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ON THE CLASS OF SQUARE PETRIE MATRICES INDUCED BY CYCLIC PERMUTATIONS

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Let $n \geq 2$ be an integer and let $P = \{1,2,\ldots,n,n+1\}$. Let Z_p denote the finite field $\{0,1,2,\ldots,p-1\}$, where $p \geq 2$ is a prime. Then every map σ on P determines a real $n \times n$ Petrie matrix A_σ which is known to contain information on the dynamical properties such as topological entropy and the Artin-Mazur zeta function of the linearization of σ. In this paper, we show that if σ is a *cyclic* permutation on P, then all such matrices A_σ are similar to one another over Z_2 (but not over Z_p for any prime $p \geq 3$) and their characteristic polynomials over Z_2 are all equal to $\sum_{k=0}^{n} x^k$. As a consequence, we obtain that if σ is a *cyclic* permutation on P, then the coefficients of the characteristic polynomial of A_σ are all odd integers and hence nonzero.

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1. Introduction. Throughout this paper, let $n \geq 2$ be a fixed integer and let $P = \{1,2,\ldots,n,n+1\}$. For every integer $1 \leq i \leq n$, let $J_i = [i,i+1]$. Let σ be a map from P into itself. The linearization of σ on P is defined as the continuous map f_σ from $[1,n+1]$ into itself such that $f_\sigma(k) = \sigma(k)$ for every integer $1 \leq k \leq n+1$ and f_σ is linear on J_i for every integer $1 \leq i \leq n$. Let $A_\sigma = (a_{ij})$ be the *real* $n \times n$ matrix defined by $a_{ij} = 1$ if $f_\sigma(J_i) \supset J_j$ and $a_{ij} = 0$ otherwise. The definition of A_σ may seem opaque. But if we take J_i's as the vertices of a directed graph and draw an arrow from the vertex J_i to the vertex J_j if $f_\sigma(J_i) \supset J_j$, then A_σ will be the adjacency matrix [4, page 17] of the resulting directed graph. For example, the adjacency matrix of the cyclic permutation $\sigma : 1 \to 2 \to 5 \to 4 \to 3 \to 1$ is given as

$$A_\sigma = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{1.1}$$

In the theory of discrete dynamical systems on the interval, this adjacency matrix A_σ turns out to contain much information on the dynamical properties of the map f_σ. For example, for some special types (including cyclic permutations) of σ, if $x^n + \sum_{k=0}^{n-1} a_k x^k$ is the characteristic polynomial of A_σ, then it is shown in [6] that the Artin-Mazur zeta function $\zeta(z)$ [2] of f_σ is $\zeta(z) = 1/(1+\sum_{k=1}^{n} a_{n-k} z^k)$. On the other hand, it follows from [1, Theorem 4.4.5, page 222] or [4, Proposition 19, page 204] that the topological entropy of f_σ equals $\max\{0,\log\lambda\}$, where λ is the maximal eigenvalue of A_σ. Since every cyclic graph defines a communication channel, as defined by Shannon, we can claim that the logarithm of the largest eigenvalue of A_σ gives its channel capacity. This motivates further investigation of such matrices A_σ.

Due to the continuity of f_σ, it is clear that such matrices A_σ have entries either zeros or ones such that the ones in each *row* occur consecutively. Actually, we have $a_{ij} = 1$ for all $a_i \le j \le b_i - 1$, where $a_i = \min\{f_\sigma(i), f_\sigma(i+1)\}$ and $b_i = \max\{f_\sigma(i), f_\sigma(i+1)\}$, and $a_{ij} = 0$ elsewhere. For our purpose, we define a Petrie matrix [5] to be a matrix whose entries are either zeros or ones such that the ones in each *row* occur consecutively. So, the matrix A_σ induced by a map σ on P is a square Petrie matrix whose determinant is easily seen (by induction) [7] to be either 0 or ± 1. For any prime number $p \ge 2$, let $Z_p = \{0, 1, 2, \ldots, p-1\}$ denote the usual finite field and let $W_{Z_p} = \{\sum_{i=1}^n r_i J_i \mid r_i \in Z_p, 1 \le i \le n\}$ be the n-dimensional vector space over Z_p with $\{J_i \mid 1 \le i \le n\}$ as a set of basis. Then the matrix $A_\sigma \pmod 2$ defines a linear transformation ψ_σ on W_{Z_2} such that, for every integer $1 \le i \le n$, $\psi_\sigma(J_i) = \sum_{j=1}^n a_{ij} J_j$.

