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C^m SOLUTIONS OF SYSTEMS OF FINITE DIFFERENCE EQUATIONS

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Let $\mathbb{R}$ be the real number axis. Suppose that G, H are C^m maps from $\mathbb{R}^{2n+3}$ to $\mathbb{R}$. In this note, we discuss the system of finite difference equations $G(x,f(x),f(x+1),\dots,f(x+n),g(x),g(x+1),\dots,g(x+n))=0$ and $H(x,g(x),g(x+1),\dots,g(x+n),f(x),f(x+1),\dots,f(x+n))=0$ for all $x\in\mathbb{R}$, and give some relatively weak conditions for the above system of equations to have unique C^m solutions $(m\geq 0)$.

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1. Introduction. In [4, 5, 6], the iterative functional equations $f^2(z)(=f(f(z)))=az^2+bz+c$ and $\sum_{k=0}^{n}c_kf^k=0$ were considered, respectively. Zhang [7, 8] showed the existence and uniqueness of C^0, C^1 solutions of the equation $F(x)-\sum_{k=1}^{n}\lambda_kf^k(x)=0$. In [3], the authors studied more general iterative functional equation $G(x,f(x),\dots,f^n(x))=0$ and showed the existence, uniqueness, and stability of C^m solutions $(m\geq 0)$ of the equation. The C^m solutions $(m\geq 0)$ of the equation $\sum_{i=0}^{n}c_if(x+i)=F(x)$ were discussed in [2]. In this note, we discuss the following system of finite difference equations:

$$\begin{aligned}&G(x,f(x),f(x+1),\dots,f(x+n),g(x),g(x+1),\dots,g(x+n))=0,\\&H(x,g(x),g(x+1),\dots,g(x+n),f(x),f(x+1),\dots,f(x+n))=0,\end{aligned}\tag{1.1}$$

for all $x\in\mathbb{R}$, where $G,H\in C^m(\mathbb{R}^{2n+3},\mathbb{R})$ are given functions and $f,g\in C^m(\mathbb{R},\mathbb{R})$ are unknown functions to be solved. Using the method of approximating fixed points by a small shift of maps, we give some relatively weak conditions for the above system of equations to have unique C^m solutions for any integer $m\geq 0$.

Denote by $\mathbb{Z}_+$ the set of all nonnegative integers. For $m\in\mathbb{Z}_+$ and $k\in\mathbb{N}$, write $\mathbb{Z}_m=\{0,1,\dots,m\}$ and $\mathbb{N}_k=\{1,\dots,k\}$. For $f,g\in C^0(\mathbb{R},\mathbb{R})$ and $r,s\in\mathbb{R}$, define the map $rf+sg:\mathbb{R}\to\mathbb{R}$ by $(rf+sg)(x)=rf(x)+sg(x)$ (for any $x\in\mathbb{R}$). Then, under this operation, $C^0(\mathbb{R},\mathbb{R})$ is a linear space.

Let $m\geq k>0$. For $g\in C^m(\mathbb{R},\mathbb{R})$, denote by $g^{(k)}$ the kth derivative of g. Then $g^{(k)}\in C^{m-k}(\mathbb{R},\mathbb{R})$. Usually, $g^{(1)}$ and $g^{(2)}$ are written as g' and g''. In addition, for any $g\in C^0(\mathbb{R},\mathbb{R})$, we put $g^{(0)}=g$ and call $g^{(0)}$ the 0th derivative of g.

Now we introduce some symbols which are defined as in [3]. For any two points $x \neq y$ in $\mathbb{R}$, $(g(x)-g(y))/(x-y)$ is called a difference quotient of g. Let

$$\Lambda_g = \Lambda(g) = \{(g(x)-g(y))/(x-y) : x,y \in \mathbb{R},\ x \neq y\}. \tag{1.2}$$

The set Λ_g is called the set of difference quotients of g. If $\Lambda_g \subset [0,\infty)$, then g is increasing; and if $\Lambda_g \subset (0,\infty)$, then g is strictly increasing. For $g \in C^1(\mathbb{R},\mathbb{R})$, it is easy to verify that

$$\Lambda_g \subset g'(\mathbb{R}) \subset \overline{\Lambda}_g, \tag{1.3}$$

where $g'(\mathbb{R}) = \{g'(x) : x \in \mathbb{R}\}$ and $\overline{\Lambda}_g$ is the closure of Λ_g in $\mathbb{R}$. Write

$$\lambda_g = \lambda(g) = \sup\{|t| : t \in \Lambda_g\}. \tag{1.4}$$

If $\lambda_g < \infty$, that is, Λ_g is bounded, then g is said to be Lipschitz continuous and λ_g is called the (smallest) Lipschitz constant of g.

