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ON HYPERSURFACES IN A LOCALLY AFFINE RIEMANNIAN BANACH MANIFOLD II

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In our previous work (2002), we proved that an essential second-order hypersurface in an infinite-dimensional locally affine Riemannian Banach manifold is a Riemannian manifold of constant nonzero curvature. In this note, we prove the converse; in other words, we prove that a hypersurface of constant nonzero Riemannian curvature in a locally affine (flat) semi-Riemannian Banach space is an essential hypersurface of second order.

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1. Introduction. Let M be an infinite-dimensional Banach manifold of class C^k, $k \geq 1$, modelled on a Banach space E, and let $\overset{1}{\bar{g}}$ be a symmetric bilinear form defined on M, that is, $\overset{1}{\bar{g}} \in L_2(M;\mathbb{R})$. The metric $\overset{1}{\bar{g}}$ is said to be strongly nonsingular if $\overset{1}{\bar{g}}$ associates a mapping $\overset{1}{\bar{g}}{}^{*} : x \in M \to \overset{1}{\bar{g}}{}^{*}_{x} = \overset{1}{\bar{g}}{}^{*}(x,\cdot) \in L(M;\mathbb{R})$ which is bijective [2]. Let $\overset{1}{\bar{\Gamma}}$ be the linear connection on M. A C^k Banach manifold $(M,\overset{1}{\bar{\Gamma}})$, $k \geq 3$, is called locally affine if its curvature and torsion tensors are zero. In general, it is proved in [2] that a Banach manifold $(M,\overset{1}{\bar{\Gamma}})$ is locally affine if and only if there exists an atlas $\mathcal{A}$ on M such that for any chart $c \in \mathcal{A}$, $\overset{1}{\Gamma} \equiv 0$, where $\overset{1}{\Gamma}$ is the model of the linear connection $\overset{1}{\bar{\Gamma}}$. The hypersurface $N \subset M$ which is defined by the equation $\overset{1}{\bar{g}}_x(\bar{x},\bar{x}) = er^2$, $e = \pm 1$, $0 \neq r \in \mathbb{R}$, is called an essential hypersurface of the second order in the space M (see [2]).

2. Hypersurface of nonzero constant Riemannian curvature in a locally affine Banach manifold. Let M be a locally affine Banach manifold and assume that $\overset{1}{\bar{g}}$ is a strongly nonsingular metric on M, then the pair $(M,\overset{1}{\bar{g}})$ is a Riemannian Banach manifold. Denote by $\bar{i} : \bar{x} \in N \to \bar{i}(\bar{x}) = \bar{x} \in M$ the inclusion mapping. Let $c = (U,\Phi,E)$ be a chart at $\bar{x} \in M$ and let $d = (V,\Psi,F \subseteq E)$ be a chart at $\bar{x} \in N$, where the Banach spaces E and F are the models of the manifolds M and N with respect to the charts c, and d, respectively. Furthermore, we have that $\Psi(\bar{x}) = x$ is the model of the point $\bar{x}$ with respect to the chart $d, z = \Phi(\bar{x})$ is the model of $\bar{x}$ with respect to the chart c, and i is the model of $\bar{i}$ with respect to the charts c and d. Then we have an inclusion

$$i : x = \Psi(\bar{x}) \in \Psi(V) \subset F \longrightarrow i(x) = z = \Phi(\bar{x}) \in \Phi(V) \subset E \tag{2.1}$$

of a hypersurface of a semi-Riemannian Banach space E.

In this case, (2.1) is called the local equation of the submanifold $N \subset M$ with respect to the charts c and d. Also N will be a Riemannian submanifold of M with induced metric $\overset{2}{\bar{g}}$, which is defined by the rule

$$\overset{2}{\bar{g}}_x(\bar{X}_1,\bar{X}_2) = \overset{1}{\bar{g}}_{i(x)}(T_x i(\bar{X}_1), T_x i(\bar{X}_2)), \tag{2.2}$$

for all $\bar{x} \in N$ and $\bar{X}_1, \bar{X}_2 \in T_{\bar{x}}N$, where $T_{\bar{x}}\bar{i} : T_{\bar{x}}N \to T_{\bar{x}}M$ is the tangent mapping of $\bar{i}$ at the point $\bar{x} \in N$ (see [1]).

