IJMMS 2004:27, 1437–1445
PII. S0161171204203088
http://ijmms.hindawi.com

SPECTRAL PROPERTIES OF THE KLEIN-GORDON s-WAVE EQUATION WITH SPECTRAL PARAMETER-DEPENDENT BOUNDARY CONDITION

GÜLEN BAŞCANBAZ-TUNCA

Received 12 March 2002

We investigate the spectrum of the differential operator L_λ defined by the Klein-Gordon s-wave equation $y'' + (\lambda - q(x))^2 y = 0$, $x \in \mathbb{R}_+ = [0,\infty)$, subject to the spectral parameter-dependent boundary condition $y'(0) - (a\lambda + b)y(0) = 0$ in the space $L^2(\mathbb{R}_+)$, where $a \neq \pm i$, b are complex constants, q is a complex-valued function. Discussing the spectrum, we prove that L_λ has a finite number of eigenvalues and spectral singularities with finite multiplicities if the conditions $\lim_{x\to\infty} q(x) = 0$, $\sup_{x\in R_+}\{\exp(\varepsilon\sqrt{x})|q'(x)|\} < \infty$, $\varepsilon > 0$, hold. Finally we show the properties of the principal functions corresponding to the spectral singularities.

2000 Mathematics Subject Classification: 47A10, 47E05, 34B05.

1. Introduction. Let L denote the operator generated in $L^2(\mathbb{R}_+)$ by differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+, \tag{1.1}$$

and the boundary condition $y'(0) - hy(0) = 0$; here q is a complex-valued function, and $h \in \mathbb{C}$. The spectral analysis of L was first investigated by Naĭmark [6]. In this paper, he has proved that some of the poles of the resolvent's kernel of L are not the eigenvalues of the operator. Moreover, he has proved that these poles are on the continuous spectrum. (Schwartz named these poles as spectral singularities [11].) Furthermore, Naĭmark has shown that if the condition

$$\int_0^\infty e^{\varepsilon x}|q(x)|\,dx < \infty, \quad \varepsilon > 0, \tag{1.2}$$

holds, then L has a finite number of eigenvalues and spectral singularities with finite multiplicities.

The effect of spectral singularities in the spectral expansion of the operator L in terms of principal functions has been investigated in [4, 5]. The dependence of the structure of spectral singularities of L on the behavior of q at infinity has been studied in [9]. In [10], some problems of spectral theory of the operator L with real potential and complex boundary condition were studied under the condition on the potential

$$\sup_{x\in\mathbb{R}_+}\{\exp(\varepsilon\sqrt{x})|q(x)|\} < \infty, \quad \varepsilon > 0, \tag{1.3}$$

which is weaker than that considered in [6].

We consider the operator L_λ generated in $L^2(\mathbb{R}_+)$ by the Klein-Gordon s-wave equation for a particle of zero mass with static potential

$$y'' + (\lambda - q(x))^2 y = 0, \quad x \in \mathbb{R}_+, \tag{1.4}$$

and the spectral parameter-dependent boundary condition

$$y'(0) - (a\lambda + b)y(0) = 0, \tag{1.5}$$

where $a, b \in \mathbb{C}$, $a \neq \pm i$, $a \neq 0$, q is a complex-valued function and is absolutely continuous on each finite subinterval of $\mathbb{R}_+$.

Some problems of the spectral theory of the Klein-Gordon equation have been investigated in [2, 3] with real potential, and in [1] with complex potential subject to the boundary condition $y(0) = 0$.

