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NONASSOCIATIVE ALGEBRAS: A FRAMEWORK FOR DIFFERENTIAL GEOMETRY

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A nonassociative algebra endowed with a Lie bracket, called a *torsion algebra*, is viewed as an algebraic analog of a manifold with an affine connection. Its elements are interpreted as vector fields and its multiplication is interpreted as a connection. This provides a framework for differential geometry on a formal manifold with a formal connection. A torsion algebra is a natural generalization of pre-Lie algebras which appear as the "torsionless" case. The starting point is the observation that the associator of a nonassociative algebra is essentially the curvature of the corresponding Hochschild quasicomplex. It is a cocycle, and the corresponding equation is interpreted as Bianchi identity. The curvature-associator-monoidal structure relationships are discussed. Conditions on torsion algebras allowing to construct an algebra of functions, whose algebra of derivations is the initial Lie algebra, are considered. The main example of a torsion algebra is provided by the pre-Lie algebra of Hochschild cochains of a k-module, with Lie bracket induced by Gerstenhaber composition.

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1. Introduction. The differential calculus on a noncommutative algebra is by now a classical topic (see [3, 5, 7, 11, 34] and the references therein). It is a generalization of the differential calculus on a commutative algebra [26].

Replacing C^*-algebras of functions by noncommutative algebras [6, 16, 27, 28] was essentially the birth of noncommutative geometry [33, page 4]. Although Connes approach "... has had a considerable impact on mathematics ..." [29, Section 1.1, page 150], physicists rather adopted a cohomological point of view (BRST-formalism, BV-theory, and so forth) towards what Stasheff calls "cohomological physics" [31, 32].

The corresponding modern mathematical approach to noncommutative geometry [2, 25] is based on differential graded algebras (and more generally, on A_∞-algebras) as an algebraic model for a *formal manifold* (see [25, page 10]). The main difference (in our opinion), in a rough stated form, is that the motto is not to generalize functions, but rather vector fields, as it will be explained below.

Classical differential geometry is built on the notion of *space*: differential manifolds. A rough hierarchy is space, functions, and vector fields and differential forms, connections, and so forth.

Algebraic geometry starts at the *second level* (functions) by considering an arbitrary commutative algebra and then constructing the *first level*, the substitute for a spaceis its spectrum. A space (affine variety) is roughly a pair consisting of a topological space and its algebra of functions.

A natural question arises: "what can be derived starting from the *third level*—vector fields—and to what extent is it profitable?" [17].

A somewhat similar approach to algebraization of the basic concepts in differential geometry and mechanics, focusing on Hamiltonian formalism of the calculus of variations, has also been investigated in [13]. The idea of replacing the ring of functions used for constructing such a scheme, with a Lie algebra and a complex of Lie modules with a differential, is considered not only suitable for a calculus of variations, but has far reaching applications (see [13, page 241] for details).

Another related direction of research that we should mention involves loops and quasigroups (relaxing associativity) [22, 30]. For connections with web geometry (families of smooth foliations), see [1].

In this paper, we consider a (possibly) nonassociative algebra endowed with an additional Lie algebra structure $(A,\mu,[\cdot,\cdot])$, called a *torsion algebra* (Definition 4.1), and a covariant calculus is defined. A torsion algebra is based on the interpretation of its elements as vector fields with its multiplication interpreted as a connection. The above algebra generalizes pre-Lie algebras which occur as the torsionless case (Proposition 4.2).

The main motivation for the above interpretation, at a formal level, is provided by the properties of the associator of a nonassociative algebra. The associator is the curvature of the Hochschild quasicomplex (Section 2) and has the properties of the curvature of a linear connection (Theorem 3.11; see also [10, 24]).

As a motivation for the emphasis on vector fields, at a semantic level, we mention two sources. Physical understanding evolved from considering phase spaces (Poisson manifolds) rather than configuration spaces. Moreover, the actual goal is to model the *space of evolutions* of a system. The second motivation is the correspondence between Poisson-Lie group structures and Lie bialgebra structures of a Lie algebra. After quantizing its universal enveloping algebra, one has deformed vector fields, and it would be convenient to have a procedure allowing to recover an algebra of functions from it.

In deformation quantization of Poisson manifolds, one keeps the classical observables and deforms the laws of mechanics to account for the Heisenberg bracket. We consider this approach as slightly conservative since the basic conceptual level of quantum physics is rooted in the concepts of states (vectors) and their evolution (operators), which does not need a concrete configuration space. Our reconstructed functions are naturally operators on the given algebra of vector fields.

We investigate conditions on a given torsion algebra, allowing to construct an algebra of functions (Definition 4.5), whose derivations form the Lie algebra $(A,[\cdot,\cdot])$ (Theorem 4.12).

The reconstruction of the function algebra is immediate for the simple example of the real line (Example 4.4).

In the associative case, it is pleasing to be able to *represent* in this way the original associative algebra as an algebra of functions in our sense (Theorem 5.1), and to support the classical approach to noncommutative geometry: to adopt a (possibly) noncommutative algebra as an algebra of functions on a noncommutative space.

The main source of examples of torsion algebras is provided by the pre-Lie algebra of Hochschild cochains associated to a k-module, with Gerstenhaber composition $(C^\bullet(V),\bar{\circ})$ and its associated Lie bracket (Theorem 5.2).

The paper is organized as follows. In Section 3, some well-known facts about the Lie algebra structure on the Hochschild complex are generalized to the case of a nonassociative algebra. Motivated by this generic example, we consider a Lie algebra, with elements thought of as vector fields, with a multiplication (not necessarily associative) thought of as a linear connection. In Section 4, the algebra of functions is defined (second level). Conditions when the original Lie algebra is obtained as a Lie algebra of derivations are considered (Theorem 4.12). Section 5 includes some basic examples of torsion algebras.

