IJMMS 2003:28, 1783–1801
PII. S0161171203204324
http://ijmms.hindawi.com

ON SOME BK SPACES

BRUNO DE MALAFOSSE

Received 25 April 2002

We characterize the spaces $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$ and we deal with some sets generalizing the well-known sets $w_0(\lambda)$, $w_\infty(\lambda)$, $w(\lambda)$, $c_0(\lambda)$, $c_\infty(\lambda)$, and $c(\lambda)$.

2000 Mathematics Subject Classification: 46A45, 40C05.

1. Notations and preliminary results. For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$, the operators A_n, for any integer $n \geq 1$, are defined by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m, \tag{1.1}$$

where $X = (x_n)_{n\geq 1}$ is the series intervening in the second member being convergent. So, we are led to the study of the infinite linear system

$$A_n(X) = b_n, \quad n = 1,2,\ldots, \tag{1.2}$$

where $B=(b_n)_{n\geq 1}$ is a one-column matrix and X the unknown, see [2, 3, 5, 6, 7, 9]. Equation (1.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper, we will also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\,\|_E$ is a BK space if each projection $P_n : X \to P_nX$ is continuous. A BK space E is said to have AK (see [8]) if for every $B = (b_n)_{n\geq 1}$, $B = \sum_{n=1}^{\infty} b_m e_m$, that is,

$$\left\|\sum_{m=N+1}^{\infty} b_m e_m\right\|_E \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{1.3}$$

We shall write s, c, c_0, and l_∞ for the sets of all complex, convergent sequences, sequences convergent to zero, and bounded sequences, respectively. We shall write cs and l_1 for the sets of convergent and absolutely convergent series, respectively. We will use the set

$$U^{+*} = \{(u_n)_{n\geq 1} \in s \mid u_n > 0\ \forall n\}. \tag{1.4}$$

Using Wilansky's notations [12], we define, for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}$ and for any set of sequences E, the set

$$\alpha * E = \left\{(x_n)_{n\geq 1} \in s \mid \left(\frac{x_n}{\alpha_n}\right)_n \in E\right\}. \tag{1.5}$$

Writing

$$\alpha * E = \begin{cases} s_\alpha^\circ & \text{if } E = c_0, \\ s_\alpha^{(c)} & \text{if } E = c, \\ s_\alpha & \text{if } E = l_\infty, \end{cases} \tag{1.6}$$

we have for instance

$$\alpha * c_0 = s_\alpha^\circ = \{(x_n)_{n\geq 1} \in s \mid x_n = o(\alpha_n)\ n \longrightarrow \infty\}. \tag{1.7}$$

Each of the spaces $\alpha * E$, where $E \in \{c_0, c, l_\infty\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right), \tag{1.8}$$

and s_α° has AK.

Now, let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$. We shall write $S_{\alpha,\beta}$ for the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$(a_{nm}\alpha_m)_{m\geq 1} \in l^1 \quad \forall n \geq 1, \qquad \sum_{m=1}^{\infty} |a_{nm}|\alpha_m = O(\beta_n) \quad (n \longrightarrow \infty). \tag{1.9}$$

The set $S_{\alpha,\beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha,\beta}} = \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n}\right). \tag{1.10}$$

Let E and F be any subsets of s. When A maps E into F, we write $A \in (E,F)$, see [10]. So, for every $X \in E$, $AX \in F$ ($AX \in F$ will mean that for each $n \geq 1$, the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$). It has been proved in [8] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So, we can write $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$, we obtain the unital Banach algebra $S_{\alpha,\beta} = S_\alpha$, (see [2, 3, 9]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we say that $A \in \Gamma_\alpha$. The set S_α being a unital algebra, we have the useful result: if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n\geq 1}$, then Γ_α, S_α, s_α, s_α°, and $s_\alpha^{(c)}$ are replaced by Γ_r, S_r, s_r, s_r°, and $s_r^{(c)}$, respectively, (see [2, 3, 5, 6, 7, 8, 9]). When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^\circ = c_0$, and $s_1^{(c)} = c$, and putting $e = (1,1,\dots)$, we have $S_1 = S_e$. It is well known, see [10], that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1. \tag{1.11}$$

We write $e_n = (0,\dots,1,\dots)$ (where 1 is in the nth position).

For any subset E of s, we put

$$AE = \{Y \in s \mid \exists X \in E\ Y = AX\}. \tag{1.12}$$

If F is a subset of s, we denote

$$F(A) = F_A = \{X \in s \mid Y = AX \in F\}. \tag{1.13}$$

We can see that $F(A) = A^{-1}F$.

2. Sets $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$. In this section, we will give necessary and sufficient conditions permitting us to write the sets $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$ by means of the spaces s_ξ, s_ξ°, or $s_\xi^{(c)}$. For this, we need to study the sequence $C(\alpha)\alpha$.

2.1. Properties of the sequence $C(\alpha)\alpha$. Here, we will deal with the operators represented by $C(\lambda)$ and $\Delta(\lambda)$, see [2, 5, 7, 8, 9].

Let

$$U = \{(u_n)_{n\geq 1} \in s \mid u_n \neq 0\ \forall n\}. \tag{2.1}$$

We define $C(\lambda) = (c_{nm})_{n,m\geq 1}$, for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

It can be proved that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m\geq 1}$, with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1,\ n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

is the inverse of $C(\lambda)$, see [8]. If $\lambda = e$, we get the well-known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$, and Δ and Σ belong to any given space S_R with $R > 1$.

We use the following sets:

$$\begin{aligned}\widehat{C} &= \left\{\alpha \in U^{+*} \mid C(\alpha)\alpha = \left(\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right)\right)_{n\geq 1} \in c\right\},\\ \widehat{C_1} &= \left\{\alpha \in U^{+*} \mid C(\alpha)\alpha \in s_1 = l_\infty\right\},\\ \Gamma &= \left\{\alpha \in U^{+*} \mid \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}.\end{aligned} \tag{2.4}$$

Note that $\Delta \in \Gamma_\alpha$ implies $\alpha \in \Gamma$. It can be easily seen that $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n\geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1. \tag{2.5}$$

See [7].

In order to express the following results, we will denote by $[C(\alpha)\alpha]_n$ (instead of $[C(\alpha)]_n(\alpha)$) the nth coordinate of $C(\alpha)\alpha$. We get the following proposition.