Assume that $\overset{2}{\bar{g}}$ is a strongly nonsingular metric on N. Also we have that M and N are Riemannian manifolds with free-torsion connections $\overset{1}{\bar{\Gamma}}$ and $\overset{2}{\bar{\Gamma}}$, respectively, such that $\overset{1}{\bar{\nabla}}\overset{1}{\bar{g}} = 0$ and $\overset{2}{\bar{\nabla}}\overset{2}{\bar{g}} = 0$ (see [3, 4]). Let $X_1, X_2 \in F$ be the models of $\bar{X}_1$, $\bar{X}_2 \in T_{\bar{x}}N$ with respect to the chart d on N. Then $Y_1 = Di_x(X_1)$ and $Y_2 = Di_x(X_2)$ are the models of $\bar{X}_1$ and $\bar{X}_2$ with respect to the chart c on M.

In this case, the local equation of (2.2) takes the form

$$\overset{2}{g}_x(X_1,X_2) = \overset{1}{g}_x(Di_x(X_1), Di_x(X_2)). \tag{2.3}$$

Theorem 2.1. *A local hypersurface of constant nonzero Riemannian curvature in a locally affine (flat) semi-Riemannian Banach space is an essential hypersurface of second order.*

Proof. Let N be a local hypersurface of constant curvature K_0 of the Banach type in the Riemannian manifold $(M, \overset{1}{\bar{g}})$ such that $\dim N > 2$. We know that the first differential equation of the hypersurface $N \subset M$ has the form (see [5])

$$\overset{2}{\nabla} Di_x(X,Y) = eA_x(X,Y)\xi_x, \tag{2.4}$$

where $\bar{\xi}_x \in T^{1+0}_{0+0}(M) = T^1_0(M)$ is a unit vector in M orthogonal to N at the point $\bar{x} \in M$, that is,

$$\overset{1}{\bar{g}}(\bar{\xi}_x,\bar{\xi}_x) = e, \qquad \overset{1}{\bar{g}}(\bar{\xi}_x,\bar{X}) = 0, \tag{2.5}$$

for all $\bar{x} \in N \subset M$ and all $\bar{X} \in T_x N$, and A_x is the second fundamental form for the hypersurface N which is defined by the equality (see [5])

$$A_x(X,Y) = \overset{1}{g}_x(D^2 i_x(X,Y), \xi_x) = -\overset{1}{g}_x(Di_x(X), D\xi_x(Y)). \tag{2.6}$$

Taking into account that $T_x\bar{i} \in T^{1+0}_{0+1}(N)$ is a mixed tensor of type $(1+0, 0+1)$ on the submanifold N (see [7]), $\bar{\xi}_x \in T^1_0(M)$, and (2.6), we conclude that A_x is a symmetric tensor of type $(0,2)$ on N at the point $\bar{x} \in N$.

Now let $\xi : x = \Psi(\bar{x}) \in \Psi(V) \subset F \to \xi_x \in E$ be the model of the vector field

$$\bar{\xi} : \bar{x} \in N \longrightarrow \bar{\xi}_{\bar{x}} \in T_{\bar{x}}M, \tag{2.7}$$

with respect to the charts c and d at the point $\bar{x}$. Then the local equations of equalities (2.5) take the form

$$\overset{1}{g}(\xi_x,\xi_x)=e,\qquad \overset{1}{g}(Di_x(X),\xi_x)=0, \tag{2.8}$$

for all $x\in\Psi(V)\subset F$ and all $X\in F$. Furthermore, the integral condition for (2.4) takes the form

$$\overset{1}{g}\Big(Di_x\Big(\overset{2}{R}_x(Y;Z,X),Di_x(S)\Big)\Big)=\overset{2}{g}_x\Big(\overset{2}{R}_x(Y;Z,X),S\Big)=eA_x(\underline{Z},Y)A_x(\underline{X},S). \tag{2.9}$$

Remark 2.2. In formula (2.9), there exists an alternation with respect to the underlined vectors without division by 2. This convention will be used henceforth.