In this paper, using the similar technique used in [1, 12], we discuss the spectrum of L_λ and prove that this operator has a finite number of eigenvalues and spectral singularities with finite multiplicities if the conditions

$$\lim_{x\to\infty} q(x) = 0, \qquad \sup_{x\in\mathbb{R}_+} \{\exp(\varepsilon\sqrt{x})|q'(x)|\} < \infty, \quad \varepsilon > 0, \tag{1.6}$$

hold. Therefore we find the principal functions corresponding to the eigenvalues and the spectral singularities of L_λ. In the rest of the paper, we use the following notations:

$$\begin{gathered} \mathbb{C}_+ = \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda > 0\}, \qquad \mathbb{C}_- = \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda < 0\}, \\ \overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \geq 0\}, \qquad \overline{\mathbb{C}}_- = \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \leq 0\}, \\ \mathbb{R}^* = \mathbb{R}\backslash\{0\}. \end{gathered} \tag{1.7}$$

2. Preliminaries.

We suppose that the functions q and q' satisfy

$$\int_0^\infty x\{|q(x)| + |q'(x)|\}dx < \infty. \tag{2.1}$$

Obviously we have from (2.1) that $\lim_{x\to\infty} q(x) = 0$, $x|q(x)|$ is bounded and

$$\int_0^\infty x|q(x)|^2 dx < \infty \tag{2.2}$$

(see [1]). Under condition (2.1), equation (1.4) has the following solutions:

$$f^+(x,\lambda) = e^{i(\alpha(x)+\lambda x)} + \int_x^\infty K^+(x,t)e^{i\lambda t}dt \tag{2.3}$$

for $\lambda \in \overline{\mathbb{C}}_+$, and

$$f^-(x,\lambda) = e^{-i(\alpha(x)+\lambda x)} + \int_x^\infty K^-(x,t)e^{-i\lambda t}dt \tag{2.4}$$

for $\lambda \in \overline{\mathbb{C}}_-$, where $\alpha(x) = \int_x^\infty q(t)dt$ and $K^\pm(x,t)$ are solutions of integral equations of Volterra type and are continuously differentiable with respect to their arguments. Moreover, $|K^\pm(x,t)|$, $|K_x^\pm(x,t)|$, $|K_t^\pm(x,t)|$ satisfy the following inequalities:

$$\begin{gathered} |K^\pm(x,t)| \le c\omega\left(\frac{x+t}{2}\right)\exp\{\gamma(x)\}, \\ |K_x^\pm(x,t)|, |K_t^\pm(x,t)| \le c\left\{\omega^2\left(\frac{x+t}{2}\right)+\theta\left(\frac{x+t}{2}\right)\right\}, \end{gathered} \tag{2.5}$$

where

$$\begin{gathered} \omega(x) = \int_x^\infty \left\{|q(t)|^2+|q'(t)|\right\}dt, \qquad \gamma(x) = \int_x^\infty \left\{t|q(t)|^2+2|q(t)|\right\}dt, \\ \theta(x) = \frac{1}{4}\left\{2|q(x)|^2+|q'(x)|\right\}, \end{gathered} \tag{2.6}$$

and $c > 0$ is a constant [3]. Therefore $f^+(x,\lambda)$ and $f^-(x,\lambda)$ are analytic with respect to λ in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and continuous in λ up to the real axis. $f^\pm(x,\lambda)$ also satisfy the following asymptotic equalities:

$$\begin{aligned} f^\pm(x,\lambda) &= e^{\pm i\lambda x}[1+o(1)], \quad \lambda \in \overline{\mathbb{C}}_\pm,\ x \longrightarrow \infty, \\ f_x^\pm(x,\lambda) &= e^{\pm i\lambda x}[\pm i\lambda+o(1)], \quad \lambda \in \overline{\mathbb{C}}_\pm,\ x \longrightarrow \infty. \end{aligned} \tag{2.7}$$

Moreover, from (2.3) and (2.4), we have

$$f^\pm(x,\lambda) = e^{\pm i(\alpha(x)+\lambda x)} + o(1), \quad \lambda \in \overline{\mathbb{C}}_\pm,\ |\lambda| \longrightarrow \infty. \tag{2.8}$$

From (2.7), the Wronskian of the solutions of $f^+(x,\lambda)$ and $f^-(x,\lambda)$ is

$$W\{f^+(x,\lambda), f^-(x,\lambda)\} = \lim_{x\to\infty} W\{f^+(x,\lambda), f^-(x,\lambda)\} = -2i\lambda \tag{2.9}$$

for $\lambda \in \mathbb{R}$. Hence $f^+(x,\lambda)$ and $f^-(x,\lambda)$ are the fundamental solutions of (1.4) for $\lambda \in \mathbb{R}^*$.