Let $m \geq j \geq 0$ be integers and let $r \geq 0$ be a real number. Suppose that K, K_0, $K_1, \ldots, K_j$ are all connected closed subsets of $\mathbb{R}$. Write

$$\begin{gathered} C^m(\mathbb{R},K;K_0,K_1,\ldots,K_j) = \{f \in C^m(\mathbb{R},K) : \Lambda(f^{(i)}) \subset K_i,\ \text{for } i = 0,1,\ldots,j\}, \\ C^m(\mathbb{R},K;r) = \{f \in C^m(\mathbb{R},K) : |f(0)| \leq r\}, \end{gathered} \tag{1.5}$$

$$C^m(\mathbb{R},K;r,K_0,K_1,\ldots,K_j) = C^m(\mathbb{R},K;r) \cap C^m(\mathbb{R},K;K_0,K_1,\ldots,K_j), \tag{1.6}$$

$$LC^m(\mathbb{R},K) = \{f \in C^m(\mathbb{R},K) : \Lambda(f^{(i)}) \text{ is always bounded for each } i \in \mathbb{Z}_m\}. \tag{1.7}$$

Let $n \geq 1$. For any $G \in C^0(\mathbb{R}^{2n+3},\mathbb{R})$ and any $i \in \mathbb{Z}_{2n+2}$, put

$$\begin{gathered} \Lambda_{iG} = \Lambda_i(G) = \bigg\{ \frac{G(y_0,\ldots,y_i,\ldots,y_{2n+2}) - G(y_0,\ldots,y_{i-1},w_i,y_{i+1},\ldots,y_{2n+2})}{y_i - w_i}, \\ (y_0,\ldots,y_i,\ldots,y_{2n+2}) \in \mathbb{R}^{2n+3},\ w_i \in \mathbb{R} - \{y_i\} \bigg\}, \\ \lambda_{iG} = \sup\{|t| : t \in \Lambda_{iG}\}, \\ \lambda_G = \lambda_G^{(0)} = \max\{\lambda_{iG} : i = 0,1,\ldots,2n+2\}. \end{gathered} \tag{1.8}$$

If $\lambda_G < \infty$, that is, each Λ_{iG} is bounded, then G is said to be Lipschitz continuous. Let the 0th-order partial derivative $G^{(0)}$ of G be G itself. For $G \in C^m(\mathbb{R}^{2n+3},\mathbb{R})$, $k \in \mathbb{N}_m$ $(m \geq 1)$, and $(i_1,i_2,\ldots,i_k) \in \mathbb{Z}_{2n+2}^k$, denote by $G_{i_1 i_2 \cdots i_k}^{(k)}$

a kth-order partial derivative of G, the definition of which is

$$G^{(k)}_{i_1 i_2 \cdots i_k}(y_0, y_1, \dots, y_{2n+2}) = \frac{\partial^k G(y_0, y_1, \dots, y_{2n+2})}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_k}} \tag{1.9}$$

for any $(y_0, y_1, \dots, y_{2n+2}) \in \mathbb{R}^{2n+3}$. Obviously, $G^{(k)}_{i_1 i_2 \cdots i_k} \in C^{m-k}(\mathbb{R}^{2n+3}, \mathbb{R})$. In addition, we also write G'_{i_1} for $G^{(1)}_{i_1}$ and $G''_{i_1 i_2}$ for $G^{(2)}_{i_1 i_2}$. Let

$$\lambda^{(k)}_G = \max\{\lambda_H : H \text{ is a } k\text{th-order partial derivative of } G\}. \tag{1.10}$$

Let $K_0, K_1, \dots, K_{2n+2}$ be all connected closed subsets of $\mathbb{R}$ and $m \geq 0$. Write

$$C^m(\mathbb{R}^{2n+3}, \mathbb{R}; K_0, K_1, \dots, K_{2n+2}) = \{G \in C^m(\mathbb{R}^{2n+3}, \mathbb{R}) : \Lambda_{iG} \subset K_i,\ i \in \mathbb{Z}_{2n+2}\}, \tag{1.11}$$

$$LC^m(\mathbb{R}^{2n+3}, \mathbb{R}) = \{G \in C^m(\mathbb{R}^{2n+3}, \mathbb{R}) : \lambda^{(k)}_G < \infty \text{ for each } k \in \mathbb{Z}_m\}. \tag{1.12}$$

If $G \in C^1(\mathbb{R}^{2n+3}, \mathbb{R})$, then analogous to (1.3), we have

$$\Lambda_{iG} \subset G'_i(\mathbb{R}^{2n+3}) \subset \overline{\Lambda}_{iG}. \tag{1.13}$$

For convenience, we write

$$V_{fg}(x) = (x, f(x), f(x+1), \dots, f(x+n), g(x), g(x+1), \dots, g(x+n)) \tag{1.14}$$

for all $f, g \in C^0(\mathbb{R}, \mathbb{R})$ and all $x \in \mathbb{R}$.