Similarly, the second differential equation of the hypersurface $N\subset M$ will be (see [5])

$$D\xi_x(X)=Di_x(H_x(X)), \tag{2.10}$$

where $H_x\in L(F;F)$. Also by using (2.6), we find that

$$A_x(X,Y)=-\overset{1}{g}_x(Di_x(X),D\xi_x(Y))=-\overset{1}{g}_x(Di_x(X),Di_x(H_x(Y)))=-\overset{2}{g}_x(X,H_x(Y)), \tag{2.11}$$

that is,

$$\overset{2}{g}_x(X,H_x(Y))=-A_x(X,Y), \tag{2.12}$$

for all $x=\Psi(\bar{x})\in\Psi(V)\subset F$ and all $X,Y\in F$. Furthermore, the integral condition for (2.10) has the form (see [5])

$$\overset{2}{\nabla}A_x(\underline{X};\underline{Z},Y)=0, \tag{2.13}$$

for all $x=\Psi(\bar{x})\in\Psi(V)\subset F$ and all $X,Y,Z\in F$.

Now we find that

$$\overset{2}{g}_x\Big(\overset{2}{R}_x(Y;Z,X),S\Big)=\overset{1}{g}_x\Big(Di_x\Big(\overset{2}{R}_x(Y;Z,X)\Big),Di_x(S)\Big)=eA_x(\underline{Z},Y)A_x(\underline{X},S). \tag{2.14}$$

Since N is a hypersurface of constant curvature, then (2.14) takes the form (see [2])

$$\overset{2}{g}_x\Big(K_0\overset{2}{g}_x(Z,Y)X,S\Big)=eA_x(\underline{Z},Y)A_x(\underline{X},S), \tag{2.15}$$

where $K_0 \in \mathbb{R}$ is a constant independent of the choice of the point, and is called the curvature of the hypersurface N. Then, we obtain

$$\begin{aligned} &A_x(Z,Y)A_x(X,S) - A_x(X,Y)A_x(Z,S) \\ &\qquad = K\Big(\overset{2}{g}_x(Z,Y)\overset{2}{g}_x(X,S) - \overset{2}{g}_x(X,Y)\overset{2}{g}_x(Z,S)\Big), \end{aligned} \tag{2.16}$$

for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X,Y,Z,S \in F$, where $K = K_0/e$.

Now we prove that A_x is a weakly nonsingular form. Let X be a fixed vector and $A_x(X,Y) = 0$, for all $Y \in F$. Then, from (2.16) we obtain

$$\overset{2}{g}_x(Z,Y)\overset{2}{g}_x(X,S) - \overset{2}{g}_x(X,Y)\overset{2}{g}_x(Z,S) = 0, \tag{2.17}$$

for all $Y \in F$, that is, $\overset{2}{g}_x(Y,\overset{2}{g}_x(X,S)\cdot Z - \overset{2}{g}_x(Z,S)\cdot X) = 0$. By using that $\overset{2}{g}_x$ is nonsingular, we obtain $\overset{2}{g}_x(X,S)\cdot Z - \overset{2}{g}_x(Z,S)\cdot X = 0$, for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X,Z,S \in F$. Since $\dim E > 2$, then, for any S, we can choose Z which is not a multiple of X and thus $\overset{2}{g}_x(X,S) = 0$, for all $S \in F$. But $\overset{2}{g}_x$ is nonsingular, hence, $X = 0$ and this proves that A_x is a weakly nonsingular form.

Now from (2.12) and (2.16), we obtain

$$\overset{2}{g}_x(\underline{Z},H_x(Y))\overset{2}{g}_x(\underline{X},H_x(S)) = K\Big(\overset{2}{g}_x(\underline{Z},Y)\overset{2}{g}_x(\underline{X},S)\Big), \tag{2.18}$$

and then we have

$$\begin{aligned} &\overset{2}{g}_x\Big(Z,\overset{2}{g}_x(X,H_x(S))\cdot H_x(Y) - \overset{2}{g}_x(X,H_x(Y))\cdot H_x(S) \\ &\qquad - K\Big(\overset{2}{g}_x(X,S)\cdot Y - \overset{2}{g}_x(X,Y)\cdot S\Big)\Big) = 0, \quad \forall Z \in F. \end{aligned} \tag{2.19}$$

