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THE ADDITIVE APPROXIMATION ON A FOUR-VARIATE JENSEN-TYPE OPERATOR EQUATION

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We study the Hyers-Ulam stability theory of a four-variate Jensen-type functional equation by considering the approximate remainder ϕ and obtain the corresponding error formulas. We bring to light the close relation between the β-homogeneity of the norm on F^*-spaces and the approximate remainder ϕ, where we allow p,q,r, and s to be different in their Hyers-Ulam-Rassias stability.

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1. Introduction. Throughout this paper, we denote by G a linear space and by E a real or complex Hausdorff topological vector space. By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of positive integers and of reals, respectively. Let f be a mapping from G into E. We refer to the equations

$$2f\left(\frac{x+y}{2}\right)-f(x)-f(y)=\theta, \tag{1.1}$$

$$\begin{aligned}4f\left(\frac{x+y+z+w}{4}\right)+2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x+w}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+w}{2}\right)\\ -3f\left(\frac{x+y+z}{3}\right)-3f\left(\frac{y+z+w}{3}\right)-3f\left(\frac{z+w+x}{3}\right)-3f\left(\frac{w+x+y}{3}\right)=\theta\end{aligned} \tag{1.2}$$

as a Jensen equation and a four-variate Jensen-type functional equation, respectively. The approximate remainder ϕ is defined by

$$\begin{aligned}&4f\left(\frac{x+y+z+w}{4}\right)+2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x+w}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+w}{2}\right)\\ &\qquad -3f\left(\frac{x+y+z}{3}\right)-3f\left(\frac{y+z+w}{3}\right)-3f\left(\frac{z+w+x}{3}\right)-3f\left(\frac{w+x+y}{3}\right)\\ &\quad=\phi(x,y,z,w)\end{aligned} \tag{1.3}$$

for all $x,y,z,w\in G$.

In 1940, the following problem was proposed (see Ulam [11]): let G be a group and let E be a metric group with the metry $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G \to E$ satisfies the inequality $d(h(xy),h(x)h(y)) < \delta$ for all $x,y \in G$, then there exists a homomorphism $H : G \to E$ with $d(h(x),H(x)) < \varepsilon$ for all $x \in G$?

In 1941, Hyers [2] answered this question in the affirmative when G and E are Banach spaces. In 1978, Rassias [6] generalized the result of Hyers. The result was further generalized by Rassias [7], Rassias and Šemrl [9], and Găvruţa [1].

The stability problems of Jensen equations can be found in [3, 4, 5].

The author [12] considered Hyers-Ulam-Rassias stability of several functional equations under the assumption that G and E are a power-associative groupoid and a sequentially complete topological vector space, respectively. In the following, we introduce [12, Theorem 4].

Theorem 1.1. *The approximate remainder $\phi : G\times G \to E$ of Jensen equation (1.1) satisfies*

$$\begin{gathered}\lim_{n\to\infty}\frac{\phi(3^n x,3^n y)}{3^n} = \theta \quad \forall x,y \in G,\\ \sum_{k=1}^{\infty}\frac{\phi(3^{k-1}x,-3^{k-1}x)-\phi(-3^{k-1}x,3^k x)}{3^k} = \eta(x) \in E \quad \forall x \in G\end{gathered} \tag{1.4}$$

if and only if the limit $T(x) = \lim_{n\to\infty} f(3^n x)/3^n$ exists for all $x \in G$, and T is additive, where G is a real linear space and E is a real Hausdorff topological vector space. In addition,

$$T(x) - f(x) + f(\theta) = \eta(x) \quad \forall x \in G. \tag{1.5}$$

Trif [10] investigated the Hyers-Ulam-Rassias stability of the three-variate Jensen-type functional equation

$$\begin{aligned}3f\left(\frac{x+y+z}{3}\right)&+f(x)+f(y)+f(z)\\ &= 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{y+z}{2}\right)+2f\left(\frac{z+x}{2}\right)\end{aligned} \tag{1.6}$$

under the assumption that G and E are a real normed linear space and a real Banach space, respectively.

In this paper, we investigate the Hyers-Ulam stability of (1.2) by considering the approximate remainders under the assumption that G and E are a real linear space and a certain kind of F^*-space, respectively. First we solve (1.2) in Section 2. Second, in Section 3, still using the direct method, we obtain some theorems of the Hyers-Ulam stability of (1.2). Finally, we give an example that the Hyers-Ulam-Rassias stability of (1.2) does not hold.

2. Solutions of (1.2). From now we let G be a real linear space and E a real Hausdorff topological vector space, unless otherwise specified. In this section, we claim that (1.2) is equivalent to (1.1). It is well known that if G and E are real linear spaces, then a function $f : G \to E$ satisfying $f(\theta) = \theta$ is a solution of (1.1) if and only if it is additive.

Theorem 2.1. *A function $f : G \to E$ satisfies (1.2) for all $x, y, z, w \in G$ if and only if there exist a constant element $C \in E$ and a unique additive mapping $T : G \to E$ such that*

$$f(x) = T(x) + C \quad \forall x \in G. \tag{2.1}$$

Proof. The proof of the sufficiency is straightforward, so we will show only the necessity. Set $C = f(\theta)$ and $T(x) = f(x) - C$ for each $x \in G$. Then $T(\theta) = \theta$ and

$$\begin{aligned} &4T\left(\frac{x+y+z+w}{4}\right)+2T\left(\frac{x+y}{2}\right)+2T\left(\frac{x+w}{2}\right)+2T\left(\frac{y+z}{2}\right)+2T\left(\frac{z+w}{2}\right) \\ &\quad = 3T\left(\frac{x+y+z}{3}\right)+3T\left(\frac{y+z+w}{3}\right)+3T\left(\frac{z+w+x}{3}\right)+3T\left(\frac{w+x+y}{3}\right) \end{aligned} \tag{2.2}$$

for any $x, y, z, w \in G$. We will show that T is additive. Let $x \in G$. Put $y = x$ and $z = w = -x$ in (2.2) to yield

$$T(x) + T(-x) = 3\left[T\left(\frac{x}{3}\right) + T\left(-\frac{x}{3}\right)\right]. \tag{2.3}$$