If both σ and ρ are just permutations on P, then it is easy to see that A_σ may not be similar to A_ρ over Z_2. But if both σ and ρ are *cyclic* permutations on P, then we show, in this paper, that A_σ is similar to A_ρ over Z_2 (but A_σ may not be similar to A_ρ over Z_p for any prime $p \ge 3$) and their characteristic polynomials over Z_2 are all equal to $\sum_{k=0}^n x^k$. As a consequence, we obtain that if σ is a *cyclic* permutation, then the coefficients of the characteristic polynomial of A_σ are all odd integers and hence nonzero (not true in general if σ is not cyclic) with constant term ± 1.

2. On the Petrie matrix A_σ over Z_2 with any map σ on P.

In the following, we let $[x : y]$ denote the closed interval on the real line with x and y as endpoints. For integers $1 \le k < j \le n+1$, we let $[k, j]$ denote the element $\sum_{i=k}^{j-1} J_i$ of W_{Z_2} and call k and j the endpoints (this terminology will be used in the proof of Theorem 3.2 in Section 3) of the element $\sum_{i=k}^{j-1} J_i$. Part (2) of the following lemma is proved in [4, pages 22-23], which will be needed in Section 3. Here, we present a different proof (see also [3]).

Lemma 2.1. *Let n, P, J_i's, σ, f_σ, W_{Z_2}, ψ_σ, A_σ be defined as in Section 1. Let ρ be a map from P into itself and let ψ_ρ and A_ρ be defined similarly. Then the following hold.*

(1) *Let $1 \le k < j \le n+1$ be any integers. Then for any element $[k, j] = \sum_{i=k}^{j-1} J_i$ in W_{Z_2}, $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$.*

(2) $\psi_\rho \circ \psi_\sigma = \psi_{\rho\circ\sigma}$ *and* $(A_\sigma)(A_\rho) \equiv A_{\rho\circ\sigma} \pmod 2$. *Consequently, if σ is a permutation on P, then ψ_σ is invertible with inverse $\psi_{\sigma^{-1}}$ and A_σ is nonsingular with determinant ± 1.*

Proof. It follows from the definition of ψ_σ in Section 1 that $\psi_\sigma(J_i) = [f_\sigma(i) : f_\sigma(i+1)]$ for every integer $1 \le i \le n$. Thus, we obtain that $\psi_\sigma([k, j]) = \psi_\sigma(\sum_{i=k}^{j-1} J_i) = \sum_{i=k}^{j-1} \psi_\sigma(J_i) = \sum_{i=k}^{j-1} [f_\sigma(i) : f_\sigma(i+1)] = [f_\sigma(k) : f_\sigma(j)]$ since $1+1=0$ in Z_2. This proves part (1).

By part (1), $\psi_\sigma([k, j]) = [f_\sigma(k) : f_\sigma(j)]$. Similarly, $\psi_\rho([k, j]) = [f_\rho(k) : f_\rho(j)]$. So, $(\psi_\rho \circ \psi_\sigma)(J_i) = \psi_\rho([f_\sigma(i) : f_\sigma(i+1)]) = [f_\rho(f_\sigma(i)) : f_\rho(f_\sigma(i+1))] = [(\rho\circ\sigma)(i) : (\rho\circ\sigma)(i+1)] = \psi_{\rho\circ\sigma}(J_i)$ since, on the finite set P, $f_\sigma = \sigma$ and $f_\rho = \rho$. This shows that $\psi_\rho \circ \psi_\sigma = \psi_{\rho\circ\sigma}$ on W_{Z_2}. Thus, if σ is a permutation on P, then $\psi_{\sigma^{-1}} \circ \psi_\sigma = \psi_{\sigma^{-1}\circ\sigma}$ is the identity map on W_{Z_2}, and so ψ_σ is an invertible linear transformation on W_{Z_2} with inverse $\psi_{\sigma^{-1}}$. The rest of part (2) can be easily proved and is omitted. This proves part (2) and completes the proof of Lemma 2.1. □

3. On the Petrie matrix A_σ with any cyclic permutation σ on P. We will need the following elementary result. We include its proof for completeness.