Taking into account that the metric tensor $\overset{2}{g}_x$ is nonsingular, we obtain

$$\begin{aligned} &\overset{2}{g}_x(X,H_x(S))\cdot H_x(Y) - \overset{2}{g}_x(X,H_x(Y))\cdot H_x(S) \\ &\qquad - K\overset{2}{g}_x(X,S)\cdot Y + K\overset{2}{g}_x(X,Y)\cdot S = 0. \end{aligned} \tag{2.20}$$

Furthermore, we find

$$\overset{2}{g}_x(X,H_x(Y)) = A_x(X,Y) = A_x(Y,X) = \overset{2}{g}_x(Y,H_x(X)) = \overset{2}{g}_x(H_x(X),Y), \tag{2.21}$$

that is,

$$\overset{2}{g}_x(X,H_x(Y)) = \overset{2}{g}_x(H_x(X),Y), \tag{2.22}$$

and then from (2.20) and (2.22), we obtain

$$\begin{aligned}\overset{2}{g}_x(H_x(X),S)\cdot H_x(Y)-\overset{2}{g}_x(H_x(X),Y)\cdot H_x(S)\\ -K\overset{2}{g}_x(X,S)\cdot Y+K\overset{2}{g}_x(X,Y)\cdot S=0,\end{aligned} \tag{2.23}$$

for all $x=\Psi(\bar{x})\in\Psi(V)\subset F$ and all $X,Y,S\in F$.

Since $\dim F>2$, then, for every $X,Y\in F$ such that $\overset{2}{g}_x(X,Y)=0$, there exists a vector $S\in F$ orthogonal to each X and $H_x(X)$ [2]. Using this fact in (2.23) and taking into account (2.12), we obtain $A_x(X,Y)\cdot H_x(S)=0$. By using the nonsingularity of the tensor A_x, we conclude that $A_x(X,Y)=0$. Since, for any pair of vectors $X,Y\in F$, $\overset{2}{g}_x(X,Y)=0$ implies that $A_x(X,Y)=0$, then there exists a real number λ such that (see [2])

$$A_x(X,Y)=\lambda\overset{2}{g}_x(X,Y). \tag{2.24}$$

Substituting (2.24) into (2.16), we obtain

$$\lambda^2\overset{2}{g}_x(\underline{Z},Y)\overset{2}{g}_x(\underline{X},S)=K\overset{2}{g}_x(\underline{Z},Y)\overset{2}{g}_x(\underline{X},S), \tag{2.25}$$

for all $x=\Psi(\bar{x})\in\Psi(V)\subset F$ and all $X,Y,Z,S\in F$. Taking into account the nonsingularity of $\overset{2}{g}_x$, we obtain $\lambda^2=K=K_0/e$. It is convenient to put $K_0=e/r^2$, where r is a nonzero real number and $e=\pm1$, then we have $\lambda=\pm1/r$. We find that in our case, it is convenient to take $\lambda=-1/r$. Substituting λ in (2.24), we obtain

$$A_x(X,Y)=-\frac{1}{r}\overset{2}{g}_x(X,Y), \tag{2.26}$$

and in fact this equation is the unique solution, up to sign, of (2.9) and (2.13). Substituting this solution in (2.12), we have

$$\overset{2}{g}_x(X,H_x(Y))=\frac{1}{r}\overset{2}{g}_x(X,Y),\quad \forall x\in\Psi(V)\subset F,\ \forall X,Y\in F, \tag{2.27}$$

which implies that $H_x(Y)=(1/r)Y$. Hence (2.10) will be

$$D\xi_x(X)=\frac{1}{r}Di_x(X). \tag{2.28}$$

Integrating this equation gives us $\xi_x=(1/r)i(x)$. Then

$$\overset{1}{g}(i(x),i(x))=r^2\overset{1}{g}(\xi_x,\xi_x). \tag{2.29}$$

Letting $y=i(x)$ and using equalities (2.8), the above equation takes the form

$$\overset{1}{g}(y,y)=er^2,\quad \forall x\in\Psi(V)\subset F,\ e=\pm1. \tag{2.30}$$

This last equation shows that the hypersurface $N\subset M$ of constant nonzero Riemannian curvature will be locally an essential hypersurface of second order, and this completes the proof. □

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