Let $m \geq 0$ and $G, H \in C^m(\mathbb{R}^{2n+3}, \mathbb{R})$. For real number $\delta \neq 0$, define $\Psi_{\delta GH} : C^m(\mathbb{R}, \mathbb{R}) \times C^m(\mathbb{R}, \mathbb{R}) \to C^m(\mathbb{R}, \mathbb{R}) \times C^m(\mathbb{R}, \mathbb{R})$ by

$$\Psi_{\delta GH}(f, g) = (\Psi_{\delta G}(f, g), \Psi_{\delta H}(f, g)) \tag{1.15}$$

for all $(f, g) \in C^m(\mathbb{R}, \mathbb{R}) \times C^m(\mathbb{R}, \mathbb{R})$, where

$$\begin{aligned} \Psi_{\delta G}(f, g)(x) &= f(x) + \delta G(V_{fg}(x)), \\ \Psi_{\delta H}(f, g)(x) &= g(x) + \delta H(V_{gf}(x)), \end{aligned} \tag{1.16}$$

for all $x \in \mathbb{R}$. It is easy to see that (f, g) is a fixed point of the map $\Psi_{\delta GH}$ if and only if (f, g) is a C^m solution of (1.1). Thus, the problem of solutions of (1.1) can be translated into that of fixed points of $\Psi_{\delta GH}$. In order to decide the existence of the fixed points of $\Psi_{\delta GH}$, we need the following theorem which can be found in [1, page 74].

Theorem 1.1 (Schauder and Tychonoff). *Let X be a compact convex set in a locally convex linear topological space. Then each continuous map $\Psi : X \to X$ has a fixed point.*

Define a metric ρ_m on $C^m(\mathbb{R},\mathbb{R})$, for any $f,g \in C^m(\mathbb{R},\mathbb{R})$, by

$$\rho_m(f,g) = \sup\{|f^{(j)}(x)-g^{(j)}(x)|/(1+x^2) : j \in \mathbb{Z}_m,\ x \in \mathbb{R}\}. \tag{1.17}$$

Denote by 0^* the function on $\mathbb{R}$ which is identical to 0. For $f \in C^m(\mathbb{R},\mathbb{R})$, write $\|f\|_m = \rho_m(f,0^*)$. Then $0 \le \|f\|_m \le \infty$.

Now we define a metric $\rho_m \times \rho_m$ on $C^m(\mathbb{R},\mathbb{R}) \times C^m(\mathbb{R},\mathbb{R})$ by

$$\rho_m \times \rho_m((f_1,g_1),(f_2,g_2)) = \sqrt{[\rho_m(f_1,f_2)]^2 + [\rho_m(g_1,g_2)]^2} \tag{1.18}$$

for all $(f_1,g_1),(f_2,g_2) \in C^m(\mathbb{R},\mathbb{R}) \times C^m(\mathbb{R},\mathbb{R})$.

Analogous to the proof of [3, Proposition 3.2], we can obtain the following lemma.

Lemma 1.2. *Suppose that $m \ge 0$, $K_0,\dots,K_m$ are all compact intervals, and $r \ge 0$ is a real number. Let $X_r = C^m(\mathbb{R},\mathbb{R};r,K_0,\dots,K_m)$ as defined in (1.6). Then $\Psi_{\delta GH}|(X_r \times X_r, \rho_m \times \rho_m)$ is continuous.*

2. C^0 solutions of (1.1). For any $G \in C^0(\mathbb{R}^{2n+3},\mathbb{R})$, define a function $\varphi_G : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_G(x) = G(x,x,\dots,x) \quad \forall x \in \mathbb{R}. \tag{2.1}$$

Theorem 2.1. *Let $G,H \in C^0(\mathbb{R}^{2n+3},\mathbb{R})$. If the following two conditions hold:*

(i) *there exist nonnegative real numbers $\mu_0,\varepsilon_0,c_1,\dots,c_{2n+1}$ and $b \ge \mu_0+\varepsilon_0$, $c_0 \ge \mu > 0$ such that*

$$G,H \in C^0(\mathbb{R}^{2n+3},\mathbb{R};[\mu_0,b-\varepsilon_0],[-c_0,-\mu],[-c_1,c_1],\dots,[-c_{2n+1},c_{2n+1}]); \tag{2.2}$$

(ii) $\mu_0 \ge \sum_{i=1}^{2n+1} c_i(b/\mu)$, $\varepsilon_0 \ge \sum_{i=1}^{2n+1} c_i(b/\mu)$, *and* $\mu > 2\sum_{i=1}^{2n+1} c_i$,

then (1.1) has a solution $(f_0,g_0) \in C^0(\mathbb{R},\mathbb{R};r,[0,b/\mu]) \times C^0(\mathbb{R},\mathbb{R};r,[0,b/\mu])$, where

$$r = \frac{|\varphi_G(0)| + \sum_{i=1}^{n} i(c_i+c_{n+1+i})(b/\mu)}{\mu - \sum_{i=1}^{2n+1} c_i}. \tag{2.3}$$

Proof. We arbitrarily choose a constant $\delta \in (0,1/c_0]$. It follows from condition (i) that $\delta \le 1/c_0 \le 1/\mu$. Let $\Psi_{\delta GH}$ be defined as in (1.15). Let $X_r = C^0(\mathbb{R},\mathbb{R};r,[0,b/\mu])$. Consider any $(f,g) \in X_r \times X_r$. Write $\overline{f} = \Psi_{\delta G}(f,g)$. If