2. The associator: an algebraic or a geometric concept?

The question is purely rhetoric since both points of view are needed to unravel this fundamental concept. The associator (from the algebraic point of view) may be interpreted as a curvature (from the geometric point of view) or as a monoidal structure through categorification (failure to be a morphism).

From the geometric side, the main reason for attempting to interpret an algebra (A,μ) as an algebraic model for a manifold with a connection is the following. The associator of an algebra

$$\alpha(x,y,z)=\mu(\mu(x,y),z)-\mu(x,\mu(y,z)),\quad x,y,z\in A, \tag{2.1}$$

also denoted as $(xy)z-x(yz)$ for short, is formally the curvature of the left regular *quasirepresentation* $L:(A,\mu)\to(\mathrm{End}_k(A),\circ)$:

$$\alpha(x,y,z)=-(L(x)L(y)-L(xy))(z). \tag{2.2}$$

Indeed, at the infinitesimal (Lie algebra) level, the same map L, interpreted as a quasi-Lie representation $L:(A,[\cdot,\cdot]_A)\to(\mathrm{End}_k(A),[\cdot,\cdot])$, defines a formal curvature K:

$$\begin{gathered} L([x,y])\xrightarrow{K_{x,y}}[L(x),L(y)],\\ K(x,y)=[L(x),L(y)]-L([x,y]). \end{gathered} \tag{2.3}$$

Moreover, in the Hochschild quasicomplex $(C^\bullet(A), d_\mu)$ (Section 3), the differential d_μ has the properties of covariant derivative, for example, $d_\mu^2 f = [\alpha, f]$. Also $\alpha = (1/2)[\mu, \mu]$ (the curvature) is closed, that is, a Bianchi identity holds. (see Theorem 3.11.)

As a general rule, in the context of nonassociativity, it is natural to relax the action requirement as well.

Now, an action (representation) of A on M is a morphism $\rho : A \to \operatorname{End}(M)$, and an associative multiplication is an action (regular left representation) $A \to \operatorname{End}(A)$. In the nonassociative case, we will need the following definition.

Definition 2.1. A *quasiaction* (*quasirepresentation*) of A on M, in the category of k-modules, is a k-linear map $L : A \to \operatorname{End}_k(M)$.

A quasi-action/representation is the natural relaxation of the usual concept since it amounts to considering the morphisms of the underlying category, not necessarily preserving the additional structure, (e.g., commuting with the monoidal operation).

This approach models the local aspects of differential geometry. The global (cohomological) point of view should look at the associator, that is, the failure of a quasirepresentation to preserve the structure, as a 2-cocycle. In this way, quasirepresentations are a natural generalization of projective representations.

More general still, via categorification, such a 2-cocycle may be interpreted as a nonstrict monoidal structure:

$$L(xy) \xrightarrow{\sigma_{x,y}} L(x)L(y), \quad \sigma(x,y)(z) = -\alpha(x,y,z). \tag{2.4}$$

This interpretation of the associator should be understood from the perspective of the relation between non-abelian cohomology and cohomology of monoidal categories [18, 19, 20]. To further justify this point of view, and at the same time stress the geometric interpretation, recall that the associator may be used to model the monodromy of a connection [12, page 5], allowing to encode the behavior of solutions of the KZ-equations (or of its associated flat connection) in a modular category (see [12]).

3. The Hochschild quasicomplex.

We begin by generalizing Hochschild cohomology [14, 15] to the case of a possibly nonassociative algebra.

Although some of the simple statements and computations carry on without modifications from the associative case (see, e.g., [23]), they were included for the reader's convenience.

Throughout this section, A will denote a module over a commutative ring R. We will assume that 2 and 3 do not annihilate nonzero elements in A.

3.1. Pre-Lie algebra of a module.

Consider the bigraded object $C^{\bullet,\bullet}(A,A) = \oplus_{p,q\geq 0} C^{p,q}$, where

$$C^{p,q}(A,A)=\{f:A^{\otimes p}\longrightarrow A^{\otimes q}\mid fR\text{-linear}\},\quad p,q\geq 0, \tag{3.1}$$

with total degree $\deg(f^{p,q})=q-p$. As usual, $C^{0,q}$ is identified with $A^{\otimes q}$. We will be interested in the first column $C^{\bullet}(A)=\oplus_{p\in\mathbb{N}}C^{p,1}$. The grading induced by the total degree is $C^{p-1}(A)=C^{p,1}(A,A)$ with $p\geq 0$.

We recall briefly the Gerstenhaber *comp* operation and the Lie algebra structure it defines on the graded module of Hochschild cochains, as initially introduced in [14].

For simplicity, we will not use the language of operads (or PROPs).

If $f^p\in C^p(A)$ and $g^q\in C^q(A)$, define the composition into the ith place, where $i=1,\dots,p+1$, by

$$\begin{aligned}&f^p\circ_i g^q(a_1,\dots,a_{p+q-1})\\&\quad=f^p(a_1,\dots,a_{i-1},g^q(a_i,\dots,a_{i+q-1}),a_{i+q},\dots,a_{p+q-1}),\end{aligned} \tag{3.2}$$

and the *comp* operation

$$f^p\circ g^q=\sum_{i=1}^{p+1}(-1)^{(i-1)q}f^p\circ_i g^q\in C^{p+q}. \tag{3.3}$$

It is assumed that the composition is zero whenever $p=-1$. Note that the (nonassociative) composition respects the grading. Denote by $\alpha(f,g,h)=(f\circ g)\circ h-f\circ(g\circ h)$ the *associator* of $\circ$. It is a measure of the nonassociativity of the comp operation.