Let $\varphi(x,\lambda)$ denote the solution of (1.4) satisfying the initial conditions

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = a\lambda + b, \tag{2.10}$$

which is an entire function of λ.

3. Eigenvalues and spectral singularities. Let $\psi^\pm(x,\lambda)$ denote the solutions of (1.4) satisfying the following conditions:

$$\lim_{x\to\infty} e^{\pm i\lambda x}\psi^\pm(x,\lambda) = 1, \quad \lim_{x\to\infty} e^{\pm i\lambda x}\psi_x^\pm(x,\lambda) = \mp i\lambda, \quad \lambda \in \overline{\mathbb{C}}_\pm. \tag{3.1}$$

So

$$W\{f^\pm(x,\lambda), \psi^\pm(x,\lambda)\} = \mp 2i\lambda, \quad \lambda \in \overline{\mathbb{C}}_\pm \tag{3.2}$$

(see [3]). We obtain from (2.9) and (3.2) that if

$$\begin{aligned} g^{\pm}(\lambda) &= f_x^{\pm}(0,\lambda) - (a\lambda + b) f^{\pm}(0,\lambda), \\ \tilde{g}^{\pm}(\lambda) &= \psi_x^{\pm}(0,\lambda) - (a\lambda + b)\psi^{\pm}(0,\lambda), \end{aligned} \tag{3.3}$$

then we can write the solution $\varphi(x,\lambda)$ of the boundary value problem (1.4) and (1.5) as follows:

$$\varphi^{\pm}(x,\lambda) = \frac{g^{\pm}(\lambda)}{2i\lambda}\psi^{\pm}(x,\lambda) - \frac{\tilde{g}^{\pm}(x,\lambda)}{2i\lambda} f^{\pm}(x,\lambda), \quad \lambda \in \mathbb{C}_{\pm}, \tag{3.4}$$

$$\varphi(x,\lambda) = \frac{g^{+}(\lambda)}{2i\lambda} f^{-}(x,\lambda) - \frac{g^{-}(\lambda)}{2i\lambda} f^{+}(x,\lambda), \quad \lambda \in \mathbb{R}^{*}. \tag{3.5}$$

Let

$$R(x,t;\lambda) = \begin{cases} R^{+}(x,t;\lambda), & \lambda \in \mathbb{C}_{+}, \\ R^{-}(x,t;\lambda), & \lambda \in \mathbb{C}_{-}, \end{cases} \tag{3.6}$$

be Green's function of L_λ. Then, using classical methods, we easily obtain the kernel of the resolvent as follows [6]:

$$R^{\pm}(x,t;\lambda) = \frac{1}{g^{\pm}(\lambda)} \begin{cases} f^{\pm}(x,\lambda)\varphi(t,\lambda), & 0 \le t < x, \\ f^{\pm}(t,\lambda)\varphi(x,\lambda), & x \le t < \infty. \end{cases} \tag{3.7}$$

Now we denote the eigenvalues and the spectral singularities of L_λ by $\sigma_d(L_\lambda)$ and $\sigma_{ss}(L_\lambda)$, respectively. From (2.8), (3.1), (3.4), (3.5), and (3.7), we obtain that

$$\sigma_d(L_\lambda) = \{\lambda : \lambda \in \mathbb{C}_{+},\ g^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_{-},\ g^{-}(\lambda) = 0\}, \tag{3.8}$$

$$\sigma_{ss}(L_\lambda) = \{\lambda : \lambda \in \mathbb{R}^{*},\ g^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^{*},\ g^{-}(\lambda) = 0\}, \tag{3.9}$$

$$\{\lambda : \lambda \in \mathbb{R}^{*},\ g^{+}(\lambda) = 0\} \cap \{\lambda : \lambda \in \mathbb{R}^{*},\ g^{-}(\lambda) = 0\} = \phi. \tag{3.10}$$

Definition 3.1. The multiplicity of a zero of $g^{\pm}$ in $\overline{\mathbb{C}}_{\pm}$ is defined as the multiplicity of the corresponding member of the spectrum.