$G \in C^1(\mathbb{R}^{2n+3}, \mathbb{R})$ and $f, g \in C^1(\mathbb{R}, \mathbb{R})$, then for any $x \in \mathbb{R}$, we have $f'(x) \in \overline{\Lambda}_f \subset [0, b/\mu]$, $g'(x) \in \overline{\Lambda}_g \subset [0, b/\mu]$, and

$$\begin{aligned} \overline{f}'(x) = f'(x) + \delta G_0'(V_{fg}(x)) + \delta \sum_{i=1}^{n+1} G_i'(V_{fg}(x)) \cdot f'(x+i-1) \\ + \delta \sum_{i=2}^{n+2} G_{n+i}'(V_{fg}(x)) \cdot g'(x+i-2). \end{aligned} \tag{2.4}$$

Noting the upper and lower bounds of $G_i'(V_{fg}(x))$ given in condition (i), from (2.4) and condition (ii) we get

$$\begin{aligned} \overline{f}'(x) &\geq f'(x) + \delta\mu_0 - \delta c_0 f'(x) - \delta \sum_{i=1}^{2n+1} \frac{c_i b}{\mu} \\ &= (1-\delta c_0) f'(x) + \delta\left(\mu_0 - \sum_{i=1}^{2n+1} \frac{c_i b}{\mu}\right) \geq 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \overline{f}'(x) &\leq \delta(b-\varepsilon_0) + \left(1 - \delta\mu + \delta \sum_{i=1}^{2n+1} c_i\right) \frac{b}{\mu} \\ &\leq \delta(b-\varepsilon_0) + \frac{(1-\delta\mu)b}{\mu} + \delta\varepsilon_0 = \frac{b}{\mu}. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain $\Lambda(\overline{f}) \subset [0, b/\mu]$, that is,

$$\Lambda(\Psi_{\delta G}(f,g)) \subset \left[0, \frac{b}{\mu}\right]. \tag{2.7}$$

If $G \notin C^1(\mathbb{R}^{2n+3}, \mathbb{R})$, $f \notin C^1(\mathbb{R}, \mathbb{R})$, or $g \notin C^1(\mathbb{R}, \mathbb{R})$, then for any two given points $u > v$ in $\mathbb{R}$, we can take $G_1 \in C^1(\mathbb{R}^{2n+3}, \mathbb{R}; [\mu_0, b-\varepsilon_0], [-c_0, -\mu], [-c_1, c_1], \ldots, [-c_{2n+1}, c_{2n+1}])$ and $f_1, g_1 \in C^1(\mathbb{R}, \mathbb{R}; r, [0, b/\mu])$ such that for all $j \in \{0, 1, \ldots, n\}$ and $w \in \{u, v\}$,

$$\begin{gathered} f_1(w+j) = f(w+j), \qquad g_1(w+j) = g(w+j), \\ G_1(V_{fg}(w)) = G(V_{fg}(w)). \end{gathered} \tag{2.8}$$

Write $\overline{f_1}' = \Psi_{\delta G}(f_1, g_1)$. Then by (2.7), we have $\Lambda(\overline{f_1}) \subset [0, b/\mu]$; hence $(\overline{f}(u) - \overline{f}(v))/(u-v) = (\overline{f_1}(u) - \overline{f_1}(v))/(u-v) \in [0, b/\mu]$. Thus, (2.7) is still valid when $G \notin C^1(\mathbb{R}^{2n+3}, \mathbb{R})$, $f \notin C^1(\mathbb{R}, \mathbb{R})$, and $g \notin C^1(\mathbb{R}, \mathbb{R})$. Therefore, we have

$$\Psi_{\delta G}\left(C^0\left(\mathbb{R}, \mathbb{R}; \left[0, \frac{b}{\mu}\right]\right) \times C^0\left(\mathbb{R}, \mathbb{R}; \left[0, \frac{b}{\mu}\right]\right)\right) \subset C^0\left(\mathbb{R}, \mathbb{R}; \left[0, \frac{b}{\mu}\right]\right). \tag{2.9}$$

Since $f,g \in C^0(\mathbb{R},\mathbb{R};[0,b/\mu])$ are increasing, we have $f(k) \le f(0)+k(b/\mu)$ and $g(k) \le g(0)+k(b/\mu)$, for $k = 1,2,\ldots,n$. By condition (ii), we get

$$\begin{aligned}
|\overline{f}(0)| &= |f(0)+\delta[G(V_{fg}(0))-\varphi_G(0)]+\delta\varphi_G(0)| \\
&\le \left(1-\delta\mu+\delta\sum_{i=1}^{n} c_i\right)|f(0)|+\delta\sum_{i=1}^{n+1} c_{n+i}|g(0)| \\
&\quad +\delta\left[|\varphi_G(0)|+\sum_{i=1}^{n} i(c_i+c_{n+1+i})\left(\frac{b}{\mu}\right)\right] \\
&\le r+\delta\left[|\varphi_G(0)|+\sum_{i=1}^{n} i(c_i+c_{n+1+i})\left(\frac{b}{\mu}\right)-\left(\mu-\sum_{i=1}^{2n+1} c_i\right)r\right] \\
&= r.
\end{aligned} \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$\Psi_{\delta G}(X_r \times X_r) \subset X_r. \tag{2.11}$$