Proposition 2.1. *Let $\alpha \in U^{+*}$. Then*

(i) *$\alpha_{n-1}/\alpha_n \to 0$ if and only if $[C(\alpha)\alpha]_n \to 1$;*

(ii)(a) *$\alpha \in \widehat{C}$ implies that $(\alpha_{n-1}/\alpha_n)_{n\geq 1} \in c$,*

(b) *$[C(\alpha)\alpha]_n \to l$ implies that $\alpha_{n-1}/\alpha_n \to 1-1/l$;*

(iii) *if $\alpha \in \widehat{C_1}$, there are $K > 0$ and $\gamma > 1$ such that*

$$\alpha_n \geq K\gamma^n \quad \forall n; \tag{2.6}$$

(iv) *the condition $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C_1}$ and there exists a real $b > 0$ such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \textit{for } n \geq q+1,\ \chi = \gamma_q(\alpha) \in]0,1[. \tag{2.7}$$

Proof. (i) Assume that $\alpha_{n-1}/\alpha_n \to 0$. Then there is an integer N such that

$$n \geq N+1 \Longrightarrow \frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{2}. \tag{2.8}$$

So, there exists a real $K > 0$ such that $\alpha_n \geq K2^n$ for all n and

$$\frac{\alpha_k}{\alpha_n} = \frac{\alpha_k}{\alpha_{k+1}}\cdots\frac{\alpha_{n-1}}{\alpha_n} \leq \left(\frac{1}{2}\right)^{n-k} \quad \text{for } N \leq k \leq n-1. \tag{2.9}$$

Then

$$\begin{aligned}\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) &= \frac{1}{\alpha_n}\left(\sum_{k=1}^{N-1}\alpha_k\right)+\sum_{k=N}^{n-1}\frac{\alpha_k}{\alpha_n} \\ &\le \frac{1}{K2^n}\left(\sum_{k=1}^{N-1}\alpha_k\right)+\sum_{k=N}^{n-1}\left(\frac{1}{2}\right)^{n-k};\end{aligned} \tag{2.10}$$

and since

$$\sum_{k=N}^{n-1}\left(\frac{1}{2}\right)^{n-k} = 1-\left(\frac{1}{2}\right)^{n-N} \longrightarrow 1 \quad (n\longrightarrow\infty), \tag{2.11}$$

we deduce that

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) = O(1), \qquad \left([C(\alpha)\alpha]_n\right)\in l_\infty. \tag{2.12}$$

Using the identity

$$\begin{aligned}[C(\alpha)\alpha]_n &= \frac{\alpha_1+\cdots+\alpha_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n}+1 \\ &= [C(\alpha)\alpha]_{n-1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right)+1,\end{aligned} \tag{2.13}$$

we get $[C(\alpha)\alpha]_n \to 1$. This proves the necessity.

Conversely, if $[C(\alpha)\alpha]_n \to 1$, then

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{[C(\alpha)\alpha]_n-1}{[C(\alpha)\alpha]_{n-1}} \longrightarrow 0. \tag{2.14}$$

Assertion (ii) is a direct consequence of identity (2.14).

(iii) We put $\Sigma_n = \sum_{k=1}^{n}\alpha_k$. Then for a real $M>1$,

$$[C(\alpha)\alpha]_n = \frac{\Sigma_n}{\Sigma_n-\Sigma_{n-1}} \le M \quad \forall n. \tag{2.15}$$

So, $\Sigma_n \ge (M/(M-1))\Sigma_{n-1}$ and $\Sigma_n \ge (M/(M-1))^{n-1}\alpha_1$ for all n. Therefore, from

$$\frac{\alpha_1}{\alpha_n}\left(\frac{M}{M-1}\right)^{n-1} \le [C(\alpha)\alpha]_n = \frac{\Sigma_n}{\alpha_n} \le M, \tag{2.16}$$

we conclude that $\alpha_n \ge K\gamma^n$ for all n, with $K=(M-1)\alpha_1/M^2$ and $\gamma = M/(M-1)>1$.

(iv) If $\alpha \in \Gamma$, then there is an integer $q \geq 1$ for which

$$k \geq q+1 \Longrightarrow \frac{\alpha_{k-1}}{\alpha_k} \leq \chi < 1 \quad \text{with } \chi = \gamma_q(\alpha). \tag{2.17}$$

So, there is a real $M' > 0$ for which

$$\alpha_n \geq \frac{M'}{\chi^n} \quad \forall n \geq q+1. \tag{2.18}$$

Writing $\sigma_{nq} = 1/\alpha_n(\sum_{k=1}^{q} \alpha_k)$ and $d_n = [C(\alpha)\alpha]_n - \sigma_{nq}$, we get

$$\begin{aligned} d_n &= \frac{1}{\alpha_n}\left(\sum_{k=q+1}^{n} \alpha_k\right) = 1 + \sum_{j=q+1}^{n-1}\left(\prod_{k=1}^{n-j} \frac{\alpha_{n-k}}{\alpha_{n-k+1}}\right) \\ &\leq \sum_{j=q+1}^{n} \chi^{n-j} \leq \frac{1}{1-\chi}. \end{aligned} \tag{2.19}$$

And using (2.18), we get

$$\sigma_{nq} \leq \frac{1}{M'}\chi^n\left(\sum_{k=1}^{q} \alpha_k\right). \tag{2.20}$$

So

$$[C(\alpha)\alpha]_n \leq a + b\chi^n \tag{2.21}$$

with $a = 1/(1-\chi)$ and $b = (1/M')(\sum_{k=1}^{q} \alpha_k)$. $\square$

Remark 2.2. Note that $\alpha \in \widehat{C_1}$ does not imply that $\alpha \in \Gamma$.

2.2. New properties of the operator represented by Δ. Throughout this paper, we will denote by D_ξ the infinite diagonal matrix $(\xi_n\delta_{nm})_{n,m\geq 1}$ for any given sequence $\xi = (\xi_n)_{n\geq 1}$. Now, we require some lemmas.

Lemma 2.3. *The condition $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$ is equivalent to $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$ and $\Delta \in (s_\alpha^{(c)}, s_\alpha^{(c)})$ implies $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c, c)$.*

Proof. First, D_α is bijective from c_0 into s_α°. In fact, the equation $D_\alpha X = B$, for every $B = (b_n)_n \in s_\alpha^\circ$, admits a unique solution $X = D_{1/\alpha}B = (b_n/\alpha_n)_n \in c_0$. Suppose now that $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. Then for every $X \in c_0$, we get successively $X' = D_\alpha X \in s_\alpha^\circ$, $\Delta X' \in s_\alpha^\circ$, and $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$. Conversely, assume that $\Delta_\alpha \in (c_0, c_0)$ and let $X \in s_\alpha^\circ$. Then $X = D_\alpha X'$ with $X' \in c_0$. So, $\Delta X \in D_\alpha c_0 = s_\alpha^\circ$ and $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. By a similar reasoning, we get $\Delta \in (s_\alpha^{(c)}, s_\alpha^{(c)}) \Rightarrow \Delta_\alpha \in (c, c)$. $\square$

We need to recall here the following well-known results given in [12].