Take $y = -x$ and $z = w = \theta$ in (2.2) to get

$$2\left[T\left(\frac{x}{2}\right) + T\left(-\frac{x}{2}\right)\right] = 3\left[T\left(\frac{x}{3}\right) + T\left(-\frac{x}{3}\right)\right]. \tag{2.4}$$

From (2.3) and the last equality, we obtain

$$T(x) + T(-x) = 2\left[T\left(\frac{x}{2}\right) + T\left(-\frac{x}{2}\right)\right]. \tag{2.5}$$

Putting $y = x$, $z = -2x$, and $w = \theta$ in (2.2) gives

$$2[T(x) + T(-x)] + 2\left[T\left(\frac{x}{2}\right) + T\left(-\frac{x}{2}\right)\right] = 6T\left(-\frac{x}{3}\right) + 3T\left(\frac{2x}{3}\right). \tag{2.6}$$

From (2.5) and the last equality, we have

$$T(x)+T(-x)=2T\left(-\frac{x}{3}\right)+T\left(\frac{2x}{3}\right). \tag{2.7}$$

Put $y=z=x$ and $w=-3x$ in (2.2) to conclude that

$$T(x)+4T(-x)=9T\left(-\frac{x}{3}\right). \tag{2.8}$$

Replacing x by $-x$ in the above equality, we have

$$T(-x)+4T(x)=9T\left(\frac{x}{3}\right). \tag{2.9}$$

Adding the last two formulas together produces

$$5[T(x)+T(-x)]=9\left[T\left(\frac{x}{3}\right)+T\left(-\frac{x}{3}\right)\right]. \tag{2.10}$$

Hence, from (2.3) and the last equality, we conclude that

$$T(x)+T(-x)=\theta, \quad \text{that is,} \quad T(-x)=-T(x). \tag{2.11}$$

It follows from (2.7), (2.9), and (2.11) that

$$T\left(\frac{x}{3}\right)=\frac{1}{3}T(x), \qquad T\left(\frac{2x}{3}\right)=2T\left(\frac{x}{3}\right). \tag{2.12}$$

Replacing $x/3$ by x in the last equality, we obtain

$$T(2x)=2T(x), \quad \text{that is,} \quad T\left(\frac{x}{2}\right)=\frac{1}{2}T(x), \tag{2.13}$$

and so, $T(x/4)=(1/4)T(x)$. Substituting

$$T\left(\frac{x}{2}\right)=\frac{1}{2}T(x), \qquad T\left(\frac{x}{3}\right)=\frac{1}{3}T(x), \qquad T\left(\frac{x}{4}\right)=\frac{1}{4}T(x) \tag{2.14}$$

into (2.2) supplies

$$\begin{aligned} &T(x+y+z+w)+T(x+y)+T(x+w)+T(y+z)+T(z+w) \\ &\quad =T(x+y+z)+T(y+z+w)+T(z+w+x)+T(w+x+y). \end{aligned} \tag{2.15}$$

Finally, we take $z=-x-y$ and $w=\theta$ in the above equality to get from (2.11) that $T(x+y)=T(x)+T(y)$, and so, T is additive in terms of the arbitrariness of x and y. □

3. Hyers-Ulam-Rassias stability of (1.2). Next we are interested in the Hyers-Ulam stability of (1.2). For convenience, we set $\varphi(x,y) = \phi(x,y,x,y)$ for all $x,y \in G$, where ϕ is of (1.3).

Theorem 3.1. *The map $\varphi : G\times G \to E$ satisfies*

$$\lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y)}{3^n} = \theta \quad \forall x,y \in G, \tag{3.1}$$

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{\varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))}{3^{k+1}} = \eta(x) \in E \quad \forall x \in G \tag{3.2}$$

if and only if the limit $T(x) = \lim_{n\to\infty} f(3^n x)/3^n$ exists for all $x \in G$, and T is additive. In this case (1.5) holds.

Proof. We omit the easy proof of sufficiency and, like Theorem 2.1, we will show the necessity only. Let any $x,y \in G$. Putting $z = x$ and $w = y$ in (1.3), we get

$$2f\left(\frac{x+y}{2}\right) - f\left(\frac{2x+y}{3}\right) - f\left(\frac{x+2y}{3}\right) = \frac{1}{6}\varphi(x,y). \tag{3.3}$$

Let $u,v \in G$, $x = 2u - v$, and $y = -u + 2v$. Then $u = (2x+y)/3$, $v = (x+2y)/3$, and $x + y = u + v$, and so we have

$$2f\left(\frac{u+v}{2}\right) - f(u) - f(v) = \Phi(u,v), \tag{3.4}$$

where $\Phi(u,v) \stackrel{\text{def}}{=} (1/6)\varphi(2u-v, -u+2v)$.