Similarly, we can obtain that

$$\Psi_{\delta H}(X_r \times X_r) \subset X_r. \tag{2.12}$$

Therefore, it follows from (2.11) and (2.12) that

$$\Psi_{\delta GH}(X_r \times X_r) \subset X_r \times X_r. \tag{2.13}$$

By [3, Proposition 3.1], under the metric $\rho_0 \times \rho_0$, $X_r \times X_r$ is compact. By Lemma 1.2, $\Psi_{\delta GH}|(X_r \times X_r, \rho_0 \times \rho_0)$ is continuous. Since $X_r \times X_r$ is a convex subspace of $C^0(\mathbb{R},\mathbb{R}) \times C^0(\mathbb{R},\mathbb{R})$, by Theorem 1.1, $\Psi_{\delta GH}|(X_r \times X_r)$ has a fixed point. This implies that (1.1) has a solution $(f_0,g_0) \in X_r \times X_r$. Theorem 2.1 is proven. □

For any $h \in C^0(\mathbb{R},\mathbb{R})$, write $BC^0(\mathbb{R},\mathbb{R};h) = \{f \in C^0(\mathbb{R},\mathbb{R}) : \|f-h\|_0 < \infty\}$.

Theorem 2.2. *Suppose that $G,H \in C^0(\mathbb{R}^{2n+3},\mathbb{R})$ satisfy Theorem 2.1(i) and (ii), and $(f_0,g_0) \in C^0(\mathbb{R},\mathbb{R};[0,b/\mu]) \times C^0(\mathbb{R},\mathbb{R};[0,b/\mu])$ is a solution of (1.1). Then (1.1) has only a solution (f_0,g_0) in $BC^0(\mathbb{R},\mathbb{R};f_0) \times BC^0(\mathbb{R},\mathbb{R};g_0)$.*

Proof. Suppose that $(f_1,g_1) \in BC^0(\mathbb{R},\mathbb{R};f_0) \times BC^0(\mathbb{R},\mathbb{R};g_0)$ is also a solution of (1.1). Consider the following two cases.

Case 1 ($\|g_0 - g_1\|_0 \le \|f_0 - f_1\|_0$). For any $x \in \mathbb{R}$, there exists $w_i = w_i(x) \in \Lambda_{iG}$ $(i = 1,\dots,2n+2)$ such that

$$\begin{aligned}
0 &= G(V_{f_0 g_0}(x)) - G(V_{f_1 g_1}(x)) \\
&= w_1(f_0(x) - f_1(x)) + \sum_{i=2}^{n+1} w_i(f_0(x+i-1) - f_1(x+i-1)) \\
&\quad + \sum_{i=2}^{n+2} w_{n+i}(g_0(x+i-2) - g_1(x+i-2)) \\
&\ge |w_1(f_0(x) - f_1(x))| - \sum_{i=2}^{n+1} |w_i(f_0(x+i-1) - f_1(x+i-1))| \\
&\quad - \sum_{i=2}^{n+2} |w_{n+i}(g_0(x+i-2) - g_1(x+i-2))| \\
&\ge \mu |f_0(x) - f_1(x)| - \sum_{i=1}^{n} c_i \|f_0 - f_1\|_0 - \sum_{i=1}^{n+1} c_{n+i} \|g_0 - g_1\|_0 \\
&\ge \mu |f_0(x) - f_1(x)| - \sum_{i=1}^{2n+1} c_i \|f_0 - f_1\|_0.
\end{aligned} \tag{2.14}$$

By (2.14), we have $(\mu - \sum_{i=1}^{2n+1} c_i)\|f_0 - f_1\|_0 \le 0$. Since $\mu > 2\sum_{i=1}^{2n+1} c_i$, $\|f_0 - f_1\|_0 = 0$ which implies $f_1 = f_0$. It follows from $\|g_0 - g_1\|_0 \le \|f_0 - f_1\|_0 = 0$ that $g_1 = g_0$. Hence, $(f_1, g_1) = (f_0, g_0)$.

Case 2 ($\|g_0 - g_1\|_0 > \|f_0 - f_1\|_0$). Analogous to Case 1, we can also show that $(f_1, g_1) = (f_0, g_0)$. Thus, (1.1) has only a solution (f_0, g_0) in $BC^0(\mathbb{R},\mathbb{R}; f_0) \times BC^0(\mathbb{R},\mathbb{R}; g_0)$. Theorem 2.2 is proven. □

3. C^m solutions $(m \ge 1)$ of (1.1)

Theorem 3.1. *Suppose that $G, H \in LC^m(\mathbb{R}^{2n+3}, \mathbb{R})$ satisfy Theorem 2.1(i) and (ii). Then there exist positive numbers $a_1, \dots, a_m$ such that (1.1) has a solution*

$$\begin{aligned}
(f_m, g_m) \in C^m\Big(\mathbb{R},\mathbb{R}; r, \Big[0, \frac{b}{\mu}\Big], [-a_1, a_1], \dots, [-a_m, a_m]\Big) \\
\times C^m\Big(\mathbb{R},\mathbb{R}; r, \Big[0, \frac{b}{\mu}\Big], [-a_1, a_1], \dots, [-a_m, a_m]\Big).
\end{aligned} \tag{3.1}$$