Lemma 3.1. *Let $1 \le j \le n$ be any fixed integer and let b denote the greatest common divisor of j and $n+1$. Let $s = (n+1)/b$. For every integer $1 \le k \le s-1$, let $1 \le m_k \le n$ be the unique integer such that $kj \equiv m_k (\mathrm{mod}\, n+1)$. Then the m_k's are all distinct and $\{m_k \mid 1 \le k \le s-1\} = \{kb \mid 1 \le k \le s-1\}$.*

Proof. Let $B = \{m_k \mid 1 \le k \le s-1\}$ and $C = \{kb \mid 1 \le k \le s-1\}$. For every integer $1 \le k \le s-1$, since j/b and $(n+1)/b$ are relatively prime, the congruence equation $(j/b)x \equiv k(\mathrm{mod}(n+1)/b)$ has a solution in $1 \le x \le s-1 = (n+1)/b-1$. Consequently, for every integer $1 \le k \le s-1$, the congruence equation $jx \equiv kb(\mathrm{mod}\, n+1)$ has a solution in $1 \le x \le s-1$. Since $1 \le kb \le n$ for every $1 \le k \le s-1$, we obtain that $C \subset B$. Since both B and C contain exactly $s-1$ elements, we have $B = C$. That is, $\{m_k \mid 1 \le k \le s-1\} = \{kb \mid 1 \le k \le s-1\}$. This completes the proof. □

Theorem 3.2. *Let n, P, J_i's, σ, f_σ, W_{Z_2}, ψ_σ, A_σ be defined as in Section 1. Assume that σ is also a cyclic permutation on P. Then the following hold.*

(1) *For every integer $1 \le i \le n$, $\sum_{k=0}^{n} \psi_\sigma^k(J_i) = \mathbf{0}$. Consequently, $\sum_{k=0}^{n} \psi_\sigma^k(w) = \mathbf{0}$ for all $w \in W_{Z_2}$.*

(2) *Let $1 \le i \le n-1$ and $1 \le j \le n$ be two fixed integers such that $1 \le i < f_\sigma^j(i) \le n$ and let $J = [i, f_\sigma^j(i)] = \sum_{k=i}^{f_\sigma^j(i)-1} J_k$. Assume that j and $n+1$ are relatively prime. Then the set $\{\psi_\sigma^k(J) \mid 0 \le k \le n-1\}$ is a basis for W_{Z_2}.*

(3) *For any cyclic permutations σ and ρ on P, ψ_σ and ψ_ρ are similar on W_{Z_2}. Consequently, the Petrie matrices over Z_2 of all cyclic permutations on P are similar to one another and have the same characteristic polynomial $\sum_{k=0}^{n} x^k$.*

(4) *The coefficients of the characteristic polynomial of A_σ are all odd integers (and hence nonzero) with constant term ± 1.*

Remark 3.3. Part (3) of the above theorem does not hold if the Petrie matrices of cyclic permutations are over the finite field Z_p for any prime $p \ge 3$. For example, if $P = \{1,2,3,4,5\}$, σ denotes the cyclic permutation $1 \to 2 \to 5 \to 4 \to 3 \to 1$, and ρ denotes the cyclic permutation $1 \to 2 \to 3 \to 4 \to 5 \to 1$, then A_σ and A_ρ are not similar over Z_p for any prime $p \ge 3$ because the characteristic polynomials of A_σ and A_ρ are $x^4 - x^3 - 3x^2 - 3x - 1$ and $x^4 - x^3 - x^2 - x - 1$, respectively, which are distinct over Z_p for any prime $p \ge 3$.