Lemma 2.4. *The condition $A \in (c,c)$ is equivalent to the following conditions:*
(i) $A \in S_1$;
(ii) $(a_{nm})_{n\geq 1} \in c$ *for each* $m \geq 1$;
(iii) $(\sum_{m=1}^{\infty} a_{nm})_{n\geq 1} \in c$.

If for any given sequence $X = (x_n)_n \in c$, with $\lim_n x_n = l$, $A_n(X)$ is convergent for all n and $\lim_n A_n(X) = l$, it is written that

$$\lim X = A - \lim X, \tag{2.22}$$

and A is called a Toeplitz matrix. We also have the next result.

Lemma 2.5. *The operator $A \in (c,c)$ is a Toeplitz matrix if and only if*
(i) $A \in S_1$;
(ii) $\lim_n a_{nm} = 0$ *for each* $m \geq 1$;
(iii) $\lim_n (\sum_{m=1}^{\infty} a_{nm}) = 1$.

Now, we can assert the following theorem.

Theorem 2.6. *We have successively*
(i) $s_\alpha(\Delta) = s_\alpha$ *if and only if* $\alpha \in \widehat{C_1}$;
(ii) $s_\alpha^\circ(\Delta) = s_\alpha^\circ$ *if and only if* $\alpha \in \widehat{C_1}$;
(iii) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ *if and only if* $\alpha \in \hat{C}$;
(iv) $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha$ *is bijective from* c *into itself with* $\lim X = \Delta_\alpha - \lim X$ *if and only if*

$$\frac{\alpha_{n-1}}{\alpha_n} \longrightarrow 0. \tag{2.23}$$

Proof. (i) We have $s_\alpha(\Delta) = s_\alpha$ if and only if $\Delta, \Sigma \in (s_\alpha, s_\alpha)$. This means that $\Delta, \Sigma \in S_\alpha$, that is,

$$\|\Delta\|_{S_\alpha} = \sup_{n\geq 1}\left(1 + \frac{\alpha_{n-1}}{\alpha_n}\right) < \infty, \qquad \|\Sigma\|_{S_\alpha} = \sup_{n\geq 1}\left[C(\alpha)\alpha\right]_n < \infty. \tag{2.24}$$

Since $0 < \alpha_{n-1}/\alpha_n \leq [C(\alpha)\alpha]_n$, we deduce that $\Delta, \Sigma \in S_\alpha$ if and only if $\|\Sigma\|_{S_\alpha} < \infty$, that is, $\alpha \in \widehat{C_1}$.

(ii) From Lemma 2.3, if $s_\alpha^\circ(\Delta) = s_\alpha^\circ$, then $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$. So, $\Delta_\alpha \in (c_0, l_\infty) = S_1$ and since $\Delta_\alpha = (d_{nm})_{n,m\geq 1}$ with

$$d_{nm} = \begin{cases} 1 & \text{if } m = n, \\ -\dfrac{\alpha_{n-1}}{\alpha_n} & \text{if } m = n-1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.25}$$

we deduce that $\alpha_{n-1}/\alpha_n = O(1)$, $n \to \infty$. Further, $s_\alpha^\circ(\Delta) = s_\alpha^\circ$ implies $\Sigma_\alpha = D_{1/\alpha}\Sigma D_\alpha \in (c_0, c_0)$ and $\Sigma_\alpha \in (c_0, l_\infty) = S_1$. Since $\Sigma_\alpha = (\sigma_{nm})_{n,m\geq 1}$ with

$$\sigma_{nm} = \begin{cases} \dfrac{\alpha_m}{\alpha_n} & \text{if } m \leq n, \\ 0 & \text{if } m > n, \end{cases} \tag{2.26}$$

we deduce that

$$\sup_{n\geq 1}\left(\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right)\right) < \infty, \tag{2.27}$$

that is, $\alpha \in \widehat{C_1}$. Conversely, assume that $\alpha \in \widehat{C_1}$. First, $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. Indeed, from the inequality

$$\frac{\alpha_{n-1}}{\alpha_n} \leq \sup_{n\geq 1}\left(\left[C(\alpha)\alpha\right]_n\right) < \infty, \tag{2.28}$$

we deduce that for every $X \in s_\alpha^\circ$, $x_n/\alpha_n = o(1)$,

$$\frac{x_n - x_{n-1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} = o(1) \tag{2.29}$$

and $\Delta X \in s_\alpha^\circ$. Further, take $B = (b_n)_{n\geq 1} \in s_\alpha^\circ$. Then there exists $\nu = (\nu_n)_{n\geq 1} \in c_0$ such that $b_n = \alpha_n \nu_n$. We must prove that the equation $\Delta X = B$ admits a unique solution in the space s_α°. First, we obtain

$$X = \Sigma B = \left(\sum_{k=1}^{n}\alpha_k \nu_k\right)_{n\geq 1}. \tag{2.30}$$

In order to show that $X = (x_n)_{n\geq 1} \in s_\alpha^\circ$, we will consider any given $\varepsilon > 0$. From Proposition 2.1(iii), the condition $\alpha \in \widehat{C_1}$ implies that $\alpha_n \to \infty$. So, there exists an integer N such that

$$\begin{gathered} S_n = \frac{1}{\alpha_n}\left|\sum_{k=1}^{N}\alpha_k \nu_k\right| \leq \frac{\varepsilon}{2} \quad \text{for } n \geq N, \\ \sup_{n\geq N+1}\left(|\nu_k|\right) \leq \frac{\varepsilon}{2\sup_{n\geq 1}\left(\left[C(\alpha)\alpha\right]_n\right)}. \end{gathered} \tag{2.31}$$

Writing $R_n = 1/\alpha_n \left|\sum_{k=N+1}^{n}\alpha_k \nu_k\right|$, we conclude that

$$R_n \leq \left(\sup_{N+1\leq k\leq n}\left(|\nu_k|\right)\right)\left[C(\alpha)\alpha\right]_n \leq \frac{\varepsilon}{2}. \tag{2.32}$$

Finally, we obtain

$$\frac{|x_n|}{\alpha_n} = \left| \frac{1}{\alpha_n}\left(\sum_{k=1}^{N} \alpha_k \nu_k\right) + \frac{1}{\alpha_n}\left(\sum_{k=N+1}^{n} \alpha_k \nu_k\right)\right| \leq S_n + R_n \leq \varepsilon \quad \text{for } n \geq N, \tag{2.33}$$

and $X \in s_\alpha^0$.