The graded commutator is defined by

$$[f^p,g^q]=f^p\circ g^q-(-1)^{pq}g^q\circ f^p. \tag{3.4}$$

It is graded commutative:

$$[f^p,g^q]=-(-1)^{pq}[g^q,f^p] \tag{3.5}$$

and the graded Jacobi identity holds:

$$(-1)^{FH}[f,[g,h]]+(-1)^{GF}[g,[h,f]]+(-1)^{HG}[h,[f,g]]=0, \tag{3.6}$$

where F, G, and H denote the degrees of f, g, and h, respectively. It is equivalent to ad being a representation of graded Lie algebras

$$\begin{aligned}[f,[g,h]]&=[[f,g],h]+(-1)^{pq}[g,[f,h]],\\ \mathrm{ad}_f([g,h])&=[\mathrm{ad}_f(g),h]+(-1)^{pq}[g,\mathrm{ad}_f(h)].\end{aligned} \tag{3.7}$$

In order to give a short proof of the main properties of the comp operation, it is convenient to introduce the following notation.

Notation 3.1. We denote by $f\circ_{(i,j)}(g,h)$ the simultaneous insertion of two functions g and h in the ith and jth arguments of f, respectively.

Lemma 3.2. *If f, g, and h have degrees p, q, and r, respectively, then*

(i) $f\circ_i(g\circ_j h)=(f\circ_i g)\circ_{i-1+j}h$ *for* $1\le i\le p+1$ *and* $1\le j\le q+1$;

(ii) $(f\circ g)\circ h-f\circ(g\circ h)=\sum_{i\neq j}\epsilon(i,j)(-1)^{(i-1)q+(j-1)r}f\circ_{(i,j)}(g,h)$, *where* $\epsilon(i,j)=1$ *if* $1\le j<i\le p+1$ *and* $\epsilon(i,j)=(-1)^{qr}$ *if* $1\le i<j\le p+1$;

(iii) $\alpha(f,g,h)=(-1)^{qr}\alpha(f,h,g)$ *(Gerstenhaber identity).*

Proof. First two statements (i) and (ii) follow from a straightforward inspection of trees and signs. The key in (ii) is that the only trees which survive in the associator α, built out of f, g, and h, are of the type $f\circ_{(i,j)}(g,h)$. The supercommutativity sign $(-1)^{qr}$ appears when i passes over j and the order of insertion (g before h) changes. □

To introduce pre-Lie algebras, we prefer an intrinsic definition (compatible with [23, page 8]) to the generators and relations definition from [4]. In view of the Gerstenhaber identity (*pre-Jacobi* [23, page 8]), our main example, the Hochschild pre-Lie algebra, will satisfy both definitions (see Lemma 3.2 and Corollary 3.6).

Notation 3.3. Let $\mu\in C^1(A)$ and let $\mu=\mu_-+\mu_+$ be the natural decomposition, with

$$\mu_-(a,b)=\mu(a,b)-(-1)^{pq}\mu(b,a), \tag{3.8}$$

$$\mu_+(a,b)=\mu(a,b)+(-1)^{pq}\mu(b,a) \tag{3.9}$$

the graded *skew* and *symmetric* parts of μ, respectively. Alternatively, μ_- will be denoted by $[\cdot,\cdot]_\mu$ or just $[\cdot,\cdot]$ if no confusion is expected. The associator of μ will be denoted by α_μ.

Definition 3.4. A (possibly) nonassociative algebra (A,μ) is called a *pre-Lie algebra* if (A,μ_-) is a Lie algebra.

Lemma 3.5. *Let (A,μ) be an algebra and α its associator,*

(i) $\operatorname{Alt}(\alpha_{\mu_+})=0$,

(ii) $\operatorname{Alt}(\alpha_{\mu_-})=4\operatorname{Alt}(\alpha)$,

(iii) (A,μ) *is a pre-Lie algebra if and only if* $\operatorname{Alt}(\alpha)=0$.

If A is graded, then a graded alternation Alt *is assumed in (i), (ii), and (iii).*

Proof. The proof is concluded by a direct computation. □

As previously announced, from Lemmas 3.2 and 3.5, the well-known fact that the comp operation on Hochschild cochains defines a Lie bracket follows immediately.

Corollary 3.6. *$(C^\bullet,\circ)$ is a (graded) pre-Lie algebra.*

Proof. Since $\alpha(f,g,h) = (-1)^{qr}\alpha(f,h,g)$, we have

$$\operatorname{Alt}(\alpha)(f,g,h) = \sum_{\text{cycl}} \epsilon(f,g,h)\big(\alpha(f,g,h) - (-1)^{qr}\alpha(f,h,g)\big) = 0, \tag{3.10}$$

where $\epsilon(f,g,h)$, in the graded case, is not necessarily 1. For example, $\epsilon(g,h,f) = (-1)^{(q+r)p}$. □

3.2. Quasi-DGLA of a nonassociative algebra.

In this section, the differential structure is added and the special case of a coboundary quasidifferential graded Lie algebra (qDGLA) is defined.

Definition 3.7. A *quasicomplex* is a sequence of objects and morphisms in a category $\mathcal{A}$:

$$C^\bullet = \{\cdots \longrightarrow C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots\}. \tag{3.11}$$

The family of morphisms $d^\bullet$ is called a *quasidifferential*.

Now assume that an element $\mu : A \otimes A \to A$ of degree -1 is fixed so that (A,μ) is a (possibly) nonassociative R-algebra.

Definition-Theorem 3.8. *Define the following* quasidifferential *as the adjoint action corresponding to the algebra's* (A,μ) *multiplication map:*

$$d_\mu(f^p) = [\mu, f]. \tag{3.12}$$

Then $(C^\bullet(A), d_\mu, [\cdot,\cdot])$ *is qDGLA, called the* Hochschild quasicomplex *corresponding to the algebra* (A,μ).