Proof. For any fixed integer $1 \le i \le n$, let $1 \le j \le n$ be the unique integer such that $f_\sigma^j(i) = i+1$, and so $J_i = [i, i+1] = [i, f_\sigma^j(i)]$. Let b denote the greatest common divisor of j and $n+1$ and let $s = (n+1)/b$. For every integer $1 \le k \le s-1$, let $1 \le m_k \le n$ be the unique integer such that $kj \equiv m_k(\mathrm{mod}\, n+1)$. Then, by Lemma 3.1, we obtain that $\{m_k \mid 1 \le k \le s-1\} = \{kb \mid 1 \le k \le s-1\}$. Let $m_0 = 0$. Then $\{m_k \mid 0 \le k \le s-1\} = \{kb \mid 0 \le k \le s-1\}$. Hence, the set $\{0,1,2,3,\dots,n-1,n\}$ is the disjoint union of the sets $\{m_k + m \mid 0 \le k \le s-1\}$, $0 \le m \le b-1$. Therefore, $\sum_{k=0}^{s-1} \psi_\sigma^{m_k}(J_i) = \sum_{k=0}^{s-1} \psi_\sigma^{kj}(J_i)$ (since $kj \equiv m_k(\mathrm{mod}\, n+1)$) $= [i : f_\sigma^j(i)] + [f_\sigma^j(i) : f_\sigma^{2j}(i)] + [f_\sigma^{2j}(i) : f_\sigma^{3j}(i)] + \cdots + [f_\sigma^{(s-2)j}(i) : f_\sigma^{(s-1)j}(i)] + [f_\sigma^{(s-1)j}(i) : i] = \mathbf{0}$. So, $\sum_{\ell=0}^{n} \psi_\sigma^{\ell}(J_i) = \sum_{m=0}^{b-1} \psi_\sigma^m(\sum_{k=0}^{s-1} \psi_\sigma^{m_k}(J_i)) = \mathbf{0}$. This proves part (1).

For the proof of part (2), we first show that if E is a nonempty subset of $\{1,2,3,\dots,n-1,n\}$ such that $J+\sum_{k\in E}\psi_\sigma^k(J)=\mathbf{0}$, then $E=\{1,2,3,\dots,n-1,n\}$. Indeed, for every integer $1\le k\le n$, let $1\le m_k\le n$ be the unique integer such that $kj\equiv m_k(\operatorname{mod} n+1)$. Assume that $m_1=j\notin E$. Then, for any $m\in E$, $m\ne 0,j$. Since $\psi_\sigma^m(J)=\psi_\sigma^m([i,f_\sigma^j(i)])=[f_\sigma^m(i):f_\sigma^{m+j}(i)]$, the endpoints of $\psi_\sigma^m(J)$ do not contain the point $f_\sigma^j(i)$. Thus, in the expression of $\psi_\sigma^m(J)$ as a sum of the basis elements J_k's, it contains either both the basis elements $J_{f_\sigma^j(i)-1}$ and $J_{f_\sigma^j(i)}$ or none of them. But, since $J=[i,f_\sigma^j(i)]=J_i+J_{i+1}+\cdots+J_{f_\sigma^j(i)-1}$ contains the element $J_{f_\sigma^j(i)-1}$, but not the element $J_{f_\sigma^j(i)}$, in its expression as a sum of the basis elements J_k's, we obtain that in the expression of $J+\sum_{m\in E}\psi_\sigma^m(J)$ as a sum of the basis elements J_k's, the coefficient of $J_{f_\sigma^j(i)-1}$ is different from that of $J_{f_\sigma^j(i)}$ by 1. This implies that $J+\sum_{m\in E}\psi_\sigma^m(J)\ne\mathbf{0}$, which is a contradiction. Therefore, $m_1=j\in E$.