(iii) As above, $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ if and only if $\Delta_\alpha, \Sigma_\alpha \in (c,c)$; and from Lemma 2.4, we have $\Delta_\alpha \in (c,c)$ if and only if $(\alpha_{n-1}/\alpha_n)_n \in c$. In fact, we have $\Delta_\alpha \in S_1$ and $\sum_{m=1}^{n} d_{nm} = 1 + \alpha_{n-1}/\alpha_n$ tends to a limit as $n \to \infty$. Afterwards, $\Sigma_\alpha \in (c,c)$ is equivalent to

(a) $\Sigma_\alpha \in S_1$, that is, $\alpha \in \widehat{C_1}$;
(b) $\lim_n(\alpha_m/\alpha_n) = 0$ for all $m \geq 1$;
(c) $\alpha \in \widehat{C}$.

From Proposition 2.1(iii), (c) implies that α_n tends to infinity, so (c) implies (a) and (b). Finally, from Proposition 2.1(ii), we conclude that $\alpha \in \widehat{C}$ implies $(\alpha_{n-1}/\alpha_n)_n \in c$. This completes the proof of (iii).

(iv) From Lemma 2.5, it can be easily verified that $\Delta_\alpha \in (c,c)$ and $\lim X = \Delta_\alpha - \lim X$ if and only if $\alpha_{n-1}/\alpha_n \to 0$. We conclude, using (iii), since $\alpha_{n-1}/\alpha_n = o(1)$ implies that $\alpha \in \widehat{C}$. □

Remark 2.7. In Theorem 2.6(iv), we see that $\Sigma_\alpha \in (c,c)$ and $\lim X = \Sigma_\alpha - \lim X$ if and only if $\alpha_{n-1}/\alpha_n \to 0$. In fact, we must have for each $m \geq 1$, $\sigma_{nm} = \alpha_m/\alpha_n = o(1)$ $(n \to \infty)$ and

$$\lim_n \left(\sum_{m=1}^{n} \sigma_{nm}\right) = \lim_n \left(1 + \sum_{m=1}^{n-1} \frac{\alpha_m}{\alpha_n}\right) = 1, \tag{2.34}$$

and from Proposition 2.1(i), the previous property is satisfied if and only if $\alpha_{n-1}/\alpha_n \to 0$.

Remark 2.8. It can be seen that the condition $(\alpha_{n-1}/\alpha_n)_n \in c$ does not imply that $\alpha \in \widehat{C_1}$. It is enough to consider $C(e)e = (n)_n \notin c_0$.

The next corollary is a direct consequence of the previous results.

Corollary 2.9. *Consider the following properties:*

(i) $\alpha \in \widehat{C}$;
(ii) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$;
(iii) $\alpha \in \Gamma$;
(iv) $\alpha \in \widehat{C_1}$;
(v) $s_\alpha(\Delta) = s_\alpha$;
(vi) $s_\alpha^\circ(\Delta) = s_\alpha^\circ$.

Then (i)⇔(ii)⇒(iii)⇒(iv)⇔(v)⇔(vi).

We obtain from the precedent the following corollary.

Corollary 2.10. (i) $(s_\alpha - s_\alpha^\circ)(\Delta) = s_\alpha - s_\alpha^\circ$ *if and only if* $\alpha \in \widehat{C_1}$,
(ii) $(s_\alpha^{(c)} - s_\alpha^\circ)(\Delta) = s_\alpha^{(c)} - s_\alpha^\circ$ *if and only if* $\alpha \in \widehat{C}$,
(iii) $\alpha \in \widehat{C}$ *implies* $(s_\alpha - s_\alpha^{(c)})(\Delta) = s_\alpha - s_\alpha^{(c)}$.

Proof. (i) If Δ is bijective from $s_\alpha - s_\alpha^\circ$ into itself, then for every $B \in s_\alpha - s_\alpha^\circ$, we have $X = \Sigma B \in s_\alpha - s_\alpha^\circ$. Since $\alpha \in s_\alpha - s_\alpha^\circ$, we conclude that $\Sigma\alpha \in s_\alpha$, that is, $C(\alpha)\alpha \in l_\infty$. Conversely, from Theorem 2.6(i) and (ii), it can be easily seen that Δ is bijective from s_α to s_α and from s_α° to s_α°, since $\alpha \in \widehat{C_1}$. So, Δ is bijective from $s_\alpha - s_\alpha^\circ$ to $s_\alpha - s_\alpha^\circ$.

(ii) Suppose that Δ is bijective from $s_\alpha^{(c)} - s_\alpha^\circ$ into itself. Reasoning as above, we have $\alpha \in s_\alpha^{(c)} - s_\alpha^\circ$ and $\Sigma\alpha \in s_\alpha^{(c)}$, so $D_{1/\alpha}\Sigma\alpha = C(\alpha)\alpha \in c$. Conversely, using Theorem 2.6(i) and (iii), we see that Δ is bijective from $s_\alpha^{(c)}$ to $s_\alpha^{(c)}$ and from s_α° to s_α° since $\alpha \in \widehat{C}$ and $\widehat{C} \subset \widehat{C_1}$. So, $(s_\alpha^{(c)} - s_\alpha^\circ)(\Delta) = s_\alpha^{(c)} - s_\alpha^\circ$.

Similarly, (iii) comes from the fact that Δ is bijective from $s_\alpha^{(c)}$ into itself and from s_α into itself, since $\alpha \in \widehat{C}$. □

Remark 2.11. Assume that $\lim_{n\to\infty}[C(\alpha)\alpha]_n = l$. Then

$$\frac{x_n}{\alpha_n} \longrightarrow L \text{ implies } \frac{x_n - x_{n-1}}{\alpha_n} \longrightarrow \frac{L}{l}. \tag{2.35}$$

Indeed, from Proposition 2.1(ii) (b), $\alpha_{n-1}/\alpha_n \to 1-(1/l)$ and

$$\frac{x_n - x_{n-1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} \longrightarrow L - L\left(1 - \frac{1}{l}\right) = \frac{L}{l}. \tag{2.36}$$

3. Generalization to the sets $s_r(\Delta^h)$ and $s_\alpha(\Delta^h)$ for h real. In this section, we consider the operator Δ^h, where h is a real, and give among other things a necessary and sufficient condition to have $s_\alpha(\Delta^h) = s_\alpha$.

First, recall that we can associate to any power series $f(z) = \sum_{k=0}^\infty a_k z^k$, defined in the open disk $|z| < R$, the upper triangular infinite matrix $A = \varphi(f) \in \bigcup_{0<r<R} S_r$ defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot \\ & a_0 & a_1 & \cdot \\ 0 & & a_0 & \cdot \\ & & & \cdot \end{pmatrix} \tag{3.1}$$

(see [3, 4, 5]). Practically, we will write $\varphi[f(z)]$ instead of $\varphi(f)$. We have the following lemma.

