TOTAL CHARACTERS AND CHEBYSHEV POLYNOMIALS

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The total character τ of a finite group G is defined as the sum of all the irreducible characters of G. K. W. Johnson asks when it is possible to express τ as a polynomial with integer coefficients in a single irreducible character. In this paper, we give a complete answer to Johnson's question for all finite dihedral groups. In particular, we show that, when such a polynomial exists, it is unique and it is the sum of certain Chebyshev polynomials of the first kind in any faithful irreducible character of the dihedral group G.

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1. Introduction. A *Gel'fand model M* for a group G was defined in [8] as any complex representation of G which is isomorphic to the direct sum of all irreducible representations of G. We refer to the character of such a representation as the *total character τ* of G. A Gel'fand model afforded by a (generalized) permutation representation is referred to as a (weakly) geometric Gel'fand model. When a Gel'fand model M is the representation afforded by a nonnegative integer linear combination of powers of a genuine G-set X, then the total character τ of G can be expressed as a polynomial in the character χ afforded by X. A question related to this idea was posed by Johnson [5].

QUESTION 1.1. For a finite group G, do there necessarily exist an irreducible character χ and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$, where τ is the total character of G?

Johnson's question arose in the context of character sharpness which we will briefly explain. Let G be a finite group, χ a generalized character of G of degree n, $L = \{\chi(g) \mid g \neq 1\}$, and $f_L(x) = \prod_{l \in L}(x - l)$. It was discovered by Blichfeldt [2] and rediscovered in a modern context by Kiyota [6] that $f_L(x) \in \mathbb{Z}[x]$ and $|G|$ divides $f_L(n)$. In the special case that $f_L(n) = |G|$, the character χ is said to be *sharp*. Another way to characterize a sharp character is to notice that the class function $f_L(\chi) = \prod_{l \in L}(\chi - l1_G) = \rho$, where ρ is the regular character and 1_G is the trivial character of G. In other words, every irreducible character of G appears as a constituent of $f_L(\chi)$.

A partial answer to Johnson's question was given in [7] for certain dihedral groups. In this paper, we give a complete treatment for all dihedral groups and we show that the right polynomials, when they exist, are integer sums of Chebyshev polynomials of the first kind.

We prove the following main theorem.

THEOREM 1.2. *Let $G \cong D_{2n}$ and let τ be the total character of G.*

(1) *If n is odd or if $n = 2m$, where $m \equiv 2, 3 \pmod 4$, then there exists a unique monic polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\chi) = \tau$ for any faithful $\chi \in \mathrm{Irr}(G)$. Moreover,*

$$P(x) = \begin{cases} 2\sum_{k=0}^{m} T_k\left(\dfrac{x}{2}\right) - 2T_{m-1}\left(\dfrac{x}{2}\right), & \text{if } n = 2m - 1,\ n \equiv 1, 3 \pmod 8, \\[4mm] 2\sum_{k=0}^{m-1} T_k\left(\dfrac{x}{2}\right), & \text{if } n = 2m - 1,\ n \equiv 5, 7 \pmod 8, \\[4mm] 2\sum_{k=0}^{m} T_k\left(\dfrac{x}{2}\right), & \text{if } n = 2m,\ m \equiv 2, 3 \pmod 4, \end{cases} \tag{1.1}$$

where $T_k(x)$ is the kth Chebyshev polynomial of the first kind.

(2) *If $n = 2m$ and $m \equiv 1 \pmod 4$, then there exists a unique monic polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\chi) = 2\tau$ for any faithful $\chi \in \mathrm{Irr}(G)$, where*

$$P(x) = 2\left[T_{m+1}\left(\dfrac{x}{2}\right) + 2\sum_{k=0}^{m} T_k\left(\dfrac{x}{2}\right) - T_{m-1}\left(\dfrac{x}{2}\right) \right]. \tag{1.2}$$

Moreover, there does not exist $P(x) \in \mathbb{Z}[x]$ such that $P(\chi) = \tau$ for any $\chi \in \mathrm{Irr}(G)$.

