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ON BOUNDED COHOMOLOGY OF AMALGAMATED PRODUCTS OF GROUPS

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We investigate the structure of the singular part of the second bounded cohomology group of amalgamated products of groups by constructing an analog of the initial segment of the Mayer-Vietoris exact cohomology sequence for the spaces of pseudocharacters.

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1. Introduction. We recall that bounded cohomology $H_b^*(G)$ of a group G (we will be considering only cohomology with coefficients in the additive group of reals $\mathbb{R}$ with trivial action, so, in our notations for cohomology, the coefficient module will be omitted) is defined using the complex

$$\cdots \longleftarrow C_b^{n+1}(G) \xleftarrow{\delta_b^n} C_b^n(G) \longleftarrow \cdots \longleftarrow C_b^2(G) \xleftarrow{\delta_b^1} C_b^1(G) \xleftarrow{\delta_b^0=0} \mathbb{R} \xleftarrow{\delta_b^{-1}=0} 0 \tag{1.1}$$

of bounded cochains $f : G\times\cdots\times G \to \mathbb{R}$, and $\delta_b^n = \delta^n|_{C_b^n(G)}$ is the bounded differential operator. Since $H_b^0(G) = \mathbb{R}$ and $H_b^1(G) = 0$ for any group G, investigation of bounded cohomology starts in dimension 2. One observes that $H_b^2(G)$ contains a subspace $H_{b,2}^2(G)$ (called the singular part of the second bounded cohomology group), which has a simple algebraic description in terms of quasicharacters and pseudocharacters, and the quotient space $H_b^2(G)/H_{b,2}^2(G)$ is canonically isomorphic to the bounded part of the ordinary cohomology group $H^2(G)$. See [6] for background and available results on bounded cohomology of groups. (For bounded cohomology of topological spaces, see [8].)

We recall that a function $F : G \to \mathbb{R}$ is called a *quasicharacter* if there exists a constant $C_F \geqslant 0$ such that

$$\left|F(xy)-F(x)-F(y)\right| \leqslant C_F \quad \forall x,y \in G. \tag{1.2}$$

A function $f : G \to \mathbb{R}$ is called a *pseudocharacter* if f is a quasicharacter and, in addition,

$$f(g^n) = nf(g) \quad \forall g \in G,\ n \in \mathbb{Z}. \tag{1.3}$$

The notions of a quasicharacter and a pseudocharacter originally arose from the questions of stability of solutions of functional equations [9, 10, 11] and continuous representations of groups [12]. We use the following notation:

(i) $X(G)$ is the space of additive characters $G \to \mathbb{R}$;
(ii) $QX(G)$ is the space of quasicharacters;
(iii) $PX(G)$ is the space of pseudocharacters;
(iv) $B(G)$ is the space of bounded functions.

Then

$$H^2_{b,2}(G) \cong QX(G)/\big(X(G) \oplus B(G)\big) \cong PX(G)/X(G) \tag{1.4}$$

as vector spaces (cf. [6, Proposition 3.2 and Theorem 3.5]). Special interest in $H^2_{b,2}$ is motivated in part by its connections with other structural properties of groups such as commutator length [1] and bounded generation [6]. (See [3] for a simple proof of triviality of $H^2_{b,2}$ for Chevalley groups over rings of S-integers in algebraic number fields using bounded generation.) For example, Grigorchuk [7] (cf. also [5]) proved that the amalgamated product $A_1 *_H A_2$ does not have bounded generation provided that the number of double cosets of A_1 modulo H is at least 3 and $[A_2 : H] \geqslant 2$ by showing that $\dim H^2_{b,2}(A_1 *_H A_2) = \infty$ in this case. The proof is based on the explicit construction (see Example 4.4) of an infinite family of linearly independent quasicharacters which naturally generalize the construction of quasicharacters for free groups. The quasicharacters for free groups were first constructed by Brooks [2], and Faĭziev showed that they can be used to find a basis for the space of pseudocharacters of a free group [4] (cf. [6, Theorem 5.7] for a shorter and more conceptual proof).