Lemma 3.1. (i) *The map $\varphi : f \to A$ is an isomorphism from the algebra of the power series defined in $|z| < R$ into the algebra of the corresponding matrices $\bar{A}$.*

(ii) *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *with* $a_0 \neq 0$, *and assume that* $1/f(z) = \sum_{k=0}^{\infty} a'_k z^k$ *admits* $R' > 0$ *as radius of convergence. Then*

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in \bigcup_{0<r<R'} S_r. \tag{3.2}$$

Now, for $h \in R - N$, we define (see [13])

$$\begin{aligned} \binom{-h+k-1}{k} &= \frac{-h(-h+1)\cdots(-h+k-1)}{k!} \quad \text{if } k > 0, \\ \binom{-h+k-1}{k} &= 1 \quad \text{if } k = 0, \end{aligned} \tag{3.3}$$

and putting $\Delta^+ = \Delta^t$, we get for any $h \in R$,

$$(\Delta^+)^h = \varphi[(1-z)^h] = \varphi\left[\sum_{k=0}^{\infty} \binom{-h+k-1}{k} z^k\right] \quad \text{for } |z| < 1. \tag{3.4}$$

Then if $\Delta^h = (\tau_{nm})_{n,m}$,

$$\tau_{nm} = \begin{cases} \binom{-h+n-m-1}{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{3.5}$$

Using the isomorphism φ, we get the following proposition.

Proposition 3.2 (see [5]). (i) *The operator represented by* Δ *is bijective from* s_r *into itself for every* $r > 1$, *and* Δ^+ *is bijective from* s_r *into itself for all* r, $0 < r < 1$.

(ii) *The operator* Δ^+ *is surjective and not injective from* s_r *into itself for all* $r > 1$.

(iii) *For all* $r \neq 1$ *and for every integer* $\mu \geq 1$, $(\Delta^+)^h s_r = s_r$.

(iv) *We have successively*

(α) *if* h *is a real greater than* 0 *and* $h \notin N$, *then* Δ^h *maps* s_r *into itself when* $r \geq 1$, *but not for* $0 < r < 1$; *if* $-1 < h < 0$, *then* Δ^h *maps* s_r *into itself when* $r > 1$, *but not for* $r = 1$;

(β) *if* $h > 0$ *and* $h \notin N$, *then* $(\Delta^+)^h$ *maps* s_r *into itself when* $0 < r \leq 1$, *but not if* $r > 1$; *if* $-1 < h < 0$, *then* $(\Delta^+)^h$ *maps* s_r *into itself for* $0 < r < 1$, *but not for* $r = 1$.

(v) *Let* h *be any given integer* ≥ 1, *Then*

$$\begin{aligned} A \in (s_r(\Delta^h), s_r) &\iff \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty \quad \forall r > 1, \\ A \in \left(s_r(\Delta^+)^h, s_r\right) &\iff \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty \quad \forall r \in]0,1[. \end{aligned} \tag{3.6}$$

(vi) *For every integer* $h \geq 1$,

$$s_1 \subset s_1(\Delta^h) \subset s_{(n^h)_{n\geq 1}} \subset \bigcap_{r>1} s_r. \tag{3.7}$$

(vii) *If* $h > 0$ *and* $h \notin N$, *then* q *is the greatest integer strictly less than* $(h+1)$. *For all* $r > 1$,

$$\operatorname{Ker}\left((\Delta^+)^h\right) \bigcap s_r = \operatorname{span}(V_1, V_2, \ldots, V_q), \tag{3.8}$$

where

$$\begin{gathered} V_1 = e^t, \qquad V_2 = (A_1^1, A_2^1, \ldots)^t, \\ V_3 = (0, A_2^2, A_3^2, \ldots)^t, \ldots, \qquad V_q = \left(0, 0, \ldots, A_{q-1}^{q-1}, A_q^{q-1}, \ldots, A_n^{q-1}, \ldots\right)^t; \end{gathered} \tag{3.9}$$

$A_i^j = i!/(i-j)!$, *with* $0 \leq j \leq i$, *being the number of permutations of* i *things taken* j *at a time.*

We give here an extension of the previous results, where s_r is replaced by s_α.

Proposition 3.3. *Let* h *be a real greater than 0. The condition* $s_\alpha(\Delta^h) = s_\alpha$ *is equivalent to*

$$\gamma_n(h) = \frac{1}{\alpha_n}\left[\sum_{k=1}^{n-1}\binom{h+n-k-1}{n-k}\alpha_k\right] = O(1) \quad (n \longrightarrow \infty). \tag{3.10}$$

Proof. The operator Δ^h is bijective from s_α into itself if and only if Δ^h, $\Sigma^h \in (s_\alpha, s_\alpha)$. We have $\Delta^h \in (s_\alpha, s_\alpha)$ if and only if

$$D_{1/\alpha}\Delta^h D_\alpha \in S_1, \tag{3.11}$$

and using (3.5), we deduce that $\Delta^h \in (s_\alpha, s_\alpha)$ if and only if

$$\frac{1}{\alpha_n}\sum_{k=1}^{n}\left|\binom{-h+n-k-1}{n-k}\right|\alpha_k = O(1). \tag{3.12}$$

Further, $(\Sigma^t)^h = \varphi[(1-z)^{-h}]$, where

$$\varphi(z) = 1 + \sum_{n=1}^{\infty}\binom{h+n-1}{n} z^n \quad \text{with } |z| < 1. \tag{3.13}$$

So, $D_{1/\alpha}\Sigma^h D_\alpha \in S_1$ if and only if (3.10) holds. Finally, since $h > 0$, we have

$$\left|\binom{-h+n-k-1}{n-k}\right| \leq \binom{h+n-k-1}{n-k} \quad \text{for } k = 1,2,\dots,n-1, \tag{3.14}$$

and we conclude since (3.10) implies (3.12). □

We deduce immediately the next result.

Corollary 3.4. *Let h be an integer greater than or equal to 1. The following properties are equivalent:*

(i) $\alpha \in \widehat{C_1}$;

(ii) $s_\alpha(\Delta) = s_\alpha$;

(iii) $s_\alpha(\Delta^h) = s_\alpha$;

(iv) $C(\alpha)(\Sigma^{h-1}\alpha) \in l_\infty$.