On the one hand, clearly,

$$\lim_{n\to\infty} \frac{\Phi(3^n u, 3^n v)}{3^n} = \frac{1}{6}\lim_{n\to\infty} \frac{\varphi(3^n(2u-v), 3^n(-u+2v))}{3^n}. \tag{3.5}$$

This yields from assumption (3.1) that

$$\lim_{n\to\infty} \frac{\Phi(3^n u, 3^n v)}{3^n} = \theta. \tag{3.6}$$

On the other hand, using the definition of $\Phi(u,v)$, we compute

$$\begin{aligned} \Phi(3^{k-1}u, 3^{k-1}u) &= \frac{1}{6}\varphi(3^{k-1}u, -3^{k-1}u), \\ \Phi(-3^{k-1}u, 3^k u) &= \frac{1}{6}\varphi(-3^{k-1}(5u), 3^{k-1}(7u)), \end{aligned} \tag{3.7}$$

then we conclude from (3.2) that

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{\Phi(3^{k-1}u,-3^{k-1}u)-\Phi(-3^{k-1}u,3^{k}u)}{3^{k}}\\&\quad=\frac{1}{2}\sum_{k=1}^{\infty}\frac{\varphi(3^{k}u,-3^{k}u)-\varphi(-3^{k-1}(5u),3^{k-1}(7u))}{3^{k+1}}=\eta(u)\in E.\end{aligned}\tag{3.8}$$

Thus, by Theorem 1.1, the limit $T(u)=\lim_{n\to\infty}f(3^{n}u)/3^{n}$ exists, T is additive, and the equality $T(u)-f(u)+f(\theta)=\eta(u)$ holds for each $u\in G$.

The proof is complete. □

For abbreviation, we set

$$\begin{aligned}B(x,-x)&=\mathrm{co}\left(\{\theta\}\cup\{\varphi(3^{i}x,-3^{i}x)\}_{i=1}^{\infty}\right)\quad\forall x\in G,\\B(-5x,7x)&=\mathrm{co}\left(\{\theta\}\cup\{\varphi(-3^{i-1}(5x),3^{i-1}(7x))\}_{i=1}^{\infty}\right)\quad\forall x\in G.\end{aligned}\tag{3.9}$$

By Theorem 3.1 and [12, Corollary 6], we conclude the following corollary.

Corollary 3.2. *Let E be sequentially complete and let (3.1) hold. If $B(x,-x)$ and $B(-5x,7x)$ are bounded for any $x\in G$, then there exists a unique additive mapping $T:G\to E$ such that*

$$T(x)-f(x)+f(\theta)\in\frac{1}{6}\left[\overline{B}^{s}(x,-x)-\overline{B}^{s}(-5x,7x)\right]\quad\forall x\in G,\tag{3.10}$$

where $\mathrm{co}(A)$ is the convex hull of a set A, and $\overline{A}^{s}$ denotes the sequential closure of set A. If E is also locally convex, then the boundedness of $\{\varphi(3^{i}x,-3^{i}x)\}_{i=1}^{\infty}$ and $\{\phi(-3^{i-1}(5x),3^{i-1}(7x))\}_{i=1}^{\infty}$ ensures the boundedness of $B(x,-x)$ and $B(-5x,7x)$, respectively.

Next we derive the Hyers-Ulam-Rassias stability of (1.2), which is an application of Theorem 3.1. Note that it is close correlative with the β-homogeneity of the norm on F^*-spaces. Simultaneously, we allow p,q,r, and s to be different.

Let X be a linear space. A nonnegative-valued function $\|\cdot\|$ defined on X is called an F-norm if it satisfies the following conditions:

(n1) $\|x\|=0$ if and only if $x=0$;
(n2) $\|ax\|=\|x\|$ for all a, $|a|=1$;
(n3) $\|x+y\|\le\|x\|+\|y\|$;
(n4) $\|a_nx\|\to0$ provided $a_n\to0$;
(n5) $\|ax_n\|\to0$ provided $x_n\to0$.

A space X with an F-norm is called an F^*-space. An F-pseudonorm ($\|x\|=0$ does not necessarily imply that $x=0$ in (n1)) is called β-homogeneous ($\beta>0$) if $\|tx\|=|t|^{\beta}\|x\|$ for all $x\in X$ and all $t\in\mathbb{R}$. A complete F^*-space is said to be an F-space.

Corollary 3.3. *Suppose that G is an F^*-space and E a β-homogeneous F-space $(0 < \beta \le 1)$. Given $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \delta \ge 0$ and $0 \le p, q, r, s < \beta$, if ϕ satisfies*

$$\begin{aligned} &\|\phi(x,y,z,w)\| \\ &\quad \le \delta + \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q + \varepsilon_3\|z\|^r + \varepsilon_4\|w\|^s \quad \forall x,y,z,w \in G, \end{aligned} \tag{3.11}$$

then there exists a unique additive mapping $T : G \to E$ such that

$$\begin{aligned} \|T(x) - f(x) + f(\theta)\| \le A\delta + \varepsilon_1 B_1\|x\|^p + \varepsilon_2 B_2\|x\|^q \\ + \varepsilon_3 B_3\|x\|^r + \varepsilon_4 B_4\|x\|^s \end{aligned} \tag{3.12}$$

for all $x \in G$, where

$$\begin{aligned} &A \overset{\text{def}}{=} \frac{2}{6^\beta(3^\beta - 1)}, \qquad B_1 \overset{\text{def}}{=} \frac{(3^p + 5^p)}{6^\beta(3^\beta - 3^p)}, \qquad B_2 \overset{\text{def}}{=} \frac{(3^q + 7^q)}{6^\beta(3^\beta - 3^q)}, \\ &\qquad B_3 \overset{\text{def}}{=} \frac{(3^r + 5^r)}{6^\beta(3^\beta - 3^r)}, \qquad B_4 \overset{\text{def}}{=} \frac{(3^s + 7^s)}{6^\beta(3^\beta - 3^s)}. \end{aligned} \tag{3.13}$$

Proof. Let any $x, y \in G$. Firstly, put $z = x$ and $w = y$ in (3.11) to get according to the definition of φ that

$$\begin{aligned} \|\varphi(x,y)\| = \|\phi(x,y,x,y)\| \le \delta + \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q \\ + \varepsilon_3\|x\|^r + \varepsilon_4\|y\|^s \quad \forall x,y \in G. \end{aligned} \tag{3.14}$$