(3) *If $G \cong D_{2n}$, where $n \equiv 0 \pmod 8$, then there does not exist a polynomial $P(x) \in \mathbb{C}[x]$ such that $P(\chi) = \tau$ for any $\chi \in \mathrm{Irr}(G)$.*

2. Preliminaries. The total character for all dihedral groups was computed in [7, Proposition 2.1]. A dihedral group of order $2n$ with $n \geq 3$ will be presented as usual as

$$D_{2n} = G = \langle a, b : a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle. \tag{2.1}$$

Using the notation in [4], we use g_i and h_i to denote a representative and the size of the ith conjugacy class, respectively. The character table and total character τ of D_{2n} are given below.

CASE 1 (n odd). The conjugacy classes of D_{2n} (n odd) are

$$\{1\}, \quad \{a^r, a^{-r}\} \quad \left(1 \leq r \leq \frac{n-1}{2}\right), \quad \{a^s b \mid 0 \leq s \leq n-1\}. \tag{2.2}$$

The character table of D_{2n} (n odd) and the total character τ, where $\epsilon = e^{2\pi i/n}$, is presented in Table 2.1.

TABLE 2.1

g_i	1	a^r $(1 \le r \le (n-1)/2)$	b
h_i	1	2	n
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \le j \le (n-1)/2)$	2	$\epsilon^{jr} + \epsilon^{-jr}$	0
τ	$n+1$	1	0

CASE 2 (*n* even). If n is even, say $n = 2m$, then the conjugacy classes of D_{2n} are

$$\{1\}, \quad \{a^m\}, \quad \{a^r, a^{-r}\} \quad (1 \le r \le m-1),$$
$$\{a^s b \mid s \text{ even}\}, \quad \{a^s b \mid s \text{ odd}\}. \tag{2.3}$$

The character table of D_{2n} (*n* even, $n = 2m$, and the total character τ, where $\epsilon = e^{2\pi i/n}$) is given in Table 2.2.

TABLE 2.2

g_i	1	a^m	a^r $(1 \le r \le m-1)$	b	ab
h_i	1	1	2	$n/2$	$n/2$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j $(1 \le j \le m-1)$	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0
τ	$2(m+1)$	0, m odd, 2, m even,	0, r odd, 2, r even,	0	0

The nth Chebyshev polynomial of the first kind is defined as $T_n(x) = \cos(n\cos^{-1}(x))$ for $|x| \le 1$. Chebychev polynomials can be expressed recursively as $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $T_0(x) = 1$, and $T_1(x) = x$. Before we proceed with the proof of Theorem 1.2 we need the following lemmas.

LEMMA 2.1. *If $P(\psi_1) = \tau$ for some $P(x) \in \mathbb{Z}[x]$, then $P(\psi_j) = \tau$ for any faithful character ψ_j of D_{2n}.*

PROOF. We first observe that ψ_j is faithful if and only if $(j, n) = 1$, for $\epsilon^{jr} + \epsilon^{-jr} = 2\cos(2\pi r j/n) = 2$ if and only if $n | rj$ if and only if $(n, j) \ne 1$.

Hence for $(j,n) = 1$, the character ψ_j is faithful and a Galois conjugate of ψ_1 with the same set of character values. Since for n odd $P(\psi_1[a^r]) = 1$ for all r, it follows that $P(\psi_j[a^r]) = P(\psi_1[a^r]) = 1$. When n is even, we have that $P(\psi_1[a^r]) = 2$ when r is even and $P(\psi_1[a^r]) = 0$ when r is odd. For a faithful ψ_j, the set of character values is the same as those of ψ_1; and since j is necessarily odd, it follows that $P(\psi_j[a^r]) = 2$ when r is even, and $P(\psi_j[a^r]) = 0$ when r is odd as required.   □

LEMMA 2.2. *Let the nth Chebyshev polynomial of the first kind be expressed as $T_n(x) = \sum_{k=1}^{n} c_{n,k} x^k + c_{n,0}$. Then $c_{n,n} = 2^{(n-1)}$ and $2^{(k-1)} | c_{n,k}$ for $1 \le k \le n$.*

