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ON NON-MIDPOINT LOCALLY UNIFORMLY ROTUND NORMABILITY IN BANACH SPACES

A. K. MIRMOSTAFAEE

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We will show that if X is a tree-complete subspace of ℓ_∞, which contains c_0, then it does not admit any weakly midpoint locally uniformly convex renorming. It follows that such a space has no equivalent Kadec renorming.

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1. Introduction. It is known that ℓ_∞ has an equivalent strictly convex renorming [2]; however, by a result due to Lindenstrauss, it cannot be equivalently renormed in locally uniformly convex manner [10]. In this note, we will show that every tree-complete subspace of ℓ_∞, which contains c_0, does not admit any equivalent weakly midpoint locally uniformly convex norm. This can be considered as an extension of [1, 8]. Since every strictly convexifiable Banach space with Kadec property admits an equivalent midpoint locally uniformly convex renorming [9], it follows that every subspace of ℓ_∞ with the tree-completeness property has no equivalent Kadec renorming. The existence of such a (nontrivial) subspace, which does not contain any copy of ℓ_∞, has already been proved by Haydon and Zizler (see [5, 7]).

2. Results. We recall that a norm $\|\cdot\|$ on a Banach space X is said to be *midpoint locally uniformly rotund* (MLUR) if, whenever $\{x_n\}$, $\{y_n\}$, and x are in X with $\|x_n\| \to \|x\|$, $\|y_n\| \to \|x\|$, and $\|(x_n+y_n)/2-x\| \to 0$, we necessarily have $\|x_n-y_n\| \to 0$. If at the end of the last sentence, we replace norm with weak, the definition of *weakly midpoint locally uniformly rotund* (wMLUR) will be obtained [3]. Let T be the set of all finite (possible empty) strings of 0's and 1's. The empty string ( ) is the unique string of length 0; the *length* $|t|$ of a string t is n if $t \in \{0,1\}^n$. The *tree order* is defined by $s \prec t$ if $|s| < |t|$ and $t(m) = s(m)$ for $m \le |s|$. Each $t \in T$ has exactly two immediate successors, that is, $t0$ and $t1$.

A lattice L is said to be *tree-complete* if, whenever $\{f_t\}_{t\in T}$ is a bounded disjoint family in L, there exists $b \in \{0,1\}^N$, such that $\sum_{n\in N} f_{b|n}$ is in L.

Haydon and Zizler [7] constructed a closed linear subspace of ℓ_∞ (which is a tree-complete sublattice of ℓ_∞) such that it contains c_0 but does not contain any subspace isomorphic to ℓ_∞. Notice that in this space X every infinite subset M of N has an infinite subset $M_0 \subset M$ such that $\mathbf{1}_{M_0} \in X$ [7].

Theorem 2.1. *Let X be a tree-complete sublattice of ℓ_∞. If X contains c_0, then X does not admit any equivalent wMLUR renorming.*

Proof. Let $|||\cdot|||$ be an equivalent norm on X. We will show that this norm is not wMLUR. Let

$$\begin{gathered} A_{(\,)} = \{f \in X : \|f\|_\infty = 1,\ N \setminus \operatorname{supp}(f) \text{ is infinite}\}, \\ M_{(\,)} = \sup\{|||f||| : f \in A_{(\,)}\}, \qquad m_{(\,)} = \inf\{|||f||| : f \in A_{(\,)}\}. \end{gathered} \tag{2.1}$$

Choose an element $f_{(\,)}$ of X such that $|||f_{(\,)}||| > (3M_{(\,)} + m_{(\,)})/4$. Then select two disjoint infinite subsets N_0' and N_1' of $N \setminus \operatorname{supp}(f_{(\,)})$ with $\mathbf{1}_{N_i'} \in X$ for some $k_i \in N_i'$, define $N_i = N_i' \setminus \{k_i\}$, and let

$$A_i = \{f \in A_{(\,)} : f(n) = f_{(\,)}(n) \text{ for each } n \notin N_i\} \quad (i = 0,1). \tag{2.2}$$

Suppose that for some $t \in T$, with $|t| < n$, A_t is specified. Put

$$M_t = \sup\{|||f||| : f \in A_t\}, \qquad m_t = \inf\{|||f||| : f \in A_t\}. \tag{2.3}$$

Let $f_t \in A_t$ satisfy $|||f_t||| > (3M_t + m_t)/4$ and take two disjoint infinite subsets N_{t0}' and N_{t1}' of $N_t \setminus \operatorname{supp}(f_t)$ with $\mathbf{1}_{N_{ti}'} \in X$, put $N_{ti} = N_{ti}' \setminus \{k_{ti}\}$, and define

$$A_{ti} = \{f \in A_t : f(n) = f_t(n) n \notin N_{ti}\} \quad (i = 0,1). \tag{2.4}$$

Thus, by induction on $|t|$, we can obtain a family $\{A_t\}_{t\in T}$ of subsets of X, a family $\{f_t\}$ of elements of X, a family $\{N_t\}$ of infinite subsets of N, and a family of integers $\{k_t\}$ with the following properties.