It follows from $p, q, r, s < \beta$ that

$$\begin{aligned} \lim_{n\to\infty}\left\|\frac{\varphi(3^n x, 3^n y)}{3^n}\right\| \le \lim_{n\to\infty}\Bigg[\frac{\delta}{3^n} + \frac{\varepsilon_1}{3^{n(\beta-p)}}\|x\|^p + \frac{\varepsilon_2}{3^{n(\beta-q)}}\|y\|^q \\ + \frac{\varepsilon_3}{3^{n(\beta-r)}}\|x\|^r + \frac{\varepsilon_4}{3^{n(\beta-s)}}\|y\|^s\Bigg] = 0. \end{aligned} \tag{3.15}$$

Secondly, in light of the triangle inequality of F-norm and $p, q, r, s \ge 0$, we have, for any $i \in \mathbb{N}$,

$$\begin{aligned} &\|\varphi(3^i x, -3^i x)\| \le \delta + \varepsilon_1 3^{ip}\|x\|^p + \varepsilon_2 3^{iq}\|x\|^q + \varepsilon_3 3^{ir}\|x\|^r + \varepsilon_4 3^{is}\|x\|^s, \\ &\|\varphi(-3^{i-1}(5x), 3^{i-1}(7x))\| \le \delta + \varepsilon_1 3^{(i-1)p}5^p\|x\|^p + \varepsilon_2 3^{(i-1)q}7^q\|x\|^q \\ &\qquad + \varepsilon_3 3^{(i-1)p}5^r\|x\|^r + \varepsilon_4 3^{(i-1)q}7^s\|x\|^s. \end{aligned} \tag{3.16}$$

As in the proof of [12, Theorem 3], we infer from (3.4) that

$$\frac{1}{3^n}f(3^n x) - f(x) = \sum_{k=1}^{n}\frac{\Psi(3^{k-1}x)}{3^k} \tag{3.17}$$

holds for any $n \in \mathbb{N}$, where $\Psi(x) = \Phi(x, -x) - \Phi(-x, 3x) - 2f(\theta)$.

Consequently, for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{3^n} f(3^n x) - f(x) + 2\sum_{k=1}^{n} \frac{f(\theta)}{3^k} \\
&\quad = \sum_{k=1}^{n} \frac{\Phi(3^{k-1}x, -3^{k-1}x) - \Phi(-3^{k-1}x, 3^k x)}{3^k} \\
&\quad = \frac{1}{2}\sum_{k=1}^{n} \frac{\varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))}{3^{k+1}}.
\end{aligned} \tag{3.18}$$

It is easy to see that

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{\varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))}{3^{k+1}} \tag{3.19}$$

exists for every $x \in G$. Indeed, from the above, we conclude that

$$\begin{aligned}
&\frac{f(3^m x)}{3^m} - \frac{f(3^n x)}{3^n} \\
&\quad = \frac{1}{3^n}\left[\frac{f(3^{m-n}(3^n x))}{3^{m-n}} - f(3^n x)\right] \\
&\quad = \frac{1}{3^n}\sum_{k=1}^{m-n} \frac{\Psi(3^{n+k-1}x)}{3^k} = \sum_{k=n+1}^{m} \frac{\Psi(3^{k-1}x)}{3^k} \\
&\quad = \frac{1}{2}\sum_{k=n+1}^{m} \frac{\varphi(3^k x, -3^k x) - \varphi(-3^{k-1}(5x), 3^{k-1}(7x))}{3^{k+1}} - 2\sum_{k=n+1}^{m} \frac{f(\theta)}{3^k}
\end{aligned} \tag{3.20}$$

for any $m > n$, where $m, n \in \mathbb{N}$, and so

$$\begin{aligned}
&\left\|\frac{f(3^m x)}{3^m} - \frac{f(3^n x)}{3^n}\right\| \\
&\quad \le \frac{1}{2^\beta}\sum_{k=n+1}^{m} \frac{2\delta + \varepsilon_1(3^{kp} + 3^{(k-1)p}5^p)\|x\|^p + \varepsilon_2(3^{kq} + 3^{(k-1)q}7^q)\|x\|^q}{3^{(k+1)\beta}} \\
&\qquad + \frac{1}{2^\beta}\sum_{k=n+1}^{m} \frac{\varepsilon_3(3^{kr} + 3^{(k-1)r}5^r)\|x\|^r + \varepsilon_4(3^{ks} + 3^{(k-1)s}7^s)\|x\|^s}{3^{(k+1)\beta}} \\
&\qquad + 2\|f(\theta)\|\sum_{k=n+1}^{m} \frac{1}{3^k} \\
&\quad \le \sum_{k=n+1}^{m} \frac{2^{1-\beta}\delta}{3^{(k+1)\beta}} + \frac{\varepsilon_1}{2^\beta}\sum_{k=n+1}^{m}\left[\frac{1}{3^\beta}3^{k(p-\beta)} + \frac{5^p}{3^{2\beta}}3^{(k-1)(p-\beta)}\right]\|x\|^p \\
&\qquad + \frac{\varepsilon_2}{2^\beta}\sum_{k=n+1}^{m}\left[\frac{1}{3^\beta}3^{k(q-\beta)} + \frac{7^q}{3^{2\beta}}3^{(k-1)(q-\beta)}\right]\|x\|^q
\end{aligned}$$

$$+\frac{\varepsilon_3}{2^\beta}\sum_{k=n+1}^{m}\left[\frac{1}{3^\beta}3^{k(r-\beta)}+\frac{5^r}{3^{2\beta}}3^{(k-1)(r-\beta)}\right]\|x\|^r$$
$$+\frac{\varepsilon_4}{2^\beta}\sum_{k=n+1}^{m}\left[\frac{1}{3^\beta}3^{k(s-\beta)}+\frac{7^s}{3^{2\beta}}3^{(k-1)(s-\beta)}\right]\|x\|^s+2\|f(\theta)\|\sum_{k=n+1}^{m}\frac{1}{3^k} \tag{3.21}$$

for any $m > n$, where $m,n \in \mathbb{N}$. Since $p,q,r,s < \beta$, $\{f(3^n x)/3^n\}$ is a Cauchy sequence of E. By the completeness of E, $\{f(3^n x)/3^n\}$ converges to an element of E.