However, no systematic study of bounded cohomology of amalgamated products of groups has been undertaken. The goal of this paper is to provide the first step in an attempt to obtain general information about bounded cohomology of amalgamated products of groups. Since the main technical tool used to compute cohomology of the amalgamated product $A_1 *_H A_2$ is the Mayer-Vietoris exact sequence (see [14, Theorem 2.3])

$$\cdots \longrightarrow H^n(A_1 *_H A_2) \longrightarrow H^n(A_1) \oplus H^n(A_2) \longrightarrow H^n(H) \longrightarrow H^{n+1}(A_1 *_H A_2) \longrightarrow \cdots, \tag{1.5}$$

it is natural to try to exhibit an analog of this sequence for bounded cohomology. We construct an initial segment of this sequence for bounded cohomology (it starts in dimension 2) and we formulate our results in terms of spaces of pseudocharacters.

We begin by considering the case where the amalgamated subgroup is normal in both factors (Theorem 2.1). In the general case, we restrict our attention to the special class of pseudocharacters which we call H-spherical (Theorem 4.6); see Section 4 for relevant definitions and discussion.

The sequences (2.11) and (4.14) constructed in Theorems 2.1 and 4.6, respectively, reduce the problem of computation of spaces of pseudocharacters for amalgamated products of groups to that for free products of groups (terms on the left); the structure of the latter spaces is known [6, Proposition 4.3 and Remark 4.4].

We conclude this section with two easy facts which will be used throughout the paper without special reference.

Lemma 1.1. *Any pseudocharacter is constant on conjugacy classes; a bounded pseudocharacter is trivial.*

Proof. Let $f \in PX(G)$ and suppose that $f(yxy^{-1}) - f(x) = a \neq 0$ for some x, $y \in G$. Then the difference $f(yx^ny^{-1}) - f(x^n) = na$ is unbounded when $n \to \infty$. On the other hand,

$$|f(yx^ny^{-1}) - f(x^n)| = |f(yx^ny^{-1}) - f(y) - f(x^n) - f(y^{-1})| \leqslant 2C_f, \tag{1.6}$$

a contradiction. The second assertion is obvious. □

2. The case of a normal subgroup. In this section, we will establish an analog of the initial segment of the Mayer-Vietoris sequence for spaces of pseudocharacters assuming that the amalgamated subgroup N is normal in both factors A_1 and A_2 (in which case it is also normal in the amalgamated product). To describe this sequence, we need to introduce some natural linear maps. First, we define

$$\beta : PX(A_1 *_N A_2) \longrightarrow PX(A_1) \oplus PX(A_2) \tag{2.1}$$

as $\beta = (\beta_1, \beta_2)$, where

$$\beta_i : PX(A_1 *_N A_2) \longrightarrow PX(A_i), \quad i = 1, 2, \tag{2.2}$$

is the restriction map associated with the natural embedding $A_i \hookrightarrow A_1 *_N A_2$. Next, let

$$\gamma : PX(A_1) \oplus PX(A_2) \longrightarrow PX(N) \tag{2.3}$$

be defined by

$$\gamma(f_1, f_2) = f_1|_N - f_2|_N. \tag{2.4}$$

In contrast to the usual Mayer-Vietoris sequence for the spaces of characters,

$$0 \longrightarrow X(A_1 *_N A_2) \xrightarrow{\tilde{\beta}} X(A_1) \oplus X(A_2) \xrightarrow{\tilde{\gamma}} X(N), \tag{2.5}$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ are analogous to β and γ introduced above, the sequence for pseudocharacters will contain, at the extreme left, one extra term which is typically an infinite-dimensional vector space. To define it, we consider the embedding

$$\alpha : PX((A_1/N) * (A_2/N)) \longrightarrow PX(A_1 *_N A_2) \tag{2.6}$$

induced by the natural surjective homomorphism

$$A_1 *_N A_2 \longrightarrow (A_1/N) * (A_2/N) \tag{2.7}$$

and let $PX_0((A_1/N)*(A_2/N))$ denote the kernel of the linear map

$$\bar{\beta}: PX((A_1/N)*(A_2/N)) \longrightarrow PX(A_1/N)\oplus PX(A_2/N), \tag{2.8}$$

