1}. \tag{3.12}$$

Therefore, for all $s \in \Gamma_{2n+1/2}$, $n \geq n_0$,

$$\left|\frac{f(s)}{g(s)}\right| < 1. \tag{3.13}$$

Moreover, $f(s)$ and $g(s)$ are analytic on the square contour $\Gamma_{2n+1/2}$ and on the region bounded by the square contour. The number of roots of function $f(s)$ is $4n+2$ inside the square contour $\Gamma_{2n+1/2}$. By Rouche's theorem, $\Phi^+(s)$ has $4n+2$ roots inside the square contour $\Gamma_{2n+1/2}$. Denote these roots by

$$-s_{2n}^+, -s_{2n}^-, \ldots, -s_2^+, -s_2^-, -s_0^+, s_0^+, s_2^-, s_2^+, \ldots, s_{2n}^-, s_{2n}^+. \tag{3.14}$$

Similarly, $\Phi^+(s)$ must have $4n-2$ roots inside the square contour $\Gamma_{2n-1/2}$. Therefore, there are 4 roots between $\Gamma_{2n+1/2}$ and $\Gamma_{2n-1/2}$. Two of these roots belong to region

$$D_n = \Big\{ s \in \mathbb{C} : \Big(2n - \frac{1}{2}\Big)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} < \operatorname{Re} s < \Big(2n + \frac{1}{2}\Big)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1}, \quad |\operatorname{Im} s| \le \Big(2n + \frac{1}{2}\Big)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} \Big\}. \tag{3.15}$$

Consider circles $|s - 2n\pi/A \pm (1/A)\arccos(2k/(k^2+1))| = \rho$, where ρ is any number such that $0 < \rho < \pi/2A$. Then functions $f(s)$ and $g(s)$ are analytic on these circles and region bounded by these circles.

Lemma 3.3. *Consider circles* $|s - 2n\pi/A \pm (1/A)\arccos(2k/(k^2+1))| = \rho$ $(0 < \rho < \pi/2A)$. *Then, for all* s *on these circles, there is a positive number* K *such that*

$$|f(s)| \ge K\rho^2 e^{A|\operatorname{Im} s|}, \tag{3.16}$$

where K *does not depend on* s *and* $n \in \mathbb{N}$.

Proof. Let $s = u + iv$ and consider $|s - 2n\pi/A \pm (1/A)\arccos(2k/(k^2+1))| = \rho$. Then

$$u = \frac{2n\pi}{A} \mp \frac{1}{A}\arccos\frac{2k}{k^2+1} \mp \sqrt{\rho^2 - v^2}. \tag{3.17}$$

On the other hand,

$$|f(s)| = \Big(\frac{k^2+1}{2k}\Big)\Big[e^{vA} + e^{-vA} - \frac{4k}{k^2+1}\cos uA\Big] \times \Bigg[1 - \frac{4(k^2-1)^2}{(k^2+1)^2} \cdot \frac{\sin^2 uA}{\big(e^{vA} + e^{-vA} - (4k/(k^2+1))\cos uA\big)^2}\Bigg]^{1/2}. \tag{3.18}$$

Let

$$F(u,v) = \frac{\sin uA}{e^{vA} + e^{-vA} - (4k/(k^2+1))\cos uA}. \tag{3.19}$$

This function has a maximum value and a minimum value at the points

$$\begin{gathered}\left(\frac{2n\pi}{A}+\frac{1}{A}\arccos\frac{2k}{k^2+1},0\right),\\ \left(\frac{2n\pi}{A}-\frac{1}{A}\arccos\frac{2k}{k^2+1},0\right),\end{gathered}\tag{3.20}$$

respectively. So, for all u and v on the circles, we have

$$2\left|F(u,v)\right| < \frac{k^2+1}{\left|k^2-1\right|}\tag{3.21}$$

and hence

$$0<1-\left[2\cdot\frac{k^2-1}{k^2+1}\cdot F(u,v)\right]^2.\tag{3.22}$$

Therefore, there is a positive number ϵ_0 not depending on ρ and n such that

$$0<\epsilon_0\le 1-\left[2\cdot\frac{k^2-1}{k^2+1}\cdot F(u,v)\right]^2.\tag{3.23}$$

Thus, for all s on the circles, we get

$$\left|f(s)\right| \ge L\left[e^{vA}+e^{-vA}-\frac{4k}{k^2+1}\cos uA\right],\tag{3.24}$$

where $L=((k^2+1)/2k)\sqrt{\epsilon_0}$. Let $G(u,v)=e^{vA}+e^{-vA}-(4k/(k^2+1))\cos uA$.