Thus, by Theorem 3.1, $T(x) = \lim_{n\to\infty}(f(3^n x)/3^n)$ and it is additive. In addition, from (3.18), inequality (3.12) holds for all $x \in G$.

In order to prove the uniqueness of T, suppose that $U : G \to E$ is another additive mapping which satisfies

$$\begin{aligned}\|U(x)-f(x)+f(\theta)\| \le A\delta+\varepsilon_1B_1\|x\|^p+\varepsilon_2B_2\|x\|^q\\+\varepsilon_3B_3\|x\|^r+\varepsilon_4B_4\|x\|^s\end{aligned} \tag{3.22}$$

for all $x \in G$. On account of the last two inequalities, we conclude that, for all $x \in G$,

$$\begin{aligned}&\|U(x)-T(x)\|\\&=\frac{1}{n^\beta}\|U(nx)-T(nx)\|\\&=\frac{1}{n^\beta}\|U(nx)-f(nx)+f(\theta)-T(nx)+f(nx)-f(\theta)\|\\&\le\frac{1}{n^\beta}\left[\|U(nx)-f(nx)+f(\theta)\|+\|T(nx)-f(nx)+f(\theta)\|\right]\\&\le\frac{2}{n^\beta}(A\delta+\varepsilon_1B_1\|nx\|^p+\varepsilon_2B_2\|nx\|^q+\varepsilon_3B_3\|nx\|^r+\varepsilon_4B_4\|nx\|^s)\\&=2\left[\frac{A\delta}{n^\beta}+\frac{\varepsilon_1B_1}{n^{\beta-p}}\|x\|^p+\frac{\varepsilon_2B_2}{n^{\beta-q}}\|x\|^q+\frac{\varepsilon_3B_3}{n^{\beta-r}}\|x\|^r+\frac{\varepsilon_4B_4}{n^{\beta-r}}\|x\|^s\right],\end{aligned} \tag{3.23}$$

and so, $\|U(x)-T(x)\| \to 0$ as $n \to \infty$ since $p,q,r,s < \beta$. As a consequence, $U(x) = T(x)$ for all $x \in G$.

Therefore, the result holds. □

In order to show that Corollary 3.3 is valid in the case that $p,q,r,s > 1/\beta$, we need the following theorem, which can be proved in the same manner as Theorem 1.1.

Theorem 3.4. *The approximate remainder $\phi : G\times G \to E$ of (1.1) satisfies*

$$\lim_{n\to\infty} 3^n \phi(3^{-n}x, 3^{-n}y) = \theta \quad \forall x, y \in G,$$
$$\sum_{k=1}^{\infty} 3^{k-1}[\phi(3^{-k}x, -3^{-k}x) - \phi(-3^{-k}x, 3^{-k+1}x)] = \eta(x) \in E \quad \forall x \in G \tag{3.24}$$

if and only if the limit $T(x) = \lim_{n\to\infty} 3^n[f(3^{-n}x) - f(\theta)]$ exists for all $x \in G$, and T is additive. In this case (1.5) holds.

Proof. Note that if set $g(x) = f(x) - f(\theta)$ for any $x \in G$, then $g(\theta) = \theta$ and the approximate remainders ϕ_g and ϕ_f of (1.1) with respect to g and f, respectively, are equal. We still write it as ϕ. As in the proof of Theorem 1.1, we can conclude that, for every x in G with $x \neq 0$ and every n in $\mathbb{N}$,

$$g(x) - 3^n(3^{-n}x) = \sum_{k=1}^{n} [\phi(3^{-k}x, -3^{-k}x) - \phi(-3^{-k}x, 3^{-k+1}x)]. \tag{3.25}$$

□

We may see that it is possible that $T(x) = \lim_{n\to\infty} 3^n[f(3^{-n}x) - f(\theta)]$ exists, in particular, if f is differentiable at θ in G.

Corollary 3.5. *Suppose that G is a β-homogeneous F^*-space $(0 < \beta \leq 1)$ and E an F-space with a nondecreasing F-norm. Given $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, +\infty)$ and $p, q, r, s \in (1/\beta, +\infty)$, if ϕ satisfies*

$$\begin{aligned} \|\phi(x,y,z,w)\| &\leq \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q \\ &\quad + \varepsilon_3\|z\|^r + \varepsilon_4\|w\|^s \quad \forall x, y, z, w \in G, \end{aligned} \tag{3.26}$$

then there exists a unique additive mapping $T : G \to E$ such that

$$\begin{aligned} \|T(x) - f(x) + f(\theta)\| &\leq \varepsilon_1 B_1\|x\|^p + \varepsilon_2 B_2\|x\|^q \\ &\quad + \varepsilon_3 B_3\|x\|^r + \varepsilon_4 B_4\|x\|^s \quad \forall x \in G, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} B_1 &\overset{\text{def}}{=} \frac{(3^{p\beta} + 5^{p\beta})}{(3^{p\beta} - 3)}, \qquad B_2 \overset{\text{def}}{=} \frac{(3^{q\beta} + 7^{q\beta})}{(3^{q\beta} - 3)}, \\ B_3 &\overset{\text{def}}{=} \frac{(3^{r\beta} + 5^{r\beta})}{(3^{r\beta} - 3)}, \qquad B_4 \overset{\text{def}}{=} \frac{(3^{s\beta} + 7^{s\beta})}{(3^{s\beta} - 3)}. \end{aligned} \tag{3.28}$$