Proof. From the proof of Proposition 3.3, $s_\alpha(\Delta^h) = s_\alpha$ is equivalent to $D_{1/\alpha}\Sigma^h D_\alpha = C(\alpha)\Sigma^{h-1}D_\alpha \in S_1$, that is, $C(\alpha)(\Sigma^{h-1}\alpha) \in l_\infty$. So, (iii) and (iv) are equivalent. It remains to prove that (ii)⇔(iii). If $s_\alpha(\Delta) = s_\alpha$, Δ and consequently Δ^h are bijective from s_α into itself and condition (iii) holds. Conversely, assume that $s_\alpha(\Delta^h) = s_\alpha$ holds. Then (3.10) holds, and since

$$\binom{h+n-k-1}{n-k} \geq 1 \quad \text{for } k = 1,2,\dots,n-1, \tag{3.15}$$

we deduce that

$$[C(\alpha)\alpha]_n \leq \gamma_n(h) = O(1), \quad n \longrightarrow \infty. \tag{3.16}$$

So, (i) holds and (ii) is satisfied. □

4. Generalization of well-known sets. In this section, we see that under some conditions, the spaces $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, $\widetilde{w_\alpha^*}(\lambda)$, $\widetilde{c_\alpha}(\lambda,\mu)$, $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, and $\widetilde{c_\alpha^*}(\lambda,\mu)$ can be written by means of the sets s_ξ or s_ξ°.

4.1. Sets $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda)$. We recall some definitions and properties of some spaces. For every sequence $X = (x_n)_n$, we define $|X| = (|x_n|)_n$ and

$$\begin{aligned} \widetilde{w_\alpha}(\lambda) &= \{X \in s \mid C(\lambda)(|X|) \in s_\alpha\}, \\ \widetilde{w_\alpha^\circ}(\lambda) &= \{X \in s \mid C(\lambda)(|X|) \in s_\alpha^\circ\}, \\ \widetilde{w_\alpha^*}(\lambda) &= \{X \in s \mid X - le^t \in \widetilde{w_\alpha^\circ}(\lambda) \text{ for some } l \in C\}. \end{aligned} \tag{4.1}$$

For instance, we see that

$$\widetilde{w_\alpha}(\lambda) = \left\{ X = (x_n)_n \in s \mid \sup_{n\geq 1} \left(\frac{1}{|\lambda_n| \alpha_n} \sum_{k=1}^{n} |x_k| \right) < \infty \right\}. \tag{4.2}$$

If there exist $A, B > 0$ such that $A < \alpha_n < B$ for all n, we get the well-known spaces $\widetilde{w_\alpha}(\lambda) = w_\infty(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda) = w_0(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda) = w(\lambda)$ (see [12]). It has been proved that if λ is a strictly increasing sequence of reals tending to infinity, $w_0(\lambda)$ and $w_\infty(\lambda)$ are BK spaces and $w_0(\lambda)$ has AK, with respect to the norm

$$\|X\| = \left\|C(\lambda)(|X|)\right\|_{l^\infty} = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |x_k| \right) \tag{4.3}$$

(see [1]).

We have the next result.

Theorem 4.1. *Let α and λ be any sequences of U^{+*}.*

(i) *Consider the following properties:*

(a) $\alpha_{n-1}\lambda_{n-1}/\alpha_n\lambda_n \to 0$;

(b) $s_\alpha^{(c)}(C(\lambda)) = s_{\alpha\lambda}^{(c)}$;

(c) $\alpha\lambda \in \widehat{C_1}$;

(d) $\widetilde{w_\alpha}(\lambda) = s_{\alpha\lambda}$;

(e) $\widetilde{w_\alpha^\circ}(\lambda) = s_{\alpha\lambda}^\circ$;

(f) $\widetilde{w_\alpha^*}(\lambda) = s_{\alpha\lambda}^\circ$.

Then (a)⇒(b), (c)⇔(d), and (c)⇒(e) and (f).

(ii) *If $\alpha\lambda \in \widehat{C_1}$, $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda)$ are BK spaces with respect to the norm*

$$\|X\|_{s_{\alpha\lambda}} = \sup_{n\geq 1} \left(\frac{|x_n|}{\alpha_n\lambda_n} \right), \tag{4.4}$$

and $\widetilde{w_\alpha^\circ}(\lambda) = \widetilde{w_\alpha^*}(\lambda)$ has AK.

Proof. (i) First, we prove that (a)⇒(b). We have

$$s_\alpha^{(c)}(C(\lambda)) = \Delta(\lambda) s_\alpha^{(c)} = \Delta D_\lambda s_\alpha^{(c)} = \Delta s_{\alpha\lambda}^{(c)}, \tag{4.5}$$

and from Proposition 2.1(i) and Theorem 2.6(iii), we get successively $\alpha\lambda \in \widehat{C}$, $\Delta s_{\alpha\lambda}^{(c)} = s_{\alpha\lambda}^{(c)}$, and (b) holds.

(c)⇔(d). Assume that (c) holds. Then

$$\widetilde{w_\alpha}(\lambda) = \{X \mid |X| \in \Delta(\lambda) s_\alpha\}. \tag{4.6}$$

Since $\Delta(\lambda) = \Delta D_\lambda$, we get $\Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda}$. Now, using (c), we see that Δ is bijective from $s_{\alpha\lambda}$ into itself and $w_\alpha(\lambda) = s_{\alpha\lambda}$. Conversely, assume that $w_\alpha(\lambda) = s_{\alpha\lambda}$. Then $\alpha\lambda \in s_{\alpha\lambda}$ implies that $C(\lambda)(\alpha\lambda) \in s_\alpha$, and since $D_{1/\alpha}C(\lambda)(\alpha\lambda) \in s_1 = l_\infty$, we conclude that $C(\alpha\lambda)(\alpha\lambda) \in l_\infty$. The proof of (c)⇒(e) follows on the same lines of the proof of (c)⇒(d) replacing $s_{\alpha\lambda}$ by $s^\circ_{\alpha\lambda}$.

We prove that (c) implies (f). Take $X \in \widetilde{w^*_\alpha}(\lambda)$. There is a complex number l such that

$$C(\lambda)\left(\left|X - le^t\right|\right) \in s^\circ_\alpha. \tag{4.7}$$

So

$$\left|X - le^t\right| \in \Delta(\lambda)s^\circ_\alpha = \Delta s^\circ_{\alpha\lambda}, \tag{4.8}$$

and from Theorem 2.6(ii), $\Delta s^\circ_{\alpha\lambda} = s^\circ_{\alpha\lambda}$. Now, since (c) holds, we deduce from Proposition 2.1(iii) that $\alpha_n\lambda_n \to \infty$ and $le^t \in s^\circ_{\alpha\lambda}$. We conclude that $X \in \widetilde{w^*_\alpha}(\lambda)$ if and only if $X \in le^t + s^\circ_{\alpha\lambda} = s^\circ_{\alpha\lambda}$.