It is clear from (3.8) and (3.9) that, in order to investigate the quantitative properties of the members of the spectrum of L_λ, we must investigate the zeros of $g^{+}(\lambda)$ in $\overline{\mathbb{C}}_{+}$ and $g^{-}(\lambda)$ in $\overline{\mathbb{C}}_{-}$. We will consider the zeros of $g^{+}(\lambda)$ in $\overline{\mathbb{C}}_{+}$. The zeros of $g^{-}(\lambda)$ in $\overline{\mathbb{C}}_{-}$ will be similar then.

We define the following sets:

$$N_1^{+} = \{\lambda : \lambda \in \mathbb{C}_{+},\ g^{+}(\lambda) = 0\}, \qquad N_2^{+} = \{\lambda : \lambda \in \mathbb{R},\ g^{+}(\lambda) = 0\}. \tag{3.11}$$

Lemma 3.2. *Under condition (2.1),*

(a) *the set N_1^{+} is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis,*

(b) *the set N_2^{+} is compact.*

Proof. From (2.3) we get that g^+ is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and it has the form

$$g^+(\lambda) = (i-a)\lambda e^{i\alpha(0)} - (iq(0)+b)e^{i\alpha(0)} + (ai-1)K^+(0,0) + \int_0^\infty A^+(t)e^{i\lambda t}dt, \quad (3.12)$$

where

$$A^+(t) = K_x^+(0,t) - aiK_t^+(0,t) - bK^+(0,t). \quad (3.13)$$

It is clear from (2.5) and (3.12) that $A^+ \in L^1(\mathbb{R}_+)$ and

$$g^+(\lambda) = (i-a)\lambda e^{i\alpha(0)} + O(1), \quad \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \longrightarrow \infty. \quad (3.14)$$

Hence the proof is obtained from (3.14) by the assumption $i \neq a$. □

Now we suppose that the following conditions hold:

$$\lim_{x\to\infty} q(x) = 0, \qquad |q'(x)| \le c\exp(-\varepsilon x), \quad c > 0,\ \varepsilon > 0. \quad (3.15)$$

We then have

$$|q(x)| \le c\exp(-\varepsilon x). \quad (3.16)$$

Using (2.5) and (2.6) we find

$$|K^+(x,t)|, |K_x^+(x,t)|, |K_t^+(x,t)| \le c\exp\left\{\frac{-\varepsilon}{2}(x+t)\right\}. \quad (3.17)$$

From (3.17) we get that the functions $f_x^+(0,\lambda)$ and $f^+(0,\lambda)$ can be continued analytically from $\mathbb{C}_+$ into the half-plane $\operatorname{Im}\lambda > -\varepsilon/2$. Hence $g^+(\lambda)$ can be continued analytically from $\mathbb{C}_+$ into the half-plane $\operatorname{Im}\lambda > -\varepsilon/2$. Therefore the sets N_1^+ and N_2^+ cannot have limit points on the real axis. From Lemma 3.2, we find that N_1^+ and N_2^+ have a finite number of points. Moreover, multiplicities of the zeros of $g^+(\lambda)$ in $\overline{\mathbb{C}}_+$ are finite. (Similarly, we can prove the finiteness of the zeros, and their multiplicities, of $g^-(\lambda)$ in $\overline{\mathbb{C}}_-$.)

From (3.8) and (3.9), we get the following theorem.