Case 1. Let $u=2n\pi/A+(1/A)\arccos(2k/(k^2+1))+\sqrt{\rho^2-v^2}$ and $u=2n\pi/A-(1/A)\arccos(2k/(k^2+1))-\sqrt{\rho^2-v^2}$. Then,

$$\begin{aligned}G(u,v)&=e^{vA}+e^{-vA}-\frac{4k}{k^2+1}\cos uA\\ &\ge e^{vA}+e^{-vA}-\frac{8k^2}{\left(k^2+1\right)^2}\cos\left(A\sqrt{\rho^2-v^2}\right)\\ &\ge e^{vA}+e^{-vA}-2\cos\left(A\sqrt{\rho^2-v^2}\right).\end{aligned}\tag{3.25}$$

Let $g(v)=e^{vA}+e^{-vA}-2\cos(A\sqrt{\rho^2-v^2})$. So, we can work on interval $[0,\rho]$ because the function g is even for all $v\in[-\rho,\rho]$. First, take the derivative of g on the variable v and we get

$$g'(v)\ge A\left[e^{vA}-e^{-vA}-2Av\right]>0.\tag{3.26}$$

So, g is increasing and then $g(0) \le g(v)$ for $0 \le v \le \rho$. Moreover,

$$0 \le A\rho \le \frac{\pi}{2} \Longrightarrow \frac{\sqrt{2}A\rho}{\pi} \le \sin\frac{A\rho}{2} \tag{3.27}$$

and hence

$$8\left(\frac{A\rho}{\pi}\right)^2 \le 4\sin^2\frac{A\rho}{2} \le g(v). \tag{3.28}$$

Since $|f(s)| \ge 8A^2\rho^2L/\pi^2$, we have

$$\frac{e^{A|v|}}{|f(s)|} \le \frac{e^{A|v|}}{8A^2\rho^2L/\pi^2} \le \frac{1}{K\rho^2}, \tag{3.29}$$

where $K = 8A^2\rho^2L/\pi^2e^{\pi/2}$. Hence, $|f(s)| \ge K\rho^2e^{A|v|}$.

Case 2. Similarly, when we take

$$\begin{aligned} u &= \frac{2n\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} - \sqrt{\rho^2 - v^2}, \\ u &= \frac{2n\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} + \sqrt{\rho^2 - v^2}, \end{aligned} \tag{3.30}$$

the inequality is verified. □

From Lemma 3.3 and definition of $g(s)$, we have

$$\left|\frac{g(s)}{f(s)}\right| \le \frac{C_1(e^{A|y|}/|s|)}{K\rho^2e^{A|y|}}. \tag{3.31}$$

We can choose $\rho = \sqrt{2C_2/((2n-1/2)\pi/A + (1/A)\arccos(2k/(k^2+1)))}$, where $C_2 = C_1/K$ because there is a positive number m such that for all $n \ge m$, $\rho < \pi/2A$. Moreover, for all s on these circles, we have

$$|s| \ge \left(2n - \frac{1}{2}\right)\frac{\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1}. \tag{3.32}$$

Thus,

$$\left|\frac{g(s)}{f(s)}\right| \le \frac{C_2}{\rho^2[(2n-1/2)\pi/A + (1/A)\arccos(2k/(k^2+1))]} = \frac{1}{2} < 1. \tag{3.33}$$

By Rouche's theorem, the function $\Phi^+(s)$ has one root inside the circle

$$\left| s - \frac{2n\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} \right| = \rho \tag{3.34}$$

and one root inside the circle

$$\left| s - \frac{2n\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} \right| = \rho. \tag{3.35}$$

Denote these roots by s_{2n}^- and s_{2n}^+, respectively. Let

$$\begin{aligned} s_{2n}^- &= \frac{2n\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} + \frac{r_{2n}^-}{\sqrt{n}}, \\ s_{2n}^+ &= \frac{2n\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} + \frac{r_{2n}^+}{\sqrt{n}}. \end{aligned} \tag{3.36}$$

Since $|r_{2n}^{\mp}| < \sqrt{2nC_2/((2n-1/2)\pi/A + (1/A)\arccos(2k/(k^2+1)))}$, we have $\operatorname{Sup} r_{2n}^{\mp} < \infty$. Hence, for periodic boundary value problem, we have

$$\begin{aligned} s_{2n}^- &= \frac{2n\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right), \\ s_{2n}^+ &= \frac{2n\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{3.37}$$