PROOF. The first half follows easily by an inductive argument on n and by using the recursive relation mentioned above. For divisibility of the coefficients, observe that the result is true for $n = 1$ and $n = 2$. Now assume that the result is true for all Chebyshev polynomials of degree less that n and consider the nth degree Chebyshev polynomial $T_n(x)$. We have by the recursive relation that $c_{n,k} = 2c_{n-1,k-1} - c_{n-2,k}$. By inductive hypothesis, $2^{(k-1)} | 2c_{n-1,k-1} - c_{n-2,k}$ for any $1 \le k \le n$. Hence the result is true for all the coefficients of $T_n(x)$ and hence for all n.   □

LEMMA 2.3 [3, 134.2]. *For $m \in \mathbb{Z}^+$,*

$$\sum_{k=0}^{m} \cos kx = \frac{1}{2}\left[1 + \frac{\sin\left((m+1/2)x\right)}{\sin\left((1/2)x\right)} \right]. \tag{2.4}$$

LEMMA 2.4. *Let $p_m(x) = 2\sum_{k=0}^{m} T_k(x/2)$. The following equations hold:*

$$p_m(2) = 2(m+1); \tag{2.5}$$

$$p_m\left(2\cos\frac{2\pi r}{2m} \right) = \begin{cases} 0, & \text{if } r \text{ odd,} \\ 2, & \text{if } r \text{ even;} \end{cases} \tag{2.6}$$

$$p_m\left(2\cos\left(\frac{2\pi r}{(2m-1)}\right) \right) = 1 + (-1)^r 2\cos\left(\frac{\pi r}{(2m-1)}\right); \tag{2.7}$$

$$p_m(0) = 1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right). \tag{2.8}$$

PROOF. Equation (2.5) follows immediately since $T_k(1) = 1$, for all $k \ge 0$. For the other equations, we use Lemma 2.3,

$$p_m\left(2\cos\left(\frac{2\pi r}{2m}\right) \right) = 2\sum_{k=0}^{m} \cos\left(\frac{\pi k r}{m}\right)$$

$$= 1 + \frac{\sin\left((m+1/2)(\pi r/m)\right)}{\sin(\pi r/2m)}$$

$$= 1 + \cos(\pi r) = \begin{cases} 0, & \text{if } r \text{ odd,} \\ 2, & \text{if } r \text{ even,} \end{cases}$$

$$p_m\left(2\cos\left(\frac{2\pi r}{2m-1}\right)\right) = 1 + \frac{\sin\left(((2m-1)+2)/2\right)(2\pi r/(2m-1))}{\sin\left(\pi r/(2m-1)\right)}$$

$$= 1 + \cos(\pi r)\frac{\sin 2\left(\pi r/(2m-1)\right)}{\sin\left(\pi r/(2m-1)\right)}$$

$$= 1 + (-1)^r 2\cos\left(\frac{\pi r}{2m-1}\right),$$

$$p_m(0) = 2\sum_{k=0}^{m} T_k(0) = 2\sum_{k=0}^{m}\cos\left(\frac{k\pi}{2}\right)$$

$$= 1 + \frac{\sin\left((m+1/2)(\pi/2)\right)}{\sin\left(\pi/4\right)}$$

$$= 1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right).$$

$$(2.9)$$

$\square$

3. Proof of main theorem. It is a well-known theorem of polynomial interpolation that there exists a unique polynomial of degree k or less that maps $k+1$ distinct points in the domain to predefined points in the range. We refer the reader to [1] for a natural proof. Thus, by presenting a polynomial of degree less than the number of distinct values of some $\chi \in \mathrm{Irr}(G)$, which matches the values of the total character when considered as a class function on χ, we immediately get that the polynomial is both minimal and unique.