$\bar{\beta}=(\bar{\beta}_1,\bar{\beta}_2)$, where

$$\bar{\beta}_i: PX((A_1/N)*(A_2/N)) \longrightarrow PX(A_i/N) \tag{2.9}$$

is the restriction map induced by the natural embedding

$$A_i/N \hookrightarrow (A_1/N)*(A_2/N), \quad i=1,2. \tag{2.10}$$

The above spaces and linear maps align in the following sequence.

Theorem 2.1. *Let N be a normal subgroup of A_1 and A_2. Then the sequence of vector spaces*

$$\begin{aligned} 0 \longrightarrow PX_0((A_1/N)*(A_2/N)) &\xrightarrow{\alpha} PX(A_1 *_N A_2) \\ \xrightarrow{\beta} PX(A_1)\oplus PX(A_2) &\xrightarrow{\gamma} PX(N) \end{aligned} \tag{2.11}$$

is exact.

We will prove the theorem in the next section and will now derive two consequences.

Corollary 2.2. *Given two arbitrary pseudocharacters f_1 and f_2 on the groups A_1 and A_2, respectively, there exists a pseudocharacter f on the free product $A_1 * A_2$ such that $f|_{A_i}=f_i$, $i=1,2$.*

Corollary 2.3. *Let A be an arbitrary group and N its normal subgroup. Then the restriction homomorphism $\rho: PX(A *_N A)\to PX(A)$, induced by embedding A into $A *_N A$ as either factor, is surjective. If, moreover, $[A:N]=2$, then ρ is an isomorphism.*

Proof. For the first assertion, one needs to observe that for any $f\in PX(A)$, the pair (f,f) belongs to $\operatorname{Ker}\gamma$, and is therefore obtained as the restriction of a pseudocharacter on $A *_N A$. If $[A:N]=2$, then $(A/N)*(A/N)$ is the infinite dihedral group. Since it is amenable, all its pseudocharacters are in fact characters [6, Theorem 2.1]. On the other hand, since it is generated by elements of order two, it does not have nonzero characters. This, in particular, implies that

$$PX_0((A_1/N)*(A_2/N))=0, \tag{2.12}$$

hence our second claim. □

3. Proof of Theorem 2.1

3.1. Exactness in the term $PX(A_1)\oplus PX(A_2)$. The inclusion $\operatorname{Im}\beta\subset\operatorname{Ker}\gamma$ being obvious, all we need to prove is that, given pseudocharacters $f_i\in PX(A_i)$, $i=1,2$, satisfying $f_1|_N=f_2|_N$, there exists a pseudocharacter $f\in PX(A_1 *_N A_2)$ such that $f|_{A_i}=f_i$. The following observation saves the (serious) trouble of verifying the condition $f(g^n)=nf(g)$.

Lemma 3.1. *In the current notation, for the existence of* a pseudocharacter f, *it suffices to construct a* quasicharacter $F \in QX(A_1 *_N A_2)$ *such that the differences* $F|_{A_i} - f_i$ *are bounded for* $i = 1, 2$.

Proof. Indeed, it follows from (1.4) that given such an F, there exists a pseudocharacter $f \in PX(A_1 *_N A_2)$ for which the difference $F - f$ is bounded. Then, for $i = 1, 2$, the difference

$$f_i - f|_{A_i} = (f_i - F|_{A_i}) + (F - f)\,|_{A_i} \tag{3.1}$$

is a bounded pseudocharacter of A_i, hence zero, proving that $f|_{A_i} = f_i$. □

The construction of such a quasicharacter $F \in QX(A_1 *_N A_2)$ rests on a specific choice of systems of representatives X_i of all left cosets $\neq N$ in A_i/N for $i = 1, 2$. Namely, it is possible to choose such systems of representatives X_i having the following property:

(P_1) if $x, y \in X_i$ and $xy \in N$, then either $x^2, y^2 \in N$ or $y = x^{-1}$.