Theorem 3.3. *Under conditions (3.15), the operator L_λ has a finite number of eigenvalues and spectral singularities with finite multiplicities.*

Now we suppose that the following conditions hold:

$$\lim_{x\to\infty} q(x) = 0, \qquad \sup_{x\in\mathbb{R}_+}\{e^{\varepsilon\sqrt{x}}|q'(x)|\} < \infty, \quad \varepsilon > 0. \quad (3.18)$$

Obviously (3.18) is weaker than (3.15) which was considered in [10] for Sturm-Liouville case with real potential. Using (3.18), (2.6), and (2.5), we obtain

$$|K^+(x,t)|, |K_x^+(x,t)|, |K_t^+(x,t)| \le c\exp\left\{\frac{-\varepsilon}{2}\sqrt{\frac{x+t}{2}}\right\}. \quad (3.19)$$

Here $c > 0$ is a constant. (3.19) cannot let $g^+(\lambda)$ continue analytically into a domain containing the real axis. So the technique of analytic continuation fails to apply here.

From (2.3), (2.5), condition (3.18) gives that the function g^+ is analytic in $\overline{\mathbb{C}}_+$, and all of its derivatives are continuous in $\overline{\mathbb{C}}_+$. So

$$\begin{gathered} \left| g^+(\lambda) - (i-a)\lambda e^{\alpha(0)} \right| < \infty, \quad \lambda \in \overline{\mathbb{C}}_+, \\ \left| \frac{d}{d\lambda} g^+(\lambda) \right| \le |i-a| e^{i\alpha(0)} + \int_0^\infty t e^{-(\varepsilon/2)\sqrt{t}} dt, \quad \lambda \in \overline{\mathbb{C}}_+, \\ \left| \frac{d^n}{d\lambda^n} g^+(\lambda) \right| \le B_n, \quad \lambda \in \overline{\mathbb{C}}_+,\ n = 2,3,\dots. \end{gathered} \tag{3.20}$$

Here

$$B_n = 2^{n+1} c \int_0^\infty t^n e^{-(\varepsilon/2)\sqrt{t}} dt, \quad n = 2,3,\dots, \tag{3.21}$$

and $c > 0$ is a constant.

Let N_3^+ and N_4^+ denote the sets of limit points of N_1^+ and N_2^+, respectively, and let N_5^+ denote the set of all zeros of g^+ with infinite multiplicity in $\overline{\mathbb{C}}_+$. Clearly,

$$N_3^+ \subset N_2^+, \qquad N_4^+ \subset N_2^+, \qquad N_5^+ \subset N_2^+. \tag{3.22}$$

Since all the derivatives of $g^+(\lambda)$ are continuous on $\mathbb{R}$, we then have

$$N_3^+ \subset N_5^+, \qquad N_4^+ \subset N_5^+. \tag{3.23}$$

In order to show the finiteness of the zeros of $g^+(\lambda)$ and their multiplicities, we need to prove that $N_5^+ = \phi$. So we will use the following uniqueness theorem given by Pavlov.

Pavlov's theorem. *Assume that the function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous on $\overline{\mathbb{C}}_+$, and there exists $T > 0$ such that*

$$\begin{gathered} \left| g^{(n)}(z) \right| \le B_n, \quad n = 0,1,\dots,\ z \in \overline{\mathbb{C}}_+,\ |z| < 2T, \\ \left| \int_{-\infty}^{-T} \frac{\ln|g(x)|}{1+x^2} dx \right| < \infty, \qquad \left| \int_T^\infty \frac{\ln|g(x)|}{1+x^2} dx \right| < \infty. \end{gathered} \tag{3.24}$$

If the set Q, with linear Lebesgue measure zero, is the set of all zeros of the function g with infinite multiplicity and if

$$\int_0^h \ln E(s) d\mu(Q_s) = -\infty, \tag{3.25}$$

where $E(s) = \inf_n (A_n s^n / n!)$, $n = 0,1,\dots$, $\mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q, and h is an arbitrary positive constant, then $g(z) \equiv 0$ [9].

Therefore we have the following lemma.

Lemma 3.4. *$N_5^+ = \phi$.*

Proof. From Lemma 3.2, (3.20), and (3.21), g^+ satisfies (3.24). Since $g^+\not\equiv 0$, then by Pavlov's theorem, N_5^+ satisfies

$$\int_0^h \ln E(s)\, d\mu(N_{5,s}^+) > -\infty. \tag{3.26}$$

Here

$$E(s) = \inf_n \frac{B_n s^n}{n!}, \tag{3.27}$$

and $\mu(N_{5,s}^+)$ is the linear Lebesgue measure of s-neighborhood of N_5^+, and the constant B_n is defined by (3.21).