Proof. For any $\delta \in (0, 1/c_0]$, let the map $\Psi_{\delta GH}$ be defined as in (1.15). Write $X_{rm} = C^m(\mathbb{R},\mathbb{R}; r, [0, b/\mu])$. By (2.13), we have

$$\Psi_{\delta GH}(X_{rm} \times X_{rm}) \subset X_{rm} \times X_{rm} \tag{3.2}$$

since $G,H \in C^m(\mathbb{R}^{2n+3},\mathbb{R})$. Consider any $f,g \in C^m(\mathbb{R},\mathbb{R};r,[0,b/\mu]) \cap LC^m(\mathbb{R},\mathbb{R})$. Let $h(x) = G(V_{fg}(x))$. Then we can calculate the derivatives of h of order $1,2,\dots,m$ as follows:

$$\begin{aligned}
h'(x) &= \sum_{i=0}^{n+1} G_i'(V_{fg}(x)) \cdot f'(x+i-1) + \sum_{i=2}^{n+2} G_{n+i}'(V_{fg}(x)) \cdot g'(x+i-2),\\
h''(x) &= \sum_{i=1}^{n+1} G_i'(V_{fg}(x)) f''(x+i-1) + \sum_{i=2}^{n+2} G_{n+i}'(V_{fg}(x)) g''(x+i-2)\\
&\quad + \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} G_{ij}''(V_{fg}(x)) f'(x+i-1) f'(x+j-1)\\
&\quad + \sum_{i=0}^{n+1}\sum_{j=2}^{n+2} G_{i,j+n}''(V_{fg}(x)) f'(x+i-1) g'(x+j-2)\\
&\quad + \sum_{i=2}^{n+2}\sum_{j=0}^{n+1} G_{n+i,j}''(V_{fg}(x)) f'(x+j-1) g'(x+i-2)\\
&\quad + \sum_{i=2}^{n+2}\sum_{j=2}^{n+2} G_{n+i,n+j}''(V_{fg}(x)) g'(x+i-2) g'(x+j-2),
\end{aligned} \tag{3.3}$$

where $dx/dx(=1)$ is written as $f'(x-1)$ for convenience.

In general, for $k = 2,\dots,m$, it is easy to see that

$$\begin{aligned}
h^{(k)}(x) = \sum_{i=1}^{n+1} & G_i'(V_{fg}(x)) f^{(k)}(x+i-1)\\
&+ \sum_{i=2}^{n+2} G_{n+i}'(V_{fg}(x)) g^{(k)}(x+i-2) + \xi_k(\{\cdot\},\{\cdot\},\{\cdot\}),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\xi_k(\{\cdot\},\{\cdot\},\{\cdot\}) = \xi_k\Big(&\Big\{G_{i_1 i_2\cdots i_p}^{(p)}(V_{fg}(x)) : p \in \mathbb{N}_k, (i_1,i_2,\dots,i_p) \in \mathbb{Z}_{2n+2}^p\Big\},\\
&\{f^{(p)}(x+q-1) : p \in \mathbb{N}_{k-1}, q \in \mathbb{Z}_{n+1}\},\\
&\{g^{(p)}(x+q-1) : p \in \mathbb{N}_{k-1}, q \in \mathbb{Z}_{n+1}\}\Big)
\end{aligned} \tag{3.5}$$

($d^p x/dx^p$ $(=1)$ is written as $f^{(p)}(x-1)$ for convenience) is a polynomial of $G_{i_1 i_2\cdots i_p}^{(p)}(V_{fg}(x))$ (where $p \in \mathbb{N}_k$; $i_1,\dots,i_p \in \mathbb{Z}_{2n+2}$), $f^{(p)}(x+q-1)$, and $g^{(p)}(x+q-1)$ (where $p \in \mathbb{N}_{k-1}$, $q \in \mathbb{Z}_{n+1}$) whose coefficients are all positive integers. The functional relation ξ_k itself is related only to the rules of partial derivatives of general functions of several variables and the rules of derivatives of compositions and products of functions, but not related to specific G, f, or g. Therefore, ξ_k is still well defined for $k > m$. If $G \in C^{m+1}(\mathbb{R}^{2n+3},\mathbb{R})$ and $f,g \in C^{m+1}(\mathbb{R},\mathbb{R})$, then (3.4) also holds for $k = m+1$.