Note the difference of sign when compared with [14, 15]:

$$d_{Ge} = -[f,\mu] = (-1)^p[\mu,f] = (-1)^p d_\mu f. \tag{3.13}$$

As an example, for $p = 1$, with $d = d_\mu$ and $\mu(x,y) = xy$,

$$\begin{aligned} df(x,y,z) &= \mu \circ f(x,y,z) + f \circ \mu(x,y,z) \\ &= \mu(f(x,y),z) - \mu(x,f(y,z)) + f(\mu(x,y),z) - f(x,\mu(y,z)) \\ &= -\{xf(y,z) - f(xy,z) + f(x,yz) - f(x,y)z\}, \end{aligned} \tag{3.14}$$

which is the usual Hochschild differential (excepting sign):

$$d_{\text{Hoch}} = (-1)^{\deg} \cdot d_\mu = d_{Ge}. \tag{3.15}$$

Note that $d_\mu : C^p \to C^{p+1}$ has degree one and $[\cdot,\cdot] : C^p \otimes C^q \to C^{p+q}$ is of degree zero. Also note that d_{Hoch} is not a graded derivation since it does not satisfy the Leibniz identity (3.7).

We state the following fact about graded Lie algebras, which is an immediate consequence of the graded Jacobi identity.

Lemma 3.9. *Let $(g,[\cdot,\cdot])$ be a graded Lie algebra over a commutative ring R. Then, if x is an even-degree element, $[x,x]=0$. If x is odd, $[x,[x,x]]=0$ and $\mathrm{ad}_{[x,x]}=2(\mathrm{ad}_x)^2$.*

If the multiplication μ is associative, then (3.14) implies that

$$d\mu(x,y,z)=2\{(xy)z-x(yz)\}=0 \tag{3.16}$$

and $[\mu,\mu]=0$. By the previous lemma, $2[\mu,[\mu,f]]=d_{[\mu,\mu]}(f)=0$, and thus $(C^\bullet(A),d_\mu)$ is a complex of R-modules.

The above particular context is captured in the following definition.

Definition 3.10. A qDGLA $(C^\bullet,d,[\cdot,\cdot])$ is called *coboundary* if there is a degree-zero element I which determines a degree-one element $\mu=dI$, which in turn determines the differential via the adjoint action $d=\mathrm{ad}_\mu$.

3.3. The geometric interpretation. In the case of a coboundary qDGLA, in the absence of a genuine multiplication, the element of degree one μ may be interpreted as the *torsion* and the associator α as the *curvature* of a formal connection on a formal manifold (Section 2).

The motivation for the above geometric terminology comes from a formal analogy with the corresponding notions in the context of a derivation law in an A-module, where A is an R-algebra (see [26]), or from the context of a linear connection ∇ on a vector bundle, where the torsion T and the curvature F of the total covariant derivative d^∇ are defined as usual:

$$T=d^\nabla I,\quad T(X,Y)=\nabla_XY-\nabla_YX-[X,Y], \tag{3.17}$$

$$F(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},\qquad (d^\nabla)^2s=[F,s]. \tag{3.18}$$

Here I denotes the identity tensor.

This geometric interpretation is captured in the following theorem and will be developed in Section 4.

Theorem 3.11. *Let (A,μ) be an R-algebra. Then $(C^\bullet(A),d_\mu)$ is a coboundary quasidifferential algebra. Under the adjoint representation, its unit I corresponds to the grading character*

$$\mathrm{ad}_I(f)=-\deg(f)f, \tag{3.19}$$

and the associator is a 3*-cocycle:*

$$d_\mu\alpha_\mu=0. \tag{3.20}$$

Moreover, the associator is formally a curvature:

$$d_\mu^2 s = [\alpha, s], \tag{3.21}$$

and Bianchi's identity (3.20) holds.

Proof. Note first that the identity map $I : A \to A$ has degree zero. If $f \in C^p(A)$, then

$$\mathrm{ad}_I(f) = I \circ f - (-1)^{p \cdot 0} f \circ I = f - (p+1) f = -\deg(f) f. \tag{3.22}$$

Since $[I, f] = -[f, I]$, the right adjoint action of the unit is a scalar multiplication by the degree map.

Obviously, $d_\mu I = [\mu, I] = \deg(\mu)\mu = \mu$, thus I is a unit. Now, by the Jacobi identity, the associator is a cocycle $d_\mu \alpha = 0$:

$$d_\mu[\mu, \mu] = [\mu, [\mu, \mu]] = 0, \tag{3.23}$$

where the assumption that 2 and 3 do not annihilate nonzero elements in A is used. The second equality follows from Lemma 3.9. A comparison with (3.18) suggests interpreting the associator as a curvature. Then (3.20) states that the curvature is closed (Bianchi's identity). □

3.4. Relation with cohomology of nonassociative algebras. At this point, we would like to note that quasicomplexes have been considered by other authors, notably by Kapranov, who generalized the homology of complexes in [21] (see also [8, 9]). This allows to define the cohomology of certain classes of nonassociative algebras.

Definition 3.12. An algebra (A, μ) is called N*-coherent* if $d_\mu^N = 0$.

Note that an algebra is associative if and only if it is a 2-coherent algebra, and a 1-coherent algebra is just the trivial one $\mu = 0$.

Of course, an algebra (A, μ) is N-coherent if and only if ad_μ is a nilpotent element of order N.

For an N-coherent algebra, the qDGLA $(C^\bullet, d_\mu)$ is an N-complex, as defined in [21], and its homology may be considered as a generalization of Hochschild cohomology (see [21]). The specialization of the present approach to such cases is postponed to a separate paper.

4. Torsion algebras.

The Hochschild differential complex is defined for an associative algebra with coefficients in a symmetric A-bimodule M. When relaxing both conditions, associativity and action requirement, one obtains formulas which are familiar in differential geometry, corresponding to a manifold with an affine connection. This algebraic framework may be thought of as a geometry of vector fields without starting from a function algebra.