Similarly, for antiperiodic boundary value problem, we have

$$\begin{aligned} s_{2n+1}^- &= \frac{(2n+1)\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right), \\ s_{2n+1}^+ &= \frac{(2n+1)\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{3.38}$$

Combining these two results, we get

$$\begin{aligned} s_n^- &= \frac{n\pi}{A} - \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right), \\ s_n^+ &= \frac{n\pi}{A} + \frac{1}{A}\arccos\frac{2k}{k^2+1} + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{3.39}$$

Theorem 3.4. *Let $r'(0) \neq r'(a)$. If the second derivative of function $r(x)$ in (1.1) is piecewise continuous on open intervals $(0,b)$ and (b,a), then*

$$I_n = \frac{4n\pi}{A^2}\arccos\frac{2k}{k^2+1} + O(1). \tag{3.40}$$

Proof. We know that eigenvalues of periodic and antiperiodic boundary value problems are real and go to infinity. So, it is sufficient to take positive values of parameters λ. By (2.7) and (2.8), we have

$$\theta(t,\lambda) = \begin{cases} \cos st + O\left(\dfrac{1}{|s|}\right), & \text{if } 0 \le t < B, \\ k\cos s(t-B)\cos sB - \dfrac{1}{k}\sin s(t-B)\sin sB + O\left(\dfrac{1}{|s|}\right), & \text{if } B < t \le A, \end{cases}$$

$$\varphi(t,\lambda) = \begin{cases} \dfrac{\sin st}{s} + O\left(\dfrac{1}{|s|^2}\right), & \text{if } 0 \le t < B, \\ k\dfrac{\cos s(t-B)\sin sB}{s} + \dfrac{1}{k}\dfrac{\sin s(t-B)\cos sB}{s} + O\left(\dfrac{1}{|s|^2}\right), & \text{if } B < t \le A. \end{cases} \tag{3.41}$$

When the values $\theta(t,\lambda)$ and $\varphi(t,\lambda)$ are written in (2.1) and in derivative of (2.2), we get

$$\begin{aligned} \theta(A,\lambda) = {} & k\cos s(A-B)\cos sB - \frac{1}{k}\sin s(A-B)\sin sB + m\frac{\cos sB\sin s(A-B)}{s} \\ & + \frac{1}{4s}k\int_0^B [\sin sA - \sin s(A-2B) + \sin s(A-2\xi) - \sin s(A-2B+2\xi)] \\ & \qquad \times Q_1(\xi)\,d\xi \\ & + \frac{1}{4s}\frac{1}{k}\int_0^B [\sin sA + \sin s(A-2B) + \sin s(A-2\xi) + \sin s(A-2B+2\xi)] \\ & \qquad \times Q_1(\xi)\,d\xi \\ & + \frac{1}{4s}k\int_B^A [\sin sA + \sin s(A-2B) + \sin s(A-2\xi) + \sin s(A+2B-2\xi)] \\ & \qquad \times Q_2(\xi)\,d\xi \\ & + \frac{1}{4s}\frac{1}{k}\int_B^A [\sin sA - \sin s(A-2B) + \sin s(A-2\xi) - \sin s(A+2B-2\xi)] \\ & \qquad \times Q_2(\xi)\,d\xi \\ & + O\left(\frac{1}{|s|^2}\right), \end{aligned}$$

$$\begin{aligned}\dot{\varphi}(A,\lambda) = &-k\sin s(A-B)\sin sB + \frac{1}{k}\cos s(A-B)\cos sB + m\frac{\sin sB\cos s(A-B)}{s}\\ &+\frac{1}{4s}k\int_0^B[\sin sA+\sin s(A-2B)-\sin s(A-2\xi)-\sin s(A-2B+2\xi)]\\ &\qquad\times Q_1(\xi)d\xi\\ &+\frac{1}{4s}\frac{1}{k}\int_0^B[\sin sA-\sin s(A-2B)-\sin s(A-2\xi)+\sin s(A-2B+2\xi)]\\ &\qquad\times Q_1(\xi)d\xi\\ &+\frac{1}{4s}k\int_B^A[\sin sA-\sin s(A-2B)-\sin s(A-2\xi)+\sin s(A+2B-2\xi)]\\ &\qquad\times Q_2(\xi)d\xi\\ &+\frac{1}{4s}\frac{1}{k}\int_B^A[\sin sA+\sin s(A-2B)-\sin s(A-2\xi)-\sin s(A+2B-2\xi)]\\ &\qquad\times Q_2(\xi)d\xi\\ &+O\left(\frac{1}{|s|^2}\right).\end{aligned} \tag{3.42}$$