Thus,

$$\begin{aligned}\mathbf{0}&=J+\sum_{m\in E}\psi_\sigma^m(J)\\&=J+\psi_\sigma^j(J)+\sum_{m\in E\setminus\{m_1\}}\psi_\sigma^m(J)\\&=[i,f_\sigma^j(i)]+[f_\sigma^j(i):f_\sigma^{2j}(i)]+\sum_{m\in E\setminus\{m_1\}}\psi_\sigma^m(J)\\&=[i:f_\sigma^{2j}(i)]+\sum_{m\in E\setminus\{m_1\}}\psi_\sigma^m(J).\end{aligned}\tag{3.1}$$

Proceeding in this manner finitely many times, we obtain that $\{m_1,m_2,\dots,m_{n-1}\}\subset E$ and

$$\begin{aligned}\mathbf{0}&=J+\sum_{m\in E}\psi_\sigma^m(J)\\&=[i:f_\sigma^{2j}(i)]+\sum_{m\in E\setminus\{m_1\}}\psi_\sigma^m(J)\\&=[i:f_\sigma^{3j}(i)]+\sum_{m\in E\setminus\{m_1,m_2\}}\psi_\sigma^m(J)\\&=\cdots=[i:f_\sigma^{nj}(i)]\\&\quad+\sum_{m\in E\setminus\{m_1,m_2,\dots,m_{n-1}\}}\psi_\sigma^m(J).\end{aligned}\tag{3.2}$$

In particular, $\mathbf{0}=[i:f_\sigma^{nj}(i)]+\sum_{m\in E\setminus\{m_1,m_2,\dots,m_{n-1}\}}\psi_\sigma^m(J)$. If $m\in E$ and $m\ne m_n$, then, as above, since $m\ne 0$ and $m\ne m_n\equiv nj(\operatorname{mod} n+1)$, the endpoints of $\psi_\sigma^m(J)$ do not contain the point $f_\sigma^{nj}(i)$. Hence, in the expression of $\psi_\sigma^m(J)$ as a sum of the basis elements J_k's, it contains either both the basis elements $J_{f_\sigma^{nj}(i)-1}$ and $J_{f_\sigma^{nj}(i)}$ or none of them. But, since $[i,f_\sigma^{nj}(i)]=J_i+J_{i+1}+\cdots+J_{f_\sigma^{nj}(i)-1}$ contains the element $J_{f_\sigma^{nj}(i)-1}$, not the element $J_{f_\sigma^{nj}(i)}$, in its expression as a sum of the basis elements J_k's, we obtain that in the expression of $[i:f_\sigma^{nj}(i)]+\sum_{m\in E\setminus\{m_1,m_2,\dots,m_{n-1}\}}\psi_\sigma^m(J)$ as a sum of the basis elements J_k's, the coefficient of $J_{f_\sigma^{nj}(i)-1}$ is different from that of $J_{f_\sigma^{nj}(i)}$ by 1. This implies that $[i : f_\sigma^{nj}(i)] + \sum_{m\in E\setminus\{m_1,m_2,\dots,m_{n-1}\}} \psi_\sigma^m(J) \neq \mathbf{0}$, which is a contradiction. Thus, $m_n = nj \in E$. Since, by assumption, j and $n+1$ are relatively prime, we see that, by Lemma 3.1, $\{m_1,m_2,\dots,m_n\} = \{1,2,\dots,n-1,n\}$. Since $\{m_1,m_2,\dots,m_n\} \subset E \subset \{1,2,\dots,n-1,n\}$, we obtain that $E = \{1,2,\dots,n-1,n\}$. This proves our assertion.

Now, assume that $\sum_{k=0}^{n-1}\alpha(k)\psi_\sigma^k(J) = \mathbf{0}$, where $\alpha(k) = 0$ or 1 in Z_2, $0 \le k \le n-1$. If $\alpha(0) = 0$ and $\alpha(\ell) \neq 0$ for some integer $1 \le \ell < n-1$, let ℓ be the smallest such integer; then, since ψ_σ is invertible by Lemma 2.1(2), we obtain that $J + \sum_{k=1}^{n-\ell-1}\alpha(k)\psi_\sigma^k(J) = \mathbf{0}$. So, without loss of generality, we may assume that $\alpha(0) \neq 0$. That is, we assume that $J + \sum_{k=1}^{n-1}\alpha(k)\psi_\sigma^k(J) = \mathbf{0}$. Let $E = \{k \mid 1 \le k \le n-1, \alpha(k) \neq 0\}$. Then, we have $J + \sum_{k\in E}\psi_\sigma^k(J) = \mathbf{0}$. But then it follows from what we have just proved above that $E = \{1,2,\dots,n-1,n\}$. This contradicts the assumption that $E \subset \{1,2,\dots,n-1\}$. So, the set $\{\psi_\sigma^k(J) \mid 0 \le k \le n-1\}$ is linearly independent and hence, by [8], is a basis for W_{Z_2}. This proves part (2).