Definition 4.1. A *torsion algebra* $\mathcal{M} = (C, D, [\cdot,\cdot]_C)$ is a k-algebra (C,μ) together with a Lie bracket $[\cdot,\cdot]_C$. Its *torsion* is $T = D_- - [\cdot,\cdot]_C$,

$$T(X,Y) = D(X,Y) - D(Y,X) - [X,Y]_C, \quad X,Y \in C, \tag{4.1}$$

where $D = D_+ + D_-$ is the decomposition of D into its *symmetric (quasi-Jordan)* and *skew-symmetric (quasi-Lie)* parts. A morphism of torsion algebras is a k-linear map preserving both algebra operations.

The notion of a torsion algebra is a natural generalization of the notion of a pre-Lie algebra (Definition 3.4).

Proposition 4.2. *The torsionless algebras are precisely the pre-Lie algebras.*

Proof. If $T = 0$, the Lie bracket is given by D_-, and therefore, according to Definition 3.4, (A,D) is a pre-Lie algebra. ☐

This generalization includes the most important classes of algebras.

Associative algebras, with the usual Lie bracket $[x,y] = xy - yx$, are pre-Lie algebras, and therefore torsion algebras, with $T = 0$. Lie algebras $(C,[\cdot,\cdot])$, with $D = (1/2)[\cdot,\cdot]$, are again torsion algebras with zero torsion.

Poisson algebras (with compatibility between D and $[\cdot,\cdot]$) and Gerstenhaber algebras (noncommutative Poisson algebras) can be interpreted as torsion algebras in several ways (see Section 5).

We think of the Lie algebra part $(C,[\cdot,\cdot])$ of a torsion algebra as a Lie algebra of vector fields on a formal manifold, with the multiplication D interpreted as a formal connection. The main issue (addressed later on) is the possibility of constructing an algebra of functions supporting this algebraic model of a manifold endowed with a connection.

Example 4.3. Obviously, any manifold V with a connection ∇ defines a torsion algebra. Take C as the Lie algebra of vector fields on V and interpret the connection as a nonassociative multiplication $D(X,Y) = \nabla_X Y$. In this *geometric example*, the torsion tensor (3.17) agrees with the torsion in the sense of Definition 4.1.

To reconstruct the algebra of functions from the torsion algebra is an easy matter when the topology is trivial.

Example 4.4. If V is the real line, then the Lie algebra of vector fields $X_f = f\partial_t$ can be identified with $(C^\infty(V),[\cdot,\cdot])$, where $[f,g] = fg' - gf'$. Also any connection D has a canonical Christoffel symbol Γ and

$$D_f g = f(g' + g\Gamma) \quad (D = d + \Gamma). \tag{4.2}$$

4.1. A meta-notation. Denote by (C,μ) a possibly nonassociative k-algebra, where k is a ring. We will write $D_X Y = D(X,Y)$ in order to emphasize the geometric interpretation. Basic definitions for the usual algebraic model are assumed following [26]. The prefix $\mathcal{M}$ will be used with notions referring to the formal context (noncommutative space), and the prefix $\mathcal{A}$ will be used to refer to the usual notions in the context of a geometric example, for example, on a manifold V.

In the geometric world, functions can be identified as k-endomorphisms (multiplication of vector fields by functions) for which the connection is linear in the first argument.

4.2. The algebra of functions. We define functions in such a way to ensure that our connection is linear with respect to function multiplication in one argument as the annihilator of the left commutator of the multiplication.

Definition 4.5. Let $(C,D,[\cdot,\cdot]_C)$ be a torsion algebra. Its elements are called $\mathcal{M}$*-vector fields*. The set of $\mathcal{M}$*-functions* is

$$A = \{\phi \in \mathrm{End}_k(C) \mid D_{\phi(X)}Y = \phi(D_X Y)\}. \tag{4.3}$$

The *multiplication* of $\mathcal{M}$-functions is the natural composition of k-endomorphisms in $\mathrm{End}_k(C)$.

Note that the multiplication of $\mathcal{M}$-functions is an internal operation

$$D_{(\phi\circ\psi)x} = D_{\phi(\psi x)} = \phi D_{\psi x} = \phi\psi D_x \tag{4.4}$$

and that the set of $\mathcal{M}$-vector fields C is a left A-module.

The multiplication D defines a k-linear map

$$\tilde{D} : C \longrightarrow \mathrm{End}_k(C), \tag{4.5}$$

called the *left regular quasirepresentation* of (C,D), as a nonassociative algebra.

We will test the notions just introduced against the simplest geometric example: the real line.

Example 4.6. In the context of Example 4.4, multiplication of vector fields $C \cong C^\infty(V)$ by functions is just the regular left representation $L : C^\infty(V) \to \mathrm{End}(C^\infty(V))$ of $C^\infty(V)$ (in the usual sense):

$$(fX_g) = f(g\partial_t) = (fg)\partial_t = X_{fg}. \tag{4.6}$$

Moreover, the $\mathcal{M}$*-functions* A are naturally identified as $\mathcal{A}$*-functions* $C^\infty(V)$. Indeed, if $\phi \in \mathrm{End}_k(C)$ left commutes with D:

$$D_{\phi(f)}g = \phi(D_f g), \tag{4.7}$$

then by (4.2),

$$\phi(f)(g' + g\Gamma) = \phi(f(g' + g\Gamma)). \tag{4.8}$$

But it is clear that $g' + g\Gamma = h$ has a solution for any $h \in C^\infty(V)$. Thus $\phi(fh) = \phi(f)h$, so $\phi(h) = \phi(1)h$ and ϕ corresponds to left multiplication by $\phi(1)$.