Moreover, we have

$$\varphi(A,\lambda) = k\frac{\cos s(A-B)\sin sB}{s} + \frac{1}{k}\frac{\sin s(A-B)\cos sB}{s} + O\left(\frac{1}{|s|^2}\right). \tag{3.43}$$

After putting the values $\theta(A,\lambda)$, $\varphi(A,\lambda)$, and $\dot{\varphi}(A,\lambda)$ in equality

$$\theta(A,\lambda)+\dot{\varphi}(A,\lambda)+(\rho-\tau)\varphi(A,\lambda) = 2, \tag{3.44}$$

we get

$$\begin{aligned}\cos s_{2n}^{\mp}A &+ \frac{1}{s_{2n}^{\mp}}\cdot\frac{km}{k^2+1}\sin s_{2n}^{\mp}A\\ &+\frac{1}{2s_{2n}^{\mp}}\sin s_{2n}^{\mp}A\int_0^B Q_1(\xi)d\xi + \frac{1}{2s_{2n}^{\mp}}\sin s_{2n}^{\mp}A\int_B^A Q_2(\xi)d\xi\\ &-\frac{1}{2s_{2n}^{\mp}}\cdot\frac{k^2-1}{k^2+1}\int_0^B\sin s_{2n}^{\mp}(A-2B+2\xi)Q_1(\xi)d\xi\\ &+\frac{1}{2s_{2n}^{\mp}}\cdot\frac{k^2-1}{k^2+1}\int_B^A\sin s_{2n}^{\mp}(A+2B-2\xi)Q_2(\xi)d\xi\\ &+\frac{\rho-\tau}{2s_{2n}^{\mp}}\sin s_{2n}^{\mp}A-\frac{\rho-\tau}{2s_{2n}^{\mp}}\cdot\frac{k^2-1}{k^2+1}\sin s_{2n}^{\mp}(A-2B)\\ &-\frac{2k}{k^2+1}+O\left(\frac{1}{|s_{2n}^{\mp}|^2}\right)=0.\end{aligned} \tag{3.45}$$

Let $w = \arccos(2k/(k^2+1))$ and $\delta_{2n}^{\mp} = O(1/\sqrt{n})$. Then

$$s_{2n}^{\mp}A = 2n\pi \mp \omega + \delta_{2n}^{\mp}. \tag{3.46}$$

Since $1/s_{2n}^{\mp} = O(1/n)$, $1/(s_{2n}^{\mp})^2 = O(1/n^2)$, $\int_0^B Q_1(\xi)d\xi < \infty$, and $\int_B^A Q_2(\xi)d\xi < \infty$, (3.45) becomes

$$\mp\delta_{2n}^{\mp}\sin w = O\left(\frac{1}{n}\right), \tag{3.47}$$

it is clear that $\sin w \neq 0$. Hence, we have $\delta_{2n}^{\mp} = O(1/n)$ and therefore

$$\begin{aligned} I_{2n} = \mu_{2n}^{+} - \mu_{2n}^{-} &= \frac{8n\pi}{A^2}\arccos\frac{2k}{k^2+1} + O(1), \\ I_{2n+1} = \mu_{2n+1}^{+} - \mu_{2n+1}^{-} &= \frac{4(2n+1)\pi}{A^2}\arccos\frac{2k}{k^2+1} + O(1). \end{aligned} \tag{3.48}$$

This completes the proof. □

Theorem 3.5. *Under hypotheses of Theorem 3.4, we have*

$$\begin{aligned} I_n = {} & 4n\pi\omega A^{-2} + A^{-1}\frac{k^2-1}{k^2+1} \\ & \cdot \frac{2}{\sin\omega}\left\{\int_0^B \cos\left[\frac{2\omega}{A}(B-\xi)-\omega\right]\cdot\sin\left[\frac{2n\pi}{A}(B-\xi)\right]Q_1(\xi)d\xi \right. \\ & \qquad + \int_B^A \cos\left[\frac{2\omega}{A}(B-\xi)+\omega\right]\cdot\sin\left[\frac{2n\pi}{A}(B-\xi)\right]Q_2(\xi)d\xi \\ & \qquad \left. + (\rho-\tau)\cos\left[\frac{2B\omega}{A}-\omega\right]\cdot\sin\left[2n\pi\frac{B}{A}\right]\right\} \\ & + O\left(\frac{1}{n}\right). \end{aligned} \tag{3.49}$$