(a) A_{ti} is of the form

$$A_{ti} = \{f \in A_t : f(n) = f_t(n),\ n \notin N_{ti}\} \quad (i = 0,1), \tag{2.5}$$

for each $t \in T$.

(b) $k_{ti} \in N_t \setminus N_{ti}$ and $f_t(k_t) = 0$ for $t \in T$ and $i = 0,1$.

(c) $|||f_t||| > (3M_t + m_t)/4$, where M_t and m_t denote the supremum and infimum of $\{|||f||| : f \in A_t\}$, respectively.

(d) $N_s \subset N_t$ whenever $t \prec s$ and $N_t \cap N_s = \varnothing$, if s and t are not comparable.

(e) $\operatorname{supp}(f_t - f_s) \subset N_t \setminus N_s$ for $t \prec s$.

By (e), $\{g_t\}_{t\in T}$, defined by

$$g_{(\,)} = f_{(\,)}, \qquad g_{ti} = f_{ti} - f_t \quad (i = 0,1), \tag{2.6}$$

is a disjoint family of elements of X. By the tree-completeness of X, there exists some $b \in \{0,1\}^N$ such that

$$f_b(x) = f_{(\,)} + \sum_{n\in N} g_{b|n} \tag{2.7}$$

is in X. Let $\{k_{\alpha(n)}\}$ be a subsequence of $\{k_{b|n}\}$ such that $\mathbf{1}_E \in X$, where $E = \{k_{\alpha(1)}, k_{\alpha(2)}, \dots\}$. Let $E_n = \{k_{\alpha(n)}, k_{\alpha(n+1)}, \dots\}$ and $h_n = \mathbf{1}_{E_n}$. By (a) and (b), $g_{n+1}^+ = f_b + h_{n+1}$ and $g_{n+1}^- = f_b - h_{n+1}$ are in $A_{b|n}$. Next, select some $\mu \in X^*$, such that $\mu(h_1) = 1$ and $\mu(g) = 0$ for each $g \in c_0$. Clearly, for such an element μ and each $n \in N$, we have $\mu(h_n) = 1$. By (a), $2f_b - f \in A_{b|n}$, thus $|||2f_{b|n} - f||| \le M_{b|n}$ for each $f \in A_{b|n}$ and $n \in N$. It follows that

$$\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \le |||2f_{b|n}||| \le M_{b|n} + |||f|||, \quad \forall f \in A_{b|n}, \tag{2.8}$$

and so

$$\frac{(3M_{b|n-1} + m_{b|n-1})}{2} \le M_{b|n} + m_{b|n} \le M_{b|n-1} + m_{b|n-1}, \quad \forall n \in N. \tag{2.9}$$

Therefore,

$$\begin{aligned} M_{b|n} - m_{b|n} &\le M_{b|n} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} \\ &\le M_{b|n-1} - \frac{(M_{b|n-1} + m_{b|n-1})}{2} \\ &= \frac{(M_{b|n-1} - m_{b|n-1})}{2}. \end{aligned} \tag{2.10}$$

The above relations show that

$$\big|\, |||g^{\pm}_{n+1}||| - |||f_b|||\, \big| \le M_{b|n} - m_{b|n} \le \frac{(M_{b|n-1} - m_{b|n-1})}{2} \le \frac{(M_{(\ )} - m_{(\ )})}{2^n}. \tag{2.11}$$

That is $\lim |||g^+_n||| = |||f_b||| = \lim |||g^-_n|||$. Moreover, $f_b = (g^+_n + g^-_n)/2$. But weak-$\lim(g^+_n - g^-_n) \ne 0$, since $\mu(h_n) = 1$ for each $n \in N$. This shows that X does not admit any wMLUR norm. □

It is known that weakly midpoint locally uniformly rotundity of a Banach space X is equivalent to saying that every point of $S(\hat{X})$ is an extreme point of $B(X^{**})$ [11]. It follows that the space considered in Theorem 2.1 has no equivalent norm such that $S(\hat{X})$ is a subset of $B(X^{**})$.

A norm on a Banach space X is said to be *strictly convex (rotund)* (R) if the unit sphere of X contains no nontrivial line segment. We say that a norm is *Kadec* if the weak and norm topologies coincide on the unit sphere. Every MLUR Banach space admits Kadec renorming (see [1]). Haydon in [6, Corollary 6.6] gives an example of a Kadec renormable space which has no equivalent R norm. The following result gives an example of a strictly convexifiable space with no equivalent Kadec norm.

Corollary 2.2. *If a tree-complete subspace X of ℓ_∞ contains c_0, then it does not admit any equivalent Kadec renorming.*

Proof. It is known that ℓ_∞ admits an equivalent strictly convex norm (see [4, page 120] or [2]). In [9] it is shown that every R Banach space with the Kadec property admits an equivalent MLUR renorming (see also [3, chapter IV]). Thus the result follows from Theorem 2.1. □

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A. K. Mirmostafaee: Department of Mathematics, Damghan University of Sciences, P.O. Box 36715-364, Damghan, Iran