We note that ϕ is a function if and only if $\tilde{D} \circ \phi = L_\phi \circ \tilde{D}$, where

$$L : \mathrm{End}_k(C) \longrightarrow \mathrm{End}_k\left(\mathrm{End}_k(C)\right) \tag{4.9}$$

is the regular representation of the associative algebra $(\mathrm{End}_k(C), \circ)$. In other words, $\tilde{D}$ intertwines ϕ and L_ϕ:

$$\tilde{D} \circ \phi = L_\phi \circ \tilde{D}. \tag{4.10}$$

To interpret $\mathcal{M}$-vector fields as derivations on the algebra of functions A, an action must be defined appropriately.

Lemma 4.7. *Let $X \in C$ and $\phi \in A$. Then any two of the following conditions imply the third:*

(i) *the action of C on functions is defined by*

$$(X \cdot \phi)(Y) = [X, \phi(Y)]_C - \phi([X,Y]_C), \quad Y \in C; \tag{4.11}$$

(ii) *D is a derivation law:*

$$D_X(\phi Y) = (X \cdot \phi)Y + \phi D_X Y, \quad X \cdot \phi = [D_X, \phi]; \tag{4.12}$$

(iii) *the torsion is A-bilinear.*

Proof. Note that the torsion T is skew-symmetric and

$$\begin{aligned} T(X, \phi Y) = D_X(\phi Y) - D_{\phi Y} X - [X, \phi Y] \\ = \{D_X(\phi Y) - \phi D_X Y - (X \cdot \phi)Y\} + \phi T(X,Y) \\ + \{(X \cdot \phi)Y + \phi[X,Y] - [X, \phi Y]\}. \end{aligned} \tag{4.13}$$

Now it is clear that any two conditions imply the third:

$$\begin{aligned} T(X, \phi Y) - \phi T(X,Y) = \{D_X(\phi Y) - \phi D_X Y - (X \cdot \phi)Y\} \\ + \{(X \cdot \phi)Y + \phi[X,Y] - [X, \phi Y]\}. \end{aligned} \tag{4.14}$$

□

We will adopt the second condition in Lemma 4.7 as a definition for the action of an $\mathcal{M}$-vector field on an $\mathcal{M}$-function.

Definition 4.8. An $\mathcal{M}$-vector field $X \in C$ acts on an $\mathcal{M}$-function $\phi \in A$ by

$$X \cdot \phi = [D_X, \phi]. \tag{4.15}$$

Note that, defined in this way, the action measures the failure of D to be right A-linear. Any $X \in C$ is a candidate to the status of "vector field," that is, a derivation on the algebra of functions, except that $X \cdot \phi$ need not be a function at this point.

Proposition 4.9. *The $\mathcal{M}$-vector fields act as external derivations on A:*

$$X\cdot(\phi\circ\psi)=(X\cdot\phi)\circ\psi+\psi\circ(X\cdot\psi),\quad X\in C,\ \phi,\psi\in A. \tag{4.16}$$

Proof. If ϕ and ψ are $\mathcal{M}$-functions, then

$$\begin{gathered}(X\cdot(\phi\circ\psi))Y=D_X(\phi(\psi(Y)))-(\phi\circ\psi)D_XY,\\ (X\cdot\phi)\circ\psi(Y)+\phi\circ(X\cdot\psi)(Y)=D_X(\phi(\psi(Y)))-\phi(\psi(D_XY)).\end{gathered} \tag{4.17}$$

□

For $X\cdot\phi$ to be again a function, so that elements of C act as derivations, note the following alternative.

Lemma 4.10. *The following conditions are equivalent:*
- (i) *for any $X\in C$ and $\phi\in A$, $X\cdot\phi$ is an $\mathcal{M}$-function;*
- (ii) *the associator α of D is A-linear in the first two variables.*

Proof. Recall that the associator is

$$\alpha(X,Y,Z)=(XY)Z-X(YZ)=D_{D_XY}Z-D_XD_YZ, \tag{4.18}$$

where multiplicative notation was alternatively used. The following are equivalent:

$$\begin{gathered}D_{(X\cdot\phi)Y}Z=(X\cdot\phi)D_YZ,\\ D_{D_X\phi Y}Z-\phi D_{D_XY}Z=D_X(\phi D_YZ)-\phi D_XD_YZ,\\ (X\circ\phi(Y))\circ Z-X\phi(Y\circ Z)=\phi(\alpha(X,Y,Z)),\\ (X\circ\phi(Y))\circ Z-X\circ(\phi(Y)\circ Z)+X\circ[\phi(Y)\circ Z-\phi(Y\circ Z)]=\phi(\alpha(X,Y,Z)),\\ \alpha(X,\phi(Y),Z)+X\circ[D_{\phi(Y)}Z-\phi(D_YZ)]=\phi(\alpha(X,Y,Z)).\end{gathered} \tag{4.19}$$

Since

$$D_{D_{\phi X}Y}Z=D_{\phi D_XY}Z=\phi(D_{D_XY}Z), \tag{4.20}$$

the linearity of the associator in the first variable is clear:

$$\alpha(\phi X,Y,Z)=D_{D_{\phi X}Y}Z-\phi(D_XD_YZ)=\phi\alpha(X,Y,Z). \tag{4.21}$$

A direct computation proves the A-linearity in the second variable:

$$\begin{aligned}\alpha(X,\phi Y,Z)&=D_{D_X\phi Y}Z-D_XD_{\phi Y}Z\\ &=D_{D_X\phi Y}Z-D_X(\phi D_YZ)\\ &=D_{((X\cdot\phi)Y+\phi D_XY)}Z-((X\cdot\phi)D_YZ+\phi D_XD_YZ)\\ &=\phi(D_{D_XY}Z-D_XD_YZ)\\ &=\phi\alpha(X,Y,Z).\end{aligned} \tag{4.22}$$

□

In order for the reconstruction of the first level (function algebra) to be complete, an additional assumption is needed. With the above lemma in mind, we suggest the following definition.