Proof. Let $g(x) = f(x) - f(\theta)$ for any $x \in G$. Using Theorem 3.4, as in the proofs of Theorem 3.1 and Corollary 3.3, we can achieve that

$$\frac{1}{6}\sum_{k=1}^{\infty} 3^{k-1}\left[\varphi\left(3^{-k+1}x, -3^{-k+1}x\right) - \varphi\left(-3^{-k}(5x), 3^{-k}(7x)\right)\right] \tag{3.29}$$

exists for every $x \in G$ and

$$\begin{aligned} &g(x) - 3^n g\left(3^{-n}x\right) \\ &\quad = \frac{1}{6}\sum_{k=1}^{n} 3^{k-1}\left[\varphi\left(3^{-k+1}x, -3^{-k+1}x\right) - \varphi\left(-3^{-k}(5x), 3^{-k}(7x)\right)\right]. \end{aligned} \tag{3.30}$$

Finally, we can evaluate the error formula. □

We may also deal with the Hyers-Ulam stability of (1.2) as usual.

Theorem 3.6. *The approximate remainder ϕ satisfies*

$$\lim_{n\to\infty} \frac{\phi\left(3^n x, 3^n y, 3^n z, 3^n w\right)}{3^n} = \theta \quad \forall x, y, z, w \in G, \tag{3.31}$$

$$\sum_{k=1}^{\infty} \frac{\psi\left(3^k x\right)}{3^k} = \eta(x) \in E \quad \forall x \in G \tag{3.32}$$

if and only if the limit $T(x) = \lim_{n\to\infty} f(3^n x)/3^n$ exists for all $x \in G$, and T is additive. Moreover, (1.5) holds, where

$$\psi(x) \overset{\text{def}}{=} \frac{1}{4}\phi(x, x, -x, -x) + \frac{1}{6}\left[\phi(-x, -x, -x, 3x) - \phi(x, x, x, -3x)\right]. \tag{3.33}$$

Proof. It is enough to show the necessity. Define g as above.

Let any $x \in G$. Put $y = x$ and $z = w = -x$ in (1.3) to yield

$$g(x) + g(-x) - 3\left[g\left(\frac{x}{3}\right) + g\left(-\frac{x}{3}\right)\right] = \frac{1}{2}\phi(x, x, -x, -x). \tag{3.34}$$

Put $y = z = x$ and $w = -3x$ in (1.3) to give

$$g(x) + 4g(-x) - 9g\left(-\frac{x}{3}\right) = \phi(x, x, x, -3x). \tag{3.35}$$

Replacing x by $-x$ in the above equality, we have

$$g(-x)+4g(x)-9g\left(\frac{x}{3}\right)=\phi(-x,-x,-x,3x). \tag{3.36}$$

Adding the last two formulas together, we conclude that

$$\begin{aligned}5[g(x)+g(-x)]-9\left[g\left(\frac{x}{3}\right)+g\left(-\frac{x}{3}\right)\right]\\ =\phi(x,x,x,-3x)+\phi(-x,-x,-x,3x).\end{aligned} \tag{3.37}$$

Hence, from (3.34) and the above equality, we know that

$$\begin{aligned}g(x)+g(-x)=&\frac{1}{2}[\phi(x,x,x,-3x)+\phi(-x,-x,-x,3x)]\\ &-\frac{3}{4}\phi(x,x,-x,-x).\end{aligned} \tag{3.38}$$

It follows from (3.36) and (3.38) that

$$\begin{aligned}g(x)-3g\left(\frac{x}{3}\right)=&\frac{1}{4}\phi(x,x,-x,-x)\\ &+\frac{1}{6}[\phi(-x,-x,-x,3x)-\phi(x,x,x,-3x)]\\ =&\psi(x).\end{aligned} \tag{3.39}$$

With $3x$ in place of x in the above equality and dividing by 3, we obtain

$$\frac{1}{3}g(3x)-g(x)=\frac{1}{3}\psi(3x). \tag{3.40}$$

We will prove by induction that

$$\frac{1}{3^n}g(3^nx)-g(x)=\sum_{k=1}^{n}\frac{\psi(3^kx)}{3^k}\quad\forall n\in\mathbb{N}. \tag{3.41}$$

For $n=1$ this is trivial according to (3.40). Suppose that (3.41) holds for a certain $m-1$. Then (3.40) and the induction hypothesis imply that

$$\begin{aligned}\frac{1}{3^m}g(3^mx)-g(x)&=\frac{1}{3}\left[\frac{1}{3^{m-1}}g(3^{m-1}(3x))-g(3x)\right]+\frac{1}{3}g(3x)-g(x)\\ &=\frac{1}{3}\sum_{k=1}^{m-1}\frac{\psi(3^k(3x))}{3^k}+\frac{1}{3}\psi(3x)=\sum_{k=1}^{m}\frac{\psi(3^kx)}{3^k},\end{aligned} \tag{3.42}$$

that is, (3.41) holds for $n=m$.

We define $T(x) = \lim_{n\to\infty} g(3^n x)/3^n$. Obviously, $T(x) = \lim_{n\to\infty} f(3^n x)/3^n$, and so, by (3.32) and (3.41), $T(x)$ exists and

$$T(x) - g(x) = \eta(x). \tag{3.43}$$

Substituting the definition of g into the last equality implies that

$$T(x) - f(x) + f(\theta) = \eta(x). \tag{3.44}$$

Finally, we verify that T is additive. Indeed, the definition of T implies that

$$T(\theta) = \lim_{n\to\infty} \frac{g(3^n\theta)}{3^n} = \theta. \tag{3.45}$$

Because of (3.31), T is a solution of (1.2). Hence $T(x) = T^*(x) + T(\theta) = T^*(x)$ by Theorem 2.1, where T^* is additive. It follows that T is additive. □

To show the following corollary, we may use a manner analogous to that used in Corollary 3.3.