PROOF OF THEOREM 1.2(1). Let $n = 2m-1$, $n \equiv 1,3 \pmod 8$. Using Lemma 2.4, we have that $P(x) = p_m(x) - 2T_{m-1}(x/2)$,

$$P(\psi_1[1]) = 2(m+1) - 2 = 2m = n+1 = \tau[1],$$

$$P(\psi_1[a^r]) = 1 + (-1)^r 2\cos\left(\frac{\pi r}{2m-1}\right) - 2\cos\left(\frac{\pi r(m-1)}{2m-1}\right)$$

$$= 1 = \tau[a^r],$$

$$P(\psi_1[b]) = 1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right) - 2 = 0 = \tau[b].$$

$$(3.1)$$

Let $n = 2m-1$, $n \equiv 5,7 \pmod 8$. Then $P(x) = p_{m-1}(x)$,

$$P(\psi_1[1]) = (2(m-1)+1) = 2m = n+1 = \tau[1],$$

$$P(\psi_1[a^r]) = 2\sum_{k=0}^{m-1}\cos\left(\frac{2\pi k r}{2m-1}\right)$$

$$= 1 + \frac{\sin(\pi r)}{\sin\left(\pi r/(2m-1)\right)} = 1 = \tau[a^r],$$

$$P(\psi_1[b]) = 1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right) = 0 = \tau[b].$$

$$(3.2)$$

Let $n = 2m$, $m \equiv 2,3 \,(\mathrm{mod}\,4)$. Then $P(x) = p_m(x)$,

$$P(\psi_1[1]) = 2(m+1) = \tau[1],$$

$$P(\psi_1[a^r]) = \begin{cases} 0, & \text{if } r \text{ odd}, \\ 2, & \text{if } r \text{ even}, \end{cases}$$

$$= \tau[a^r],$$

$$P(\psi_1[b]) = P(\psi_1[ab]) = 1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right) = 0$$

$$= \tau[b] = \tau[ab].$$

(3.3)

Thus, in each of the above three cases, $P(x)$ is minimal, and by Lemma 2.2 it is monic with integer entries. By Lemma 2.1, we have that $P(\psi_j) = \tau$ for any faithful $\psi_j \in \mathrm{Irr}(G)$. □

PROOF OF THEOREM 1.2(2). Let $n = 2m$ and $m \equiv 1 \,(\mathrm{mod}\,4)$. Then $P(x) = 2T_{m+1}(x/2) + 2p_m(x) - 2T_{m-1}(x/2)$,

$$P(\psi_1[1]) = 2[1 + 2(m+1) - 1] = 2\tau[1],$$

$$P(\psi_1[a^r]) = 2\left[\cos\frac{(m+1)\pi r}{m} + 1 + (-1)^r - \cos\frac{(m-1)\pi r}{m}\right]$$

$$= \begin{cases} 0, & \text{if } r \text{ odd}, \\ 4, & \text{if } r \text{ even}, \end{cases}$$

$$= 2\tau[a^r],$$

$$P(\psi_1[b]) = P(\psi_1[ab])$$

$$= 2\left[0 + \left(1 + \sin\left(\frac{m\pi}{2}\right) + \cos\left(\frac{m\pi}{2}\right)\right) - 0\right]$$

$$= 0 = 2\tau[b] = 2\tau[ab].$$

(3.4)

Again, we have that $P(x)$ is minimal since it matches $m+2$ distinct values with a polynomial of degree $m+1$. By Lemma 2.1, $P(\psi_j) = 2\tau$ for any faithful character of G, and by Lemma 2.2 it is monic with integer entries. Thus, the unique polynomial that maps the values of a faithful $\chi \in \mathrm{Irr}(G)$ onto τ is $(1/2)P(x)$ and has noninteger coefficients. □

PROOF OF THEOREM 1.2(3). Let $n = 2m$ and $m \equiv 0 \,(\mathrm{mod}\,4)$. When $r = m/2$, we have that for any faithful character, $\psi_j[a^{m/2}] = \psi_j(b) = 0$, but $\tau[a^{m/2}] = 2$ and $\tau(b) = 0$; thus we have an inconsistent system. □

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