Definition 4.11. A torsion algebra $\mathcal{M} = (C, D, [\cdot,\cdot]_C)$ is called *regular* if the torsion and the associator are A-bilinear in the first two variables.

Since Lemma 4.7(ii) holds by definition in a torsion algebra, any of the other two imply the third. Now we can easily prove the following theorem.

Theorem 4.12. *Let $\mathcal{M} = (C, D, [\cdot,\cdot]_C)$ be a regular torsion algebra and A its algebra of functions. Then, for all $X, Y \in C$ and $\phi \in A$ an $\mathcal{M}$-function,*

1. *$X \cdot \phi$ is an $\mathcal{M}$-function;*
2. *X acts as a derivation on $\mathcal{M}$-functions;*
3. *the associator of D is A-linear in the first two variables;*
4. *$[X, \phi Y]_C = \phi[X,Y]_C + (X\cdot\phi)Y$;*
5. *D is a connection on $\mathcal{M}$: $D_X(\phi Y) = (X\cdot\phi)Y + \phi D_X Y$;*
6. *the torsion T is A-bilinear.*

Proof. Since Lemma 4.10(ii) holds by definition, (1) follows.

Again from the definition, Lemma 4.7(ii) and (iii) hold, so (2) follows.

The other statements are clear from Lemmas 4.7 and 4.10. □

4.3. Differential forms. The exterior derivative will be defined as the differential of the Chevalley-Eilenberg quasicomplex.

Let M be a left A-module with a derivation law $D^M : C \to \mathrm{End}_k(M)$.

Definition 4.13. The *M-valued $\mathcal{M}$-differential forms* are defined as usual:

$$\Omega^n(\mathcal{M}, M) = \{\omega : C \times \cdots \times C \longrightarrow M \mid \omega \text{ alternating and } A\text{-multilinear}\}. \tag{4.23}$$

Then $\Omega^\bullet(\mathcal{M}, M)$ is just the alternate part of the Hochschild cochains $C^\bullet(C; M)$ with coefficients in M (Chevalley cochains).

To define first the Hochschild quasicomplex, consider the following C-quasi-bimodule structure on M:

$$\begin{array}{ll} \lambda : C \times M \longrightarrow M, & \lambda(X,u) = D^M_X u, \\ \rho : M \times C \longrightarrow M, & \rho(u,X) = -D^M_X u, \end{array} \qquad \begin{tikzcd} C\times M \arrow[d, "-\sigma_{(12)}"'] \arrow[dr, "\lambda=D^M"] & \\ M\times C \arrow[r, "\rho=\lambda^{\mathrm{op}}"'] & C, \end{tikzcd} \tag{4.24}$$

where λ^{op} is the opposite quasiaction using the signed braiding. In the associative case with $M = A$, the use of the signed braiding gives M a structure of (A, A_{op}) *supersymmetric bimodule* structure: $am = -ma$.

Instead of the Hochschild quasicomplex derived from the associated graded Lie algebra $(C^\bullet(C), [\cdot,\cdot])$, with $d\omega = [\mu,\omega] = \mu\circ\omega - (-1)^p\omega\circ\mu$, consider the

Hochschild quasicomplex $C^p(C;M) = \mathrm{Hom}_A(C^p, M)$, $p \geq -1$, of the Lie algebra $(C, [\cdot,\cdot]_C)$ as a nonassociative algebra, with coefficients C-quasi-bimodule M:

$$\begin{aligned} d\omega &= (-1)^p\left((\lambda,\rho)\circ\omega - (-1)^p\omega\circ[\cdot,\cdot]_C\right), \\ d\omega(a_1,\dots,a_{p+2}) &= \lambda(a_1,\omega(a_2,\dots,a_{p+2})) - \omega([a_1,a_2]_C,\dots,a_{p+2}) \\ &\quad + \cdots + (-1)^p\rho(\omega(a_1,\dots,a_{p+1}),a_{p+2}). \end{aligned} \tag{4.25}$$

Then, for $u \in C^{-1} = M$ and $\omega \in C^0$,

$$\begin{aligned} du(X) &= \lambda(X,u) - \rho(u,X) = 2D_X u, \\ d\omega(X,Y) &= D_X\omega(Y) - D_Y\omega(X) - \omega([X,Y]_C), \\ ddu(X,Y) &= D_X du(Y) - D_Y du(X) - du([\cdot,\cdot]_C) \\ &= 2\left(D_X D_Y u - D_Y D_X u - D_{[X,Y]_C}\right)u \\ &= K(X,Y)u. \end{aligned} \tag{4.26}$$

To obtain the usual formulas in geometry, consider the alternating part $\Lambda^\bullet(A;M)$ of the above complex and project the differential $d_{Ch} = \mathrm{Alt}\circ d$. A quasicomplex is obtained, $(\Lambda^\bullet(A;M), d_{Ch})$, called the associated *Chevalley-Eilenberg quasicomplex* of C with coefficients in M.

4.4. The Lie derivative. Let $\mathcal{M} = (C, D, [\cdot,\cdot]_C)$ be a regular torsion algebra. Consider the A-module $M = A$ and the corresponding differential forms $\Omega^\bullet(\mathcal{M})$. The *canonical derivation law* on A is

$$D_X\phi = X\cdot\phi. \tag{4.27}$$

As usual, extend the Lie derivative defined on functions and vector fields as a derivation on the tensor algebra commuting with contractions. It is easy to see that it is an internal operation. For example, if $\omega : C \to A$ is a 1-form, then $(\mathcal{L}_X\omega)(Z) = D_X\omega(Z) - \omega([X,Z])$ is A-linear.