Assertion (ii) is a direct consequence of (i). □

4.2. Sets $\widetilde{c_\alpha}(\lambda,\mu)$, $\widetilde{c^\circ_\alpha}(\lambda,\mu)$, and $\widetilde{c^*_\alpha}(\lambda,\mu)$. Let $\alpha = (\alpha_n)_n \in U^{+*}$ be a given sequence, we consider now for $\lambda \in U$, $\mu \in s$ the space

$$\widetilde{c_\alpha}(\lambda,\mu) = \left(w_\alpha(\lambda)\right)_{\Delta(\mu)} = \left\{X \in s \mid \Delta(\mu)X \in w_\alpha(\lambda)\right\}. \tag{4.9}$$

It is easy to see that

$$\widetilde{c_\alpha}(\lambda,\mu) = \left\{X \in s \mid C(\lambda)\left(\left|\Delta(\mu)X\right|\right) \in s_\alpha\right\}, \tag{4.10}$$

that is,

$$\widetilde{c_\alpha}(\lambda,\mu) = \left\{X = (x_n)_n \in s \mid \sup_{n\geq 2}\left(\frac{1}{|\lambda_n|\alpha_n}\sum_{k=2}^{n}\left|\mu_k x_k - \mu_{k-1}x_{k-1}\right|\right) < \infty\right\}, \tag{4.11}$$

see [1]. Similarly, we define the following sets:

$$\begin{aligned}\widetilde{c^\circ_\alpha}(\lambda,\mu) &= \left\{X \in s \mid C(\lambda)\left(\left|\Delta(\mu)X\right|\right) \in s^\circ_\alpha\right\},\\ \widetilde{c^*_\alpha}(\lambda,\mu) &= \left\{X \in s \mid X - le^t \in \widetilde{c^\circ_\alpha}(\lambda,\mu) \text{ for some } l \in C\right\}.\end{aligned} \tag{4.12}$$

Recall that if $\lambda = \mu$, it is written that $c_0(\lambda) = (w_0(\lambda))_{\Delta(\lambda)}$,

$$c(\lambda) = \left\{X \in s \mid X - le^t \in c_0(\lambda) \text{ for some } l \in C\right\}, \tag{4.13}$$

and $c_\infty(\lambda) = (w_\infty(\lambda))_{\Delta(\lambda)}$, see [11]. It can be easily seen that

$$c_0(\lambda) = \widetilde{c_e^\circ}(\lambda,\lambda), \qquad c_\infty(\lambda) = \widetilde{c_e}(\lambda,\lambda), \qquad c(\lambda) = \widetilde{c_e^*}(\lambda,\lambda). \tag{4.14}$$

These sets of sequences are called strongly convergent to 0, strongly convergent, and strongly bounded. If $\lambda \in U^{+*}$ is a sequence strictly increasing to infinity, $c(\lambda)$ is a Banach space with respect to

$$\|X\|_{c_\infty(\lambda)} = \sup_{n\geq 1}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}\left|\lambda_k x_k - \lambda_{k-1}x_{k-1}\right|\right) \tag{4.15}$$

with the convention $x_0 = 0$. Each of the spaces $c_0(\lambda)$, $c(\lambda)$, and $c_\infty(\lambda)$ is a BK space, relatively to the previous norm (see [1]). The set $c_0(\lambda)$ has AK and every $X \in c(\lambda)$ has a unique representation given by

$$X = le^t + \sum_{k=1}^{\infty}(x_k - l)e_k^t, \tag{4.16}$$

where $X - le^t \in c_0$. The scalar l is called the strong $c(\lambda)$-limit of the sequence X.

We obtain the next result.

Theorem 4.2. *Let α, λ, and μ be sequences of U^{+*}.*

(i) *Consider the following properties:*

(a) $\alpha\lambda \in \widehat{C_1}$;

(b) $\widetilde{c_\alpha}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}$;

(c) $\widetilde{c_\alpha^\circ}(\lambda,\mu) = s^\circ_{\alpha(\lambda/\mu)}$;

(d) $\widetilde{c_\alpha^*}(\lambda,\mu) = \{X \in s \mid X - le^t \in s^\circ_{\alpha(\lambda/\mu)}$ *for some* $l \in C\}$.

Then (a)⇔(b) and (a)⇒(c) and (d).

(ii) *If $\alpha\lambda \in \widehat{C_1}$, then $\widetilde{c_\alpha}(\lambda,\mu)$, $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, and $\widetilde{c_\alpha^*}(\lambda,\mu)$ are BK spaces with respect to the norm*

$$\|X\|_{s_{\alpha(\lambda/\mu)}} = \sup_{n\geq 1}\left(\mu_n\frac{|x_n|}{\alpha_n\lambda_n}\right). \tag{4.17}$$

The set $\widetilde{c_\alpha^\circ}(\lambda,\mu)$ has AK and every $X \in \widetilde{c_\alpha^*}(\lambda,\mu)$ has a unique representation given by (4.16), where $X - le^t \in s^\circ_{\alpha(\lambda/\mu)}$.

Proof. We show that (a)⇒(b). Take $X \in c_\alpha(\lambda,\mu)$. We have $\Delta(\mu)X \in w_\alpha(\lambda)$, which is equivalent to

$$X \in C(\mu)s_{\alpha\lambda} = D_{1/\mu}\Sigma s_{\alpha\lambda}, \tag{4.18}$$

and using Theorem 2.6(i), Δ and consequently Σ are bijective from $s_{\alpha\lambda}$ into itself. So, $\Sigma s_{\alpha\lambda} = s_{\alpha\lambda}$ and $X \in D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{\alpha(\lambda/\mu)}$. We conclude that (b) holds. We prove that (b) implies (a). First, put $\tilde{\alpha}_{\lambda,\mu} = ((-1)^n(\lambda_n/\mu_n)\alpha_n)_{n\geq 1}$. We have

$\widetilde{\alpha}_{\lambda,\mu} \in s_{\alpha(\lambda/\mu)} = \widetilde{c_\alpha}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}$, and since $\Delta(\mu) = \Delta D_\mu$ and $D_\mu\widetilde{\alpha}_{\lambda,\mu} = ((-1)^n\lambda_n\alpha_n)_{n\ge1}$, we get $|\Delta(\mu)\widetilde{\alpha}_{\lambda,\mu}| = (\xi_n)_{n\ge1}$, with

$$\xi_n = \begin{cases} \lambda_1\alpha_1 & \text{if } n = 1, \\ \lambda_{n-1}\alpha_{n-1} + \lambda_n\alpha_n & \text{if } n \ge 2. \end{cases} \tag{4.19}$$

From (b), we deduce that $\Sigma|\Delta(\mu)\widetilde{\alpha}_{\lambda,\mu}| \in s_{\alpha\lambda}$. This means that

$$C'_n = \frac{1}{\alpha_n\lambda_n}\left(\lambda_1\alpha_1 + \sum_{k=2}^{n}(\lambda_{k-1}\alpha_{k-1} + \lambda_k\alpha_k)\right) = O(1), \quad n \longrightarrow \infty. \tag{4.20}$$

From the inequality

$$[C(\alpha\lambda)(\alpha\lambda)]_n \le C'_n, \tag{4.21}$$

we obtain (a). The proof of (a)⇒(c) follows on the same lines of the proof of (a)⇒(b) with s_α replaced by s°_α.