Proof. Since $s_{2n}^{\mp}A = 2n\pi \mp \omega + \delta_{2n}^{\mp}$ and $\delta_{2n}^{\mp} = O(1/n)$, we have

$$\begin{aligned} \cos s_{2n}^{\mp}A = \cos\omega \pm \delta_{2n}^{\mp}\sin\omega + O\left(\left(\delta_{2n}^{\mp}\right)^2\right), &\qquad \sin s_{2n}^{\mp}A = \mp\sin\omega + O\left(\delta_{2n}^{\mp}\right), \\ \sin s_{2n}^{\mp}(A-2B+2\xi) &= \sin\left[\frac{2}{A}(-B+\xi)(2n\pi\mp\omega)\mp\omega\right] + O\left(\delta_{2n}^{\mp}\right), \\ \sin s_{2n}^{\mp}(A+2B-2\xi) &= \sin\left[\frac{2}{A}(B-\xi)(2n\pi\mp\omega)\mp\omega\right] + O\left(\delta_{2n}^{\mp}\right), \\ \sin s_{2n}^{\mp}(A-2B) &= -\sin\left[\frac{2B}{A}(2n\pi\mp\omega)\pm\omega\right] + O\left(\delta_{2n}^{\mp}\right), \\ \frac{1}{s_{2n}^{\mp}} &= \frac{A}{2n\pi} + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{3.50}$$

Using these equalities in (3.45), we get

$$\begin{aligned}\delta_{2n}^{\mp} &= \frac{A}{2n\pi}\cdot\frac{km}{k^2+1}+\frac{A}{4n\pi}\int_0^B Q_1(\xi)d\xi \\ &\mp \frac{A}{4n\pi}\cdot\frac{k^2-1}{k^2+1}\cdot\frac{1}{\sin\omega}\int_0^B \sin\left[\frac{2}{A}(B-\xi)(2n\pi\mp\omega)\pm\omega\right]Q_1(\xi)d\xi \\ &\mp \frac{A}{4n\pi}\cdot\frac{k^2-1}{k^2+1}\cdot\frac{1}{\sin\omega}\int_B^A \sin\left[\frac{2}{A}(B-\xi)(2n\pi\mp\omega)\mp\omega\right]Q_2(\xi)d\xi \\ &+\frac{A}{4n\pi}\int_B^A Q_2(\xi)d\xi\mp\frac{A}{4n\pi}(\rho-\tau)\cdot\frac{k^2-1}{k^2+1}\cdot\frac{\sin[(2B/A)(2n\pi\mp\omega)\pm\omega]}{\sin\omega} \\ &+\frac{A}{4n\pi}(\rho-\tau)+O\left(\frac{1}{n^2}\right).\end{aligned} \tag{3.51}$$

Therefore,

$$\begin{aligned} I_{2n} &= 8n\pi\omega A^{-2}+A^{-1}\frac{k^2-1}{k^2+1} \\ &\cdot\frac{2}{\sin\omega}\left\{\int_0^B\cos\left[\frac{2\omega}{A}(B-\xi)-\omega\right]\cdot\sin\left[\frac{4n\pi}{A}(B-\xi)\right]Q_1(\xi)d\xi\right. \\ &\quad+\int_B^A\cos\left[\frac{2\omega}{A}(B-\xi)+\omega\right]\cdot\sin\left[\frac{4n\pi}{A}(B-\xi)\right]Q_2(\xi)d\xi \\ &\quad\left.+(\rho-\tau)\cos\left[\frac{2B\omega}{A}-\omega\right]\cdot\sin\left[4n\pi\frac{B}{A}\right]\right\} \\ &+O\left(\frac{1}{n}\right).\end{aligned} \tag{3.52}$$

Similarly,

$$\begin{aligned} I_{2n+1} &= 4(2n+1)\pi\omega A^{-2}+A^{-1}\frac{k^2-1}{k^2+1} \\ &\cdot\frac{2}{\sin\omega}\left\{\int_0^B\cos\left[\frac{2\omega}{A}(B-\xi)-\omega\right]\cdot\sin\left[\frac{(4n+2)\pi}{A}(B-\xi)\right]Q_1(\xi)d\xi\right. \\ &\quad+\int_B^A\cos\left[\frac{2\omega}{A}(B-\xi)+\omega\right]\cdot\sin\left[\frac{(4n+2)\pi}{A}(B-\xi)\right]Q_2(\xi)d\xi \\ &\quad\left.+(\rho-\tau)\cos\left[\frac{2B\omega}{A}-\omega\right]\cdot\sin\left[(4n+2)\pi\frac{B}{A}\right]\right\} \\ &+O\left(\frac{1}{n}\right).\end{aligned} \tag{3.53}$$

This completes the proof. □

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