Let θ denote the cyclic permutation $1 \to 2 \to 3 \to \cdots \to i \to i+1 \to \cdots \to n \to n+1 \to 1$ on P and let σ be any cyclic permutation on P. Choose any fixed integer $1 \le j \le n$ such that j and $n+1$ are relatively prime and let $J = [1, f_\sigma^j(1)]$. Then, by part (2), the set $\{\psi_\sigma^k(J) \mid 0 \le k \le n-1\}$ is a basis for W_{Z_2}. Let ϕ be the linear transformation on W_{Z_2} defined by $\phi(J_k) = \psi_\sigma^{k-1}(J)$, $1 \le k \le n$. Then ϕ is an isomorphism on W_{Z_2}. Furthermore, $(\phi\circ\psi_\theta)(J_n) = \phi(\sum_{k=1}^n J_k) = \sum_{k=1}^n \phi(J_k) = \sum_{k=1}^n \psi_\sigma^{k-1}(J) = \psi_\sigma^n(J)$ (by part (1)) $= \psi_\sigma(\psi_\sigma^{n-1}(J)) = \psi_\sigma(\phi(J_n)) = (\psi_\sigma\circ\phi)(J_n)$ and, for every integer $1 \le k \le n-1$, $(\phi\circ\psi_\theta)(J_k) = \phi(\psi_\theta(J_k)) = \phi(J_{k+1}) = \psi_\sigma^k(J) = \psi_\sigma(\psi_\sigma^{k-1}(J)) = \psi_\sigma(\phi(J_k)) = (\psi_\sigma\circ\phi)(J_k)$. Thus, ψ_σ is similar to ψ_θ through ϕ. Since the property of similarity is obviously transitive, we obtain that if ρ is any cyclic permutation on P, then ψ_σ and ψ_ρ are similar on W_{Z_2}. Consequently, by [8], the Petrie matrices (over Z_2) of all cyclic permutations on P are similar to one another and so have the same characteristic polynomial $\sum_{k=0}^n x^k$ since $\sum_{k=0}^n x^k$ is easily verified to be the characteristic polynomial of the Petrie matrix A_θ over Z_2. This proves part (3).

Finally, let σ be a cyclic permutation on P. Since A_σ is a real $n\times n$ matrix with entries either zeros or ones, the coefficients of the characteristic polynomial of A_σ are all integers. By taking every entry in A_σ modulo 2 and applying part (3) and the fact that the determinants of Petrie matrices are either 0 or ±1, we obtain that the characteristic polynomial of $A_\sigma \pmod 2$ is equal to $\sum_{k=0}^n x^k$. Consequently, the coefficients of the characteristic polynomial of A_σ are all odd integers with constant term ±1. This proves part (4) and completes the proof of Theorem 3.2. □

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References

[1] L. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, 2nd ed., Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific Publishing, New Jersey, 2000.

[2] M. Artin and B. Mazur, *On periodic points*, Ann. of Math. (2) **81** (1965), 82–99.

[3] C. Bernhardt, *A proof of Sharkovsky's theorem*, J. Difference Equ. Appl. **9** (2003), no. 3-4, 373–379.
[4] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Mathematics, vol. 1513, Springer-Verlag, Berlin, 1992.
[5] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Encyclopedia of Mathematics and Its Applications, vol. 39, Cambridge University Press, Cambridge, 1991.
[6] B.-S. Du, *The linearisations of cyclic permutations have rational zeta functions*, Bull. Austral. Math. Soc. **62** (2000), no. 2, 287–295.
[7] M. Gordon and E. M. Wilkinson, *Determinants of Petrie matrices*, Pacific J. Math. **51** (1974), 451–453.
[8] I. N. Herstein, *Topics in Algebra*, Blaisdell Publishing, New York, 1964.

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