Considering (3.21), we obtain the following estimates for B_n:

$$|B_n| \le D d^n n^n n!. \tag{3.28}$$

Here D and d are constants depending on c and ε. From $E(s)$ and (3.28), we find

$$E(s) \le D \inf_n \{d^n s^n n^n\} \le D \exp\{-d^{-1}e^{-1}s^{-1}\}, \tag{3.29}$$

or by (3.26), we obtain

$$\int_0^h \frac{d\mu(N_{5,s}^+)}{s} < \infty. \tag{3.30}$$

Inequality (3.30) holds for an arbitrary s if and only if $\mu(N_{5,s}^+) = 0$ or $N_5^+ = \phi$, which proves the lemma. □

Theorem 3.5. *Under conditions (3.18), the operator L_λ has a finite number of eigenvalues and spectral singularities with finite multiplicities.*

Proof. From Lemma 3.4 and (3.23), we get $N_3^+ = N_4^+ = \phi$. Hence the sets N_1^+ and N_2^+ have no limit points. So g^+ has only a finite number of zeros in $\overline{\mathbb{C}}_+$. From Lemma 3.4, the multiplicities of these zeros are finite. (Similarly, we can show that g^- has a finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_-$.) The proof follows from (3.8) and (3.9). □

4. Principal functions. Suppose that (3.18) holds. Let $\lambda_1^+,\dots,\lambda_p^+$ and $\lambda_1^-,\dots,\lambda_k^-$ denote the zeros of the functions g^+ in $\mathbb{C}_+$ and g^- in $\mathbb{C}_-$, with multiplicities $m_1^+,\dots,m_p^+$ and $m_1^-,\dots,m_k^-$, respectively. Moreover, let $\lambda_1,\dots,\lambda_l$ and $\lambda_{l+1},\dots,\lambda_r$ be the zeros of g^+ and g^- in $\mathbb{R}^*$ with multiplicities $n_1,\dots,n_l$ and $n_{l+1},\dots,n_r$, respectively. From (3.4) we get that

$$\begin{gathered}\varphi^+(x,\lambda_i^+), \left\{\frac{\partial}{\partial\lambda}\varphi^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}, \dots, \left\{\frac{\partial^{m_i^+-1}}{\partial\lambda^{m_i^+-1}}\varphi^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}, \\ \varphi^-(x,\lambda_i^-), \left\{\frac{\partial}{\partial\lambda}\varphi^-(x,\lambda)\right\}_{\lambda=\lambda_i^-}, \dots, \left\{\frac{\partial^{m_i^--1}}{\partial\lambda^{m_i^--1}}\varphi^-(x,\lambda)\right\}_{\lambda=\lambda_i^-}\end{gathered} \tag{4.1}$$

are the principal functions corresponding to the eigenvalues

$$\lambda = \lambda_i^+, \quad i = 1,2,\dots,p, \qquad \lambda = \lambda_i^-, \quad i = 1,2,\dots,k \text{ of } L_\lambda, \tag{4.2}$$

respectively. Similarly, from (3.5),

$$\varphi(x,\lambda_i), \left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i}, \dots, \left\{\frac{\partial^{n_i-1}}{\partial\lambda^{n_i-1}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i}, \tag{4.3}$$

$i = 1,\dots,l,l+1,\dots,r$ are the principal functions corresponding to the spectral singularities of L_λ.

Obviously we find from (2.3), (3.4), and (3.8) that the principal functions corresponding to the eigenvalues of L_λ are in $L^2(\mathbb{R}_+)$.