We show that (a) implies (d). Take $X \in \widetilde{c^*_\alpha}(\lambda,\mu)$. There exists $l \in C$ such that

$$\Delta(\mu)(X - le^t) \in \widetilde{w^\circ_\alpha}(\lambda), \tag{4.22}$$

and from (c)⇒(e) in Theorem 4.1, we have $\widetilde{w^\circ_\alpha}(\lambda) = s^\circ_{\alpha\lambda}$. So

$$X - le^t \in C(\mu)s^\circ_{\alpha\lambda} = D_{1/\mu}\Sigma s^\circ_{\alpha\lambda}, \tag{4.23}$$

and from Theorem 2.6(ii), $\Sigma s^\circ_{\alpha\lambda} = s^\circ_{\alpha\lambda}$, and $D_{1/\mu}\Sigma s^\circ_{\alpha\lambda} = s^\circ_{\alpha(\lambda/\mu)}$, we conclude that $X \in \widetilde{c^*_\alpha}(\lambda,\mu)$ if and only if $X \in le^t + s^\circ_{\alpha(\lambda/\mu)}$ for some $l \in C$.

Assertion (ii) is a direct consequence of (i) and of the fact that for every $X \in \widetilde{c^*_\alpha}(\lambda)$, we have

$$\left\|X - le^t - \sum_{k=1}^{N}(x_k - l)e^t_k\right\|_{s_{\alpha(\lambda/\mu)}} = \sup_{n\ge N+1}\left(\mu_n\frac{|x_n - l|}{\alpha_n\lambda_n}\right) = o(1), \quad N \longrightarrow \infty. \tag{4.24}$$

□

We deduce immediately the following corollary.

Corollary 4.3. *Assume that α, λ, $\mu \in U^{+*}$.*
(i) *If $\alpha\lambda \in \widehat{C_1}$ and $\mu \in l_\infty$, then*

$$\widetilde{c^*_\alpha}(\lambda,\mu) = s^\circ_{\alpha(\lambda/\mu)}. \tag{4.25}$$

(ii) *Then*

$$\lambda \in \Gamma \Longrightarrow \lambda \in \widehat{C_1} \Longrightarrow c_0(\lambda) = s_\lambda^\circ, \qquad c_\infty(\lambda) = s_\lambda. \tag{4.26}$$

Proof. (i) Since $\mu \in l_\infty$, we deduce, using Proposition 2.1(iii), that there are $K > 0$ and $\gamma > 1$ such that

$$\frac{\alpha_n \lambda_n}{\mu_n} \geq K\gamma^n \quad \forall n. \tag{4.27}$$

So, $le^t \in s^\circ_{\alpha(\lambda/\mu)}$ and (4.25) holds. (ii) comes from Theorem 4.2 since $\Gamma \subset \widehat{C_1}$. □

Example 4.4. We denote by $\widetilde{e}$ the base of the natural system of logarithms. From the well-known Stirling formula, we have

$$\frac{n^{n+1/2}}{n!} \sim \widetilde{e}^n \frac{1}{\sqrt{2\pi}}, \tag{4.28}$$

so $s_{(n^{n+(1/2)}/n!)_n} = s_{\widetilde{e}}$. Further, $\lambda = (n^n/n!)_n \in \Gamma$ since

$$\frac{\lambda_{n-1}}{\lambda_n} = \widetilde{e}^{-(n-1)\ln(1+1/(n-1))} \longrightarrow \frac{1}{\widetilde{e}} < 1. \tag{4.29}$$

We conclude that

$$\widetilde{C}_e^\circ\left(\left(\frac{n^n}{n!}\right)_n, \left(\frac{1}{\sqrt{n}}\right)_n\right) = s_{\widetilde{e}}^\circ, \qquad \widetilde{C}_e\left(\left(\frac{n^n}{n!}\right)_n, \left(\frac{1}{\sqrt{n}}\right)_n\right) = s_{\widetilde{e}}. \tag{4.30}$$

References

[1] A. M. Al-Jarrah and E. Malkowsky, *BK spaces, bases and linear operators*, Rend. Circ. Mat. Palermo (2) Suppl. (1998), no. 52, 177–191.

[2] B. de Malafosse, *Some properties of the Cesàro operator in the space s_r*, Comm. Fac. Sci. Univ. Ankara Ser. A_1 Math. Statist. **48** (1999), no. 1-2, 53–71.

[3] ———, *Bases in sequence spaces and expansion of a function in a series of power series*, Mat. Vesnik **52** (2000), no. 3-4, 99–112.

[4] ———, *Application of the sum of operators in the commutative case to the infinite matrix theory*, Soochow J. Math. **27** (2001), no. 4, 405–421.

[5] ———, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002), no. 2, 283–299.

[6] ———, *Recent results in the infinite matrix theory, and application to Hill equation*, Demonstratio Math. **35** (2002), no. 1, 11–26.

[7] ———, *Variation of an element in the matrix of the first difference operator and matrix transformations*, Novi Sad J. Math. **32** (2002), no. 1, 141–158.

[8] B. de Malafosse and E. Malkowsky, *Sequence spaces and inverse of an infinite matrix*, Rend. Circ. Mat. Palermo (2) **51** (2002), no. 2, 277–294.

[9] R. Labbas and B. de Malafosse, *On some Banach algebra of infinite matrices and applications*, Demonstratio Math. **31** (1998), no. 1, 153–168.

[10] I. J. Maddox, *Infinite Matrices of Operators*, Lecture Notes in Mathematics, vol. 786, Springer-Verlag, Berlin, 1980.

[11] F. Móricz, *On Λ-strong convergence of numerical sequences and Fourier series*, Acta Math. Hungar. **54** (1989), no. 3-4, 319–327.

[12] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, North-Holland Publishing, Amsterdam, 1984.

[13] K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*, Zweite, erweiterte und verbesserte Auflage. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 15, Springer-Verlag, Berlin, 1970 (German).

Bruno de Malafosse: Laboratoire de Mathematiques Appliquées Havrais, Institut Universitaire de Technologie Le Havre, Université du Havre, BP 4006, 76610 Le Havre, France

E-mail address: bdemalaf@europost.org