Indeed, let $\bar{S}_i$ denote the set of elements of order two in A_i/N and pick an arbitrary system of representatives $S_i \subset A_i$ of the cosets from $\bar{S}_i$. Since, for each

$$x \in \bar{T}_i := (A_i/N) \setminus (\bar{S}_i \cup \{e\}) \tag{3.2}$$

(e the identity), we have $x \neq x^{-1}$, there exists a partition $\bar{T}_i = \bar{T}_i' \cup \bar{T}_i''$ such that $\bar{T}_i' \cap \bar{T}_i'' = \varnothing$ and $\bar{T}_i'' = (\bar{T}_i')^{-1}$. Choose an arbitrary system of representatives $T_i' \subset A_i$ of the cosets from $\bar{T}_i'$ and let $T_i = T_i' \cup (T_i')^{-1}$. Finally, let $X_i = S_i \cup T_i$.

Suppose that the systems of representatives X_i with property (P_1) have been chosen. We define an involutive transformation $\tau_i : X_i \to X_i$ by setting

$$\tau_i(x) = \begin{cases} x & \text{if } x \in S_i, \\ x^{-1} & \text{if } x \in T_i. \end{cases} \tag{3.3}$$

Now, we let $X = X_1 \cup X_2$ (disjoint union) and introduce a function F and an involution τ on X whose restrictions to X_i are f_i and τ_i, respectively:

$$F(x) = f_i(x), \quad \tau(x) = \tau_i(x) \quad \text{if } x \in X_i. \tag{3.4}$$

Let W be the set of all words of the form $x_1 \cdots x_n$, where $x_i \in X$ and, for every $i = 1, \ldots, n-1$, the elements x_i and x_{i+1} belong to *different* parts X_1 or X_2 of X (by convention, the empty word is included in W and corresponds to $n = 0$). Then each element $g \in G := A_1 *_N A_2$ admits a unique canonical presentation of the form

$$g = x_1 \cdots x_n h \tag{3.5}$$

for some $h \in N$ and some word $x_1 \cdots x_n \in W$ (cf., e.g., [13, Chapter I, Theorem 1]). Using the canonical form (3.5), we can extend τ to an involutive transformation of G by setting

$$\tau(g) = \tau(x_n) \cdots \tau(x_1) h. \tag{3.6}$$

Let $f_0 \in PX(N)$ denote the common restriction of f_1 and f_2 to N:

$$f_0 := f_1|_N = f_2|_N. \tag{3.7}$$

It follows from (3.6) that for every $g \in G$, we have $g\tau(g) \in N$, so the expression $f_0(g\tau(g))$ makes sense. Let $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$. We extend F to a function on $A_1 *_N A_2$ by the formula

$$F(g) = \mu(g) + \eta(g), \tag{3.8}$$

where

$$\mu(g) = \frac{1}{2} f_0(g\tau(g)), \qquad \eta(g) = \sum_{x_i \in T} F(x_i), \tag{3.9}$$

and, by convention, $\eta(g) = 0$ if $g \in N$ (in this definition, we use the unique canonical presentation (3.5)).