Corollary 3.7. *Keeping all the hypotheses of Corollary 3.3, there exists a unique additive mapping $T : G \to E$ such that (3.12) holds, where*

$$A \overset{\text{def}}{=} \frac{3^\beta + 2^{\beta+1}}{12^\beta(3^\beta - 1)}, \quad B_1 \overset{\text{def}}{=} \frac{3^p(3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^p)}, \quad B_2 \overset{\text{def}}{=} \frac{3^q(3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^q)},$$
$$B_3 \overset{\text{def}}{=} \frac{3^r(3^\beta + 2^{\beta+1})}{12^\beta(3^\beta - 3^r)}, \quad B_4 \overset{\text{def}}{=} \frac{3^s(3^\beta + 2^{\beta+1}3^s)}{12^\beta(3^\beta - 3^s)}. \tag{3.46}$$

If there exists at least one of p, q, r, and s such that it is strictly less than 0, it is supposed that (3.11) holds for all $x, y, z, w \in G\backslash\{\theta\}$. Then the domain of T is $G\backslash\{\theta\}$ instead of G.

As earlier, we consider the case of $p, q, r, s > 1/\beta$.

Theorem 3.8. *The approximate remainder ϕ satisfies*

$$\lim_{n\to\infty} 3^n\phi(3^{-n}x, 3^{-n}y, 3^{-n}z, 3^{-n}w) = \theta \quad \forall x, y, z, w \in G,$$
$$\sum_{k=1}^{\infty} 3^{k-1}\psi(3^{-(k-1)}x) = \eta(x) \in E \quad \forall x \in G \tag{3.47}$$

if and only if the limit $T(x) = \lim_{n\to\infty} 3^n[f(3^{-n}x) - f(\theta)]$ *exists for all* $x \in G$*, and* T *is additive, where* ψ *is as above. Moreover,*

$$T(x) - f(x) + f(\theta) = \eta(x) \quad \forall x \in G. \tag{3.48}$$

Proof. Let $g(x) = f(x) - f(\theta)$. Note that, by virtue of (3.39), we conclude by induction that

$$g(x) - 3^n g(3^{-n}x) = \sum_{k=1}^{n} 3^{k-1}\psi(3^{-(k-1)}x) \quad \forall x \in G,\ n \in \mathbb{N}. \tag{3.49}$$

□

Corollary 3.9. *Keeping all the hypotheses of Corollary 3.5, then there exists a unique additive mapping* $T : G \to E$ *such that*

$$\begin{aligned} ||T(x) - f(x)|| \le{}& \varepsilon_1 B_1 \|x\|^p + \varepsilon_2 B_2 \|x\|^q \\ &+ \varepsilon_3 B_3 \|x\|^r + \varepsilon_4 B_4 \|x\|^s \quad \forall x \in G, \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} B_1 &\stackrel{\text{def}}{=} \frac{3^{p\beta+1}}{(3^{p\beta}-3)}, & B_2 &\stackrel{\text{def}}{=} \frac{3^{q\beta+1}}{(3^{q\beta}-3)}, \\ B_3 &\stackrel{\text{def}}{=} \frac{3^{r\beta+1}}{(3^{r\beta}-3)}, & B_4 &\stackrel{\text{def}}{=} \frac{3^{s\beta}(1+2(3^{s\beta}))}{(3^{s\beta}-3)}. \end{aligned} \tag{3.51}$$

We still mention the following immediate consequence of Corollary 3.3.

Remark 3.10. Let E be a β-homogeneous F-space $(0 < \beta \le 1)$. If ϕ satisfies the property that there exists $\delta \in [0 + \infty)$ such that $\|\phi(x,y,z,w)\| \le \delta$ for any $x, y, z, w \in G$, then there exists a unique additive mapping $T : G \to E$ such that

$$||T(x) - f(x) + f(\theta)|| \le \frac{2\delta}{6^\beta(3^\beta - 1)} \quad \forall x \in G. \tag{3.52}$$

As in [13], in the last of this section we give an example by means of Rassias and Šemrl [8] who constructed a function $f : \mathbb{R} \to \mathbb{R}$ ($f(x) \stackrel{\text{def}}{=} x\log_2(1+|x|)$) to show that (1.2) does not have Hyers-Ulam-Rassias stability property if p, q, r, and s satisfy any one condition of $(\triangle_1)$ $p = q = r = s = \beta$, $(\triangle_2)$ $p = q = r = s = 1/\beta$, and $(\triangle_3)$ $\beta \le p = q = r = s = 1 \le 1/\beta$ $(0 < \beta \le 1)$. What if p, q, r, and s satisfy that $\beta \le p, q, r, s \le 1/\beta$, where $p \ne 1$, $q \ne 1$, $r \ne 1$, and $s \ne 1$ under the assumption that G and E are β-homogeneous F-space $(0 < \beta < 1)$?

Theorem 3.11. *The function* $f:\mathbb{R}\to\mathbb{R}$ *defined by* $f(x)\overset{\text{def}}{=}x\log_2(1+|x|)$ *satisfies the inequality*

$$|\phi(x,y,z,w)|\leq 14(|x|+|y|+|z|+|w|)\quad \forall x,y,z,w\in\mathbb{R}, \tag{3.53}$$

but

$$\sup\left\{\left|\frac{f(x)-T(x)}{x}\right| : x\in\mathbb{R}\backslash\{0\}\right\}=\infty \tag{3.54}$$

for each additive mapping $T:\mathbb{R}\to\mathbb{R}$.