For $k = 0, 1, \dots, m$, let

$$b_k = \max\{\lambda(f^{(k)}), \lambda(g^{(k)})\}, \qquad B_k = \max\left\{\lambda_G^{(j)}, \lambda_H^{(j)} : j = 0, 1, \dots, k\right\}. \tag{3.6}$$

It follows from (1.4) and (1.3) that $b_k = \max\{\|f^{(k+1)}\|_0, \|g^{(k+1)}\|_0\}$, $k = 0, \dots, m-1$. Since $G, H \in LC^m(\mathbb{R}^{2n+3}, \mathbb{R})$, by (1.12), we have $B_0 \le B_1 \le \cdots \le B_m < \infty$. For $1 \le p \le k \le m$, by (1.8), (1.9), (1.10), and (1.13), we have $|G^{(p)}_{i_1 i_2 \cdots i_p}(V_{fg}(x))| \le B_{k-1}$. Now, we choose a $\delta \in (0, 1/c_0]$ such that

$$\delta < \frac{1}{2(\mu - \sum_{i=1}^{2n+1} c_i)}. \tag{3.7}$$

Write $\overline{\mu} = \mu - \sum_{i=1}^{2n+1} c_i$ and $\overline{f} = \Psi_{\delta G}(f, g)$. Replacing all $G^{(p)}_{i_1 i_2 \cdots i_p}(V_{fg}(x))$, $f^{(p)}(f(x+q-1))$, and $g^{(p)}(f(x+q-1))$ in the polynomial $\xi_k(\{\cdot\}, \{\cdot\}, \{\cdot\})$ by the upper bounds B_{k-1} and b_{p-1} of their absolute values, from (3.4) we get

$$\begin{aligned} \left|\overline{f}^{(k)}(x)\right| &\le \left(1 - \delta\mu + \delta \sum_{i=1}^{2n+1} c_i\right) b_{k-1} + \delta\eta_k(B_{k-1}, b_0, b_1, \dots, b_{k-2}) \\ &= (1 - \delta\overline{\mu}) b_{k-1} + \delta\eta_k(B_{k-1}, b_0, \dots, b_{k-2}), \quad k = 2, \dots, m, \end{aligned} \tag{3.8}$$

where $\eta_k(B_{k-1}, b_0, \dots, b_{k-2})$ is a polynomial of $B_{k-1}, b_0, \dots, b_{k-2}$, whose coefficients are all positive integers. The functional relation η_k itself is determined by ξ_k and is independent of specific G, H, f, and g. Therefore, η_k is still well defined for $k > m$.

For $k = 2, \dots, m$, noting that $\lambda(\overline{f}^{(k-1)}) = \|\overline{f}^{(k)}\|_0$, by (3.8) we obtain

$$\lambda\left(\overline{f}^{(k-1)}\right) \le (1 - \delta\overline{\mu}) b_{k-1} + \delta\eta_k(B_{k-1}, b_0, \dots, b_{k-2}). \tag{3.9}$$

If $G, H \in C^{m+1}(\mathbb{R}^{2n+3}, \mathbb{R})$ and $f, g \in C^{m+1}(\mathbb{R}, \mathbb{R})$, then $\overline{f} \in C^{m+1}(\mathbb{R}, \mathbb{R})$, (3.8) and (3.9) are also true for $k = m+1$. Adopting the method that is used in the proof of Theorem 2.1 to show that (2.7) still holds when $G \notin C^1(\mathbb{R}^{2n+3}, \mathbb{R})$, $f \notin C^1(\mathbb{R}, \mathbb{R})$, or $g \notin C^1(\mathbb{R}, \mathbb{R})$, we can verify that (3.9) still holds for $k = m+1$ even if $G, H \notin C^{m+1}(\mathbb{R}^{2n+3}, \mathbb{R})$ or $f, g \notin C^{m+1}(\mathbb{R}, \mathbb{R})$.

Let $a_0 = b/\mu$ and

$$a_{k-1} = \frac{\eta_k(B_{k-1}, a_0, a_1, \dots, a_{k-2})}{\overline{\mu}}, \quad k = 2, \dots, m+1. \tag{3.10}$$

Then a_{k-1} only depends on G and H. Since $\eta_k(B_{k-1},b_0,b_1,\dots,b_{k-2})$ is a monotone increasing function of $b_0,b_1,\dots,b_{k-2}$, and $1-\delta\overline{\mu}>\delta\overline{\mu}>0$, from (3.9) and (3.10) it follows that if $b_i\le a_i$ for all $i\in\mathbb{Z}_{k-1}$, then

$$\lambda\left(\overline{f}^{(k-1)}\right)\le(1-\delta\overline{\mu})a_{k-1}+\delta\eta_k(B_{k-1},a_0,\dots,a_{k-2})=a_{k-1},\quad k=2,\dots,m+1. \tag{3.11}$$

Write $X_m=C^m(\mathbb{R},\mathbb{R};r,[0,b/\mu],[-a_1,a_1],\dots,[-a_m,a_m])$. Then X_m is a convex subset of $C^m(\mathbb{R},\mathbb{R})$. By (3.2) and (3.11), we get

$$\Psi_{\delta G}(X_m\times X_m)\subset X_m. \tag{3.12}$$

Similarly, we can obtain that

$$\Psi_{\delta H}(X_m\times X_m)\subset X_m. \tag{3.13}$$

Therefore, it follows from (3.12) and (3.13) that

$$\Psi_{\delta GH}(X_m\times X_m)\subset X_m\times X_m. \tag{3.14}$$

By [3, Proposition 3.1], under the metric $\rho_m\times\rho_m$, $X_m\times X_m$ is compact. By Lemma 1.2, $\Psi_{\delta GH}|(X_m\times X_m,\rho_m\times\rho_m)$ is continuous. Thus, by Theorem 1.1, $\Psi_{\delta GH}|(X_m\times X_m)$ has a fixed point. This implies that (1.1) has a solution $(f_m, g_m)\in X_m\times X_m$. Theorem 3.1 is proven. □

Noting that $C^m(\mathbb{R},\mathbb{R})\subset C^0(\mathbb{R},\mathbb{R})$, we have the following proposition.