Proposition 3.2. *The function F defined by (3.8) is a quasicharacter of $A_1 *_N A_2$ such that the differences $F|_{A_i} - f_i$ are bounded for $i = 1, 2$.*

First, we observe that for $h \in N$, we have

$$\mu(h) = \frac{1}{2} f_0(h^2) = f_0(h). \tag{3.10}$$

In particular, $F|_N = \mu|_N$ is a pseudocharacter with constant $C = C_{f_0}$. Also, writing $g \in G$ in the canonical form and using the fact that f_0 is the common restriction of pseudocharacters f_1 and f_2 to N, we obtain

$$\mu(ghg^{-1}) = \mu(h) \quad \forall g \in G,\ h \in N. \tag{3.11}$$

Next, we will show that the difference $F|_{A_i} - f_i$ is a bounded function on A_i for $i = 1, 2$. For $g \in N$, we have

$$F(g) = \frac{1}{2} f_0(g^2) = f_0(g). \tag{3.12}$$

If $g \in A_i \setminus N$, we write it in the form $g = xh$ with $h \in N$, $x \in X_i$. If $x \in S_i$, then

$$\begin{aligned} |F(g) - f_i(g)| &= \left| \frac{1}{2} f_0(xhxh) - f_i(xh) \right| \\ &= \left| \frac{1}{2} f_i(x^2([x^{-1}hx]h)) - f_i(xh) \right| \\ &\leqslant \frac{1}{2} |f_i(x^2([x^{-1}hx]h)) - f_i(x^2) - 2f_i(h)| \\ &\quad + |f_i(x) + f_i(h) - f_i(xh)| \\ &\leqslant 2C_{f_i}. \end{aligned} \tag{3.13}$$

If $x \in T_i$, then

$$\begin{aligned}|F(g)-f_i(g)| &= \left|\frac{1}{2}f_0(xhx^{-1}h)+f_i(x)-f_i(xh)\right| \\ &\leqslant \frac{1}{2}|f_i((xhx^{-1})h)-2f_i(h)|+|f_i(h)+f_i(x)-f_i(xh)| \\ &\leqslant \frac{3}{2}C_{f_i}.\end{aligned} \tag{3.14}$$

In particular, we obtain

$$F|_{A_i} \in QX(A_i). \tag{3.15}$$

To complete the proof of the proposition, it remains to show that F is a quasicharacter on the entire amalgamated product $A_1 *_N A_2$. For a function f on G, we define

$$(\delta f)(g_1,g_2) = f(g_1g_2)-f(g_1)-f(g_2). \tag{3.16}$$

So, we need to show that $\delta F = \delta\mu+\delta\eta$ is bounded on $G\times G$. For convenience of further reference, we will collect, in the following lemma, some properties of the functions μ and η.

Lemma 3.3. (i) *If $g \in G$, $h \in N$, then $|\mu(gh)-\mu(g)-\mu(h)| \leqslant C$.*

(ii) *If $g_1,\dots,g_k \in G$ and $\tau(g_1\cdots g_k) = \tau(g_k)\cdots\tau(g_1)$, then*

$$\left|\mu(g_1\cdots g_k)-\sum_{i=1}^{k}\mu(g_i)\right| \leqslant \frac{k-1}{2}C. \tag{3.17}$$

(iii) *$\eta(gh) = \eta(g)$ for any $g \in G$ and any $h \in N$.*

(iv) *If $x_1\cdots x_n \in W$, then $\eta(x_1\cdots x_n) = \eta(x_1\cdots x_i)+\eta(x_{i+1}\cdots x_n)$ for any $i = 1,\dots,n-1$.*

(v) *$\eta(\tau(g)) = -\eta(g)$ for any $g \in G$.*

Proof. Indeed, if $g \in G$ and $h \in N$, then $\mu(gh) = (1/2)f_0(gh\tau(g)h)$, whence

$$|\mu(gh)-\mu(g)-\mu(h)| = \frac{1}{2}|f_0(g\tau(g)[\tau(g)^{-1}h\tau(g)]h)-f_0(g\tau(g))-f_0(h^2)|, \tag{3.18}$$

and (i) follows. For (ii), one needs to observe that

$$\begin{aligned}\mu(g_1\cdots g_k) = \frac{1}{2}f_0\Big(&\Big[(g_1\cdots g_{k-1})(g_k\tau(g_k))(g_1\cdots g_{k-1})^{-1}\Big] \\ &\times\Big[(g_1\cdots g_{k-2})(g_{k-1}\tau(g_{k-1}))(g_1\cdots g_{k-2})^{-1}\Big]\cdots[g_1\tau(g_1)]\Big).\end{aligned} \tag{3.19}$$