Proof. For all $x,y,z,w\in\mathbb{R}$, it follows from $|f(x+y)-f(x)-f(y)|\leq |x|+|y|$ in [8] and $|f(x+y+z)-f(x)-f(y)-f(z)|\leq(5/3)(|x|+|y|+|z|)$ in [10] that

$$\begin{aligned}
\phi(x,y,z,w) &= \left[4f\left(\frac{x+y+z+w}{4}\right)-f(x+y+z+w)\right]\\
&\quad +\left[f(x+y+z+w)-f(x+z)-f(y+w)\right]\\
&\quad +\left[f(x+z)-2f\left(\frac{x+z}{2}\right)\right]+\left[f(y+w)-2f\left(\frac{y+w}{2}\right)\right]\\
&\quad -\left[3f\left(\frac{x+y+z}{3}\right)-f(x+y+z)\right]\\
&\quad -\left[f(x+y+z)-f\left(\frac{x+y}{2}\right)-f\left(\frac{y+z}{2}\right)-f\left(\frac{z+x}{2}\right)\right]\\
&\quad -\left[3f\left(\frac{y+z+w}{3}\right)-f(y+z+w)\right]\\
&\quad -\left[f(y+z+w)-f\left(\frac{y+z}{2}\right)-f\left(\frac{z+w}{2}\right)-f\left(\frac{w+y}{2}\right)\right]\\
&\quad -\left[3f\left(\frac{z+w+x}{3}\right)-f(z+w+x)\right]\\
&\quad -\left[f(z+w+x)-f\left(\frac{z+w}{2}\right)-f\left(\frac{w+x}{2}\right)-f\left(\frac{x+z}{2}\right)\right]\\
&\quad -\left[3f\left(\frac{w+x+y}{3}\right)-f(w+x+y)\right]\\
&\quad -\left[f(w+x+y)-f\left(\frac{w+x}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(\frac{y+w}{2}\right)\right].
\end{aligned} \tag{3.55}$$

Furthermore, we evaluate that

$$\begin{aligned}
|\phi(x,y,z,w)| &\le 8\left|\frac{x+y+z+w}{4}\right| + |x+z| + |y+w| + 2\left|\frac{x+z}{2}\right| \\
&\quad + 2\left|\frac{y+w}{2}\right| + \frac{5}{3}3\left|\frac{x+y+z}{3}\right| \\
&\quad + \frac{5}{3}\left[\left|\frac{x+y}{2}\right| + \left|\frac{y+z}{2}\right|\left|\frac{z+x}{2}\right|\right] \\
&\quad + \frac{15}{3}\left|\frac{y+z+w}{3}\right| + \frac{5}{3}\left[\left|\frac{y+z}{2}\right| + \left|\frac{z+w}{2}\right|\left|\frac{w+y}{2}\right|\right] \\
&\quad + \frac{15}{3}\left|\frac{z+w+x}{3}\right| + \frac{5}{3}\left[\left|\frac{z+w}{2}\right| + \left|\frac{w+x}{2}\right|\left|\frac{x+z}{2}\right|\right] \\
&\quad + \frac{15}{3}\left|\frac{w+x+y}{3}\right| + \frac{5}{3}\left[\left|\frac{w+x}{2}\right| + \left|\frac{x+y}{2}\right|\left|\frac{y+w}{2}\right|\right] \\
&\le 14(|x| + |y| + |z| + |w|)
\end{aligned} \tag{3.56}$$

for all $x, y, z, w \in \mathbb{R}$. The rest of the proof has been proved in [10]. □

Remark 3.12. Let f be as in Theorem 3.11.

(i) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the Euclidean metric $\|\cdot\|_1 = |\cdot|$, and $E = (\mathbb{R}, \|\cdot\|_2)$ with the β-homogeneous norm $\|\cdot\|_2 = |\cdot|^\beta$, then

$$\begin{aligned}
&\|\phi(x,y,z,w)\|_2 \\
&\quad \le 14^\beta\left(\|x\|_1^\beta + \|y\|_1^\beta + \|z\|_1^\beta + \|w\|_1^\beta\right) \quad \forall x,y,z,w \in G,
\end{aligned} \tag{3.57}$$

but

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^\beta} : x \in \mathbb{R}\backslash\{0\}\right\} = \infty \tag{3.58}$$

for each additive mapping $T : G \to E$.

(ii) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the β-homogeneous norm $\|\cdot\|_1 = |\cdot|^\beta$, and $E = (\mathbb{R}, \|\cdot\|_2)$ with the Euclidean metric $\|\cdot\|_2 = |\cdot|$, then

$$\begin{aligned}
&\|\phi(x,y,z,w)\|_2 \\
&\quad \le 14\left(\|x\|_1^{1/\beta} + \|y\|_1^{1/\beta} + \|z\|_1^{1/\beta} + \|w\|_1^{1/\beta}\right) \quad \forall x,y,z,w \in G,
\end{aligned} \tag{3.59}$$

but

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^{1/\beta}} : x \in \mathbb{R}\backslash\{0\}\right\} = \infty \tag{3.60}$$

for each additive mapping $T : G \to E$.

(iii) If $G = E = (\mathbb{R}, \|\cdot\|)$ with the β-homogeneous norm $\|\cdot\| = |\cdot|^\beta$, then

$$\|\phi(x,y,z,w)\| \le 14^\beta(\|x\| + \|y\| + \|z\| + \|w\|) \quad \forall x,y,z,w \in G, \tag{3.61}$$

but

$$\sup\left\{\left\|\frac{f(x)-T(x)}{x}\right\| : x \in \mathbb{R}\backslash\{0\}\right\} = \infty \tag{3.62}$$

for each additive mapping $T : G \to E$.

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