Proposition 3.2. *Let $G,H\in C^0(\mathbb{R}^{2n+3},\mathbb{R})$, $\mathbf{F}\subset C^0(\mathbb{R},\mathbb{R})\times C^0(\mathbb{R},\mathbb{R})$, and $m\ge 1$. If (1.1) has at most a solution in $\mathbf{F}$, then (1.1) has at most a solution in $\mathbf{F}\cap(C^m(\mathbb{R},\mathbb{R})\times C^m(\mathbb{R},\mathbb{R}))$.*

By Proposition 3.2, after obtaining Theorem 2.2, we need not discuss the uniqueness of C^m solutions of (1.1), for $m\ge 1$, in detail unless we can give some conditions weaker than those in Theorem 2.2 (or, at least, they do not imply each other) or we can discuss the uniqueness of solutions in a subspace of $C^0(\mathbb{R},\mathbb{R})\times C^0(\mathbb{R},\mathbb{R})$ larger than the subspace of $BC^0(\mathbb{R},\mathbb{R};f_m)\times BC^0(\mathbb{R},\mathbb{R};g_m)$.

4. Example. Let $a\ge 10$ be a real number. Suppose that the system of equations is

$$\begin{gathered}\frac{3}{2}x-af(x)+\sin f(x)\cdot\cos f(x+1)+\frac{1}{2}\sin(2g(x)-1)+\arctan g(x+1)=0,\\ 2x+\sin(x-g(x+1)+f(x)+f(x+1))-\overline{a}g(x)+\arctan g(x)=0,\end{gathered} \tag{4.1}$$

for all $x \in \mathbb{R}$. Then the representatives of the corresponding $G, H : \mathbb{R}^5 \to \mathbb{R}$ are

$$G(x_0,x_1,x_2,x_3,x_4) = \frac{3}{2}x_0 - ax_1 + \sin x_1 \cdot \cos x_2 + \frac{1}{2}\sin(2x_3 - 1) + \arctan x_4,$$
$$H(x_0,x_1,x_2,x_3,x_4) = 2x_0 + \sin(x_0 - x_2 + x_3 + x_4) - \overline{a}x_1 + \arctan x_1, \tag{4.2}$$

for all $(x_0,x_1,x_2,x_3,x_4) \in \mathbb{R}^5$. We can easily calculate the derivatives of G and H, and then obtain the supremums and infimums of Λ_{iG} and Λ_{iH} $(i = 0,1,2,3,4)$; hence we can take $\mu_0 = 1$, $b = 3 + \varepsilon_0$ ($\varepsilon_0 \geq 0$ is to be selected), $c_0 = a + 1$, $\mu = a - 1$, and $c_1 = c_2 = c_3 = 1$.

When $a + 1 > \overline{a} \geq a \geq 13$, we choose $\varepsilon_0 = (a - 10)/3$. Then $b/\mu = 1/3$ and Theorem 2.1(i) and (ii) hold. Thus, by Theorems 2.1, 2.2, 3.1, and Proposition 3.2, we have the following proposition.

Proposition 4.1. *Let $m \geq 0$ be given. If $a + 1 > \overline{a} \geq a \geq 13$, then (4.1) has a solution $(f_m, g_m) \in C^m(\mathbb{R},\mathbb{R};[0,1/3]) \times C^m(\mathbb{R},\mathbb{R};[0,1/3])$, and (f_m, g_m) is the unique solution of (4.1) in $BC^0(\mathbb{R},\mathbb{R};f_m) \times BC^0(\mathbb{R},\mathbb{R};g_m)$.*

Consider the following system of equations:

$$x + \arctan x - 19f(x) + e^{-[(f(x+2))^2 + (f(x+5))^2 + (g(x))^2 + (g(x+8))^2]} = 0,$$
$$\frac{3}{2}x + 1 - 20g(x) + \ln\left[(f(x+3))^2 + (g(x+4))^2 + 1\right] \tag{4.3}$$
$$+ \sin(g(x+5)) \cdot \cos(f(x) + g(x+2)) = 0.$$

Analogous to the argument of Proposition 4.1, we can obtain the following proposition.

Proposition 4.2. *Let $m \geq 0$ be given. Equation (4.3) has a solution $(f_m, g_m) \in C^m(\mathbb{R},\mathbb{R};[0,1/6]) \times C^m(\mathbb{R},\mathbb{R};[0,1/6])$, and it is the unique solution of (4.3) in $BC^0(\mathbb{R},\mathbb{R};f_m) \times BC^0(\mathbb{R},\mathbb{R};g_m)$.*

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