Properties (iii), (iv), and (v) follow immediately from the definition of η. □

For two given elements $g_1,g_2 \in G$, we pick the canonical presentations

$$g_1 = x_1\cdots x_mh_1, \qquad g_2 = y_1\cdots y_nh_2, \tag{3.20}$$

where $x_1\cdots x_m, y_1\cdots y_n \in W$ and $h_1,h_2 \in N$. We first consider the easiest case where x_m and y_1 belong to different factors A_i, $i=1,2$. In this case, the canonical presentation of g_1g_2 is

$$g_1g_2 = x_1\cdots x_m y_1\cdots y_n h, \tag{3.21}$$

where

$$h = \left[(y_1\cdots y_n)^{-1}h_1(y_1\cdots y_n)\right]h_2 \in N. \tag{3.22}$$

It follows from Lemma 3.3(iii) and (iv) that $(\delta\eta)(g_1,g_2) = 0$, so $(\delta F)(g_1,g_2) = (\delta\mu)(g_1,g_2)$. Since

$$\tau(x_1\cdots x_m y_1\cdots y_n) = \tau(y_1\cdots y_n)\tau(x_1\cdots x_m), \tag{3.23}$$

we conclude from Lemma 3.3(i) and (ii) and (3.11) that

$$\begin{aligned}
&|\mu(g_1g_2)-\mu(h_1)-\mu(h_2)-\mu(x_1\cdots x_m)-\mu(y_1\cdots y_n)|\\
&\quad\leqslant |\mu(x_1\cdots x_m y_1\cdots y_n h)-\mu(x_1\cdots x_m y_1\cdots y_n)-\mu(h)|\\
&\qquad + |\mu(h)-\mu(h_1)-\mu(h_2)|\\
&\qquad + |\mu(x_1\cdots x_m y_1\cdots y_n)-\mu(x_1\cdots x_m)-\mu(y_1\cdots y_n)|\\
&\quad\leqslant C+C+\frac{C}{2}=\frac{5}{2}C.
\end{aligned} \tag{3.24}$$

On the other hand,

$$|[\mu(g_1)+\mu(g_2)]-[\mu(h_1)+\mu(h_2)+\mu(x_1\cdots x_m)+\mu(y_1\cdots y_n)]| \leqslant 2C. \tag{3.25}$$

It follows, in this case, that

$$|(\delta F)(g_1,g_2)| \leqslant \frac{9}{2}C. \tag{3.26}$$

To consider the general case, we need to introduce the fragments of g_1 and g_2 that cancel out in g_1g_2. Let k be the largest integer less than or equal to $\min\{m,n\}$ such that $x_{m-i+1}y_i \in N$ for all $i=1,\dots,k$. We introduce the following elements:

$$\begin{gathered}
w_1 = x_1\cdots x_{m-k-1}, \qquad u_1 = x_{m-k}, \qquad v_1 = x_{m-k+1}\cdots x_m,\\
v_2 = y_1\cdots y_k, \qquad u_2 = y_{k+1}, \qquad w_2 = y_{k+2}\cdots y_n,
\end{gathered} \tag{3.27}$$

where, by convention, $v_1 = e$ if $k=0$, $w_1 = e$ if $m=k+1$, and $w_1 = u_1 = e$ if $m=k$, with similar rules for v_2, u_2, and w_2. We observe that $v_2 = \tau(v_1)$, so, letting $v = v_1$, we have the following factorizations:

$$g_1 = w_1u_1vh_1, \qquad g_2 = \tau(v)u_2w_2h_2. \tag{3.28}$$

It