An exterior differential on forms $\Omega^\bullet(A;\mathcal{F})$ is defined by the homotopy formula $\mathcal{L}_X = di_X + i_X d$. The usual explicit formula holds for d. It coincides with d_{Ch} defined above.

5. Examples.

We will consider for the moment only torsion algebras for which $T = 0$, that is, the pre-Lie algebras (Proposition 4.2).

5.1. Associative algebras. Let (C, D) be a unital associative algebra. Consider the corresponding Lie algebra structure $[X,Y]_C = D_X Y - D_Y X$. Then the torsion is $T = 0$. The associator is zero and $(C, D, [\cdot,\cdot])$ is a regular torsion algebra. If $\phi \in \mathrm{End}_k(C)$ is a function, then $D_{\phi(X)}Y = \phi(D_X Y)$ in multiplicative notation is just $\phi(X)Y = \phi(XY)$. Thus, $\mathcal{M}$-functions are left multiplication by elements of C and the algebra of $\mathcal{M}$-functions is isomorphic to the initial algebra. The morphism $C \to \mathrm{Der}(A)$, realizing C as derivations of A, is the usual Lie algebra representation.

Theorem 5.1. *Any associative algebra* (C,μ) *has a natural structure of a torsion algebra, which is regular. The algebra* C *is isomorphic with the algebra of* $\mathcal{M}$*-functions.*

It follows that our point of view allows to represent an associative algebra as an algebra of functions, substantiating the classical point of view of noncommutative geometry: to generalize the commutative case of the classical algebraic geometry by assuming that a noncommutative algebra is an algebra of functions on a noncommutative space.

5.2. Hochschild pre-Lie algebras. Let V be a k-module and $C = (C^\bullet(V), \bar{\circ})$ the corresponding Hochschild pre-Lie algebra (see Section 3.1). Then C is a torsion algebra with $D = \bar{\circ}$ and $T = D_- - c = 0$.

A k-endomorphism $\phi \in \mathrm{End}_k(C)$ is an $\mathcal{M}$-function if and only if

$$D_{\phi x} y = \phi(D_x y) \tag{5.1}$$

and an argument similar to the case of associative algebras gives $\phi = L_{\phi(1)}$, where $1 = \mathrm{id}_V$ and $L : C \to (\mathrm{End}_k(C), \tilde{\circ})$ is the regular quasirepresentation. Note that $\bar{\circ}$ is not associative and L_{id_V} is only a projector on the even part of C.

Denote $\phi(1)$ by f. Then (5.1) holds if and only if

$$D_{f \bar{\circ} x} y = f \bar{\circ} (D_x y), \tag{5.2}$$

that is, $(f \bar{\circ} x) \bar{\circ} y = f \bar{\circ} (x \bar{\circ} y)$ for any $x, y \in C^\bullet$. It is easy to see that this is true if and only if $f \in C^0(V)$, and thus the set of functions is $A = C^0(V)$.

The composition of functions is a composition of k-endomorphisms:

$$L_f \tilde{\circ} L_g = L_{f \bar{\circ} g}, \quad f, g \in A, \tag{5.3}$$

since $\bar{\circ}$ reduces to the usual composition $\circ$ of k-endomorphisms of $C^0(V) = \mathrm{End}_k(V)$. Thus we have the following theorem.

Theorem 5.2. *Let* V *be a* k*-module and* $(C^\bullet(V), \bar{\circ})$ *the corresponding pre-Lie algebra. Then*

(i) $C = (C^\bullet(V), D, [\cdot,\cdot])$ *is a zero torsion algebra, where* $D = \bar{\circ}$ *is called the canonical connection;*

(ii) *the algebra of functions of* C *is* $A = (C^0(V), \bar{\circ})$*, that is,* $(\mathrm{End}_k(V), \circ)$*;*

(iii) C *acts through exterior derivations on* A*:*

$$(x \cdot L_f)(y) = D_x(L_f(y)) - L_f(D_x y), \quad x, y, f \in C, \tag{5.4}$$

where $L : C \to \mathrm{End}_k(C)$ *is the regular left quasirepresentation of* C*.*

We note that the failure to be a regular torsion algebra comes from the A-nonlinearity of the associator. Recall that the associator is graded skew-symmetric in the last two variables (Lemma 3.2). Thus, being a regular torsion algebra is equivalent to α being an A-multilinear form.

5.3. Poisson algebras. Let $(C, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra, with $D = \cdot$ commutative and associative, and Lie bracket $[\cdot, \cdot]$ being the Poisson bracket $\{\cdot, \cdot\}$. Then $(C, D, [\cdot, \cdot])$ is a torsion algebra with torsion $T = -[\cdot, \cdot]$. Since D is associative, its algebra of $\mathcal{M}$-functions A is isomorphic to C, in a manner similar to the associative algebra case. In this way, a Poisson algebra is not a regular torsion algebra.

If $D = [\cdot, \cdot] = \{\cdot, \cdot\}$, then it becomes a zero torsion algebra, but it is not clear what the algebra $(A, \circ)$ of $\mathcal{M}$-functions is, and what the relation with the multiplication of functions is.

6. Conclusions and further developments.

The potential applications of an algebraic point of view of what a manifold is include the Hamiltonian formalism of the calculus of variations, classical Yang-Baxter equation, and cohomology and deformations of Lie algebras [13]. To be of interest for gauge theory, a connection should be included in this framework.

In the present paper, we sketched such a framework and pondered on the relation with classical noncommutative geometry, an approach based on functions (observables), addressing the representation and reconstruction problem.

The implementation of the suggested approach and the investigation of its relation to other approaches (formal pointed manifolds [25], Fuchsian differential equation and CFT [12], and so forth) are deferred to another place (and time), possibly leading to applications suitable for the Wilsonian approach to QFT.

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