Now we recall the Hilbert spaces

$$\begin{aligned} H_+ &= \left\{f : \int_0^\infty (1+x)^{2n_0}|f(x)|^2 dx < \infty\right\}, \\ H_- &= \left\{g : \int_0^\infty (1+x)^{-2n_0}|g(x)|^2 dx < \infty\right\} \end{aligned} \tag{4.4}$$

with norms

$$\|f\|_+^2 = \int_0^\infty (1+x)^{2n_0}|f(x)|^2 dx, \qquad \|g\|_-^2 = \int_0^\infty (1+x)^{-2n_0}|g(x)|^2 dx, \tag{4.5}$$

respectively, where $n_0 = \max\{n_1,\dots,n_l,n_{l+1},\dots,n_r\}+1$. Obviously,

$$H_+ \subsetneqq L^2(\mathbb{R}_+) \subsetneqq H_- \tag{4.6}$$

and $H_- \sim H'_+$ (which is the dual of H_+) [7, 8].

Theorem 4.1. *The principal functions for the spectral singularities do not belong to the space $L^2(\mathbb{R}_+)$; these functions belong to the space H_-, in general,*

$$\begin{aligned} &\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} \notin L^2(\mathbb{R}_+), \quad n = 0,1,\dots,n_i-1, \ i = 1,\dots,l,l+1,\dots,r, \\ &\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} \in H_-, \quad n = 0,1,\dots,n_i-1, \ i = 1,\dots,l,l+1,\dots,r. \end{aligned} \tag{4.7}$$

Proof. Let $0 \le n \le n_i - 1$ and $i = 1,\dots,l$. From (3.5), we obtain

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} = -\sum_{k=1}^{n}\binom{n}{k}\left\{\frac{\partial^{n-k}}{\partial\lambda^{n-k}}g^-(\lambda)\right\}_{\lambda=\lambda_i}\left\{\frac{\partial^k}{\partial\lambda^k}f^+(x,\lambda)\right\}_{\lambda=\lambda_i}. \tag{4.8}$$

From (2.3), (2.8), and (4.8), the proof is obtained. Similarly, we can prove the result for $0 \le n \le n_i - 1$ and $i = l+1,\dots,r$. □

References

[1] E. Bairamov and A. O. Çelebi, *Spectral properties of the Klein-Gordon s-wave equation with complex potential*, Indian J. Pure Appl. Math. **28** (1997), no. 6, 813–824.

[2] A. Degasperis, *On the inverse problem for the Klein-Gordon s-wave equation*, J. Math. Phys. **11** (1970), no. 2, 551–567.

[3] M. Jaulent and C. Jean, *The inverse s-wave scattering problem for a class of potentials depending on energy*, Comm. Math. Phys. **28** (1972), 177–220.

[4] V. E. Lyance, *A differential operator with spectral singularities. I*, Amer. Math. Soc. Transl. Ser. 2 **60** (1967), 185–225.

[5] ———, *A differential operator with spectral singularities. II*, Amer. Math. Soc. Transl. Ser. 2 **60** (1967), 227–283.

[6] M. A. Naĭmark, *Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis*, Amer. Math. Soc. Transl. Ser. 2 **16** (1960), 103–193.

[7] ———, *Linear Differential Operators. Part I: Elementary Theory of Linear Differential Operators*, Frederick Ungar Publishing, New York, 1967.

[8] ———, *Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space*, Frederick Ungar Publishing, New York, 1968.

[9] B. S. Pavlov, *On a Non-selfadjoint Schrödinger Operator*, Topics in Math. Phys., vol. 1, Consultants Bureau, New York, 1967.

[10] ———, *On separation conditions for the spectral components of a dissipative operator*, Math. USSR-Izv. **9** (1975), no. 1, 113–137.

[11] J. Schwartz, *Some non-selfadjoint operators*, Comm. Pure Appl. Math. **13** (1960), 609–639.

[12] G. B. Tunca and E. Bairamov, *Discrete spectrum and principal functions of non-selfadjoint differential operator*, Czechoslovak Math. J. **49(124)** (1999), no. 4, 689–700.

Gülen Başcanbaz-Tunca: Department of Mathematics, Faculty of Science, Ankara University, 06100 Tandogan, Ankara, Turkey

E-mail address: tunca@science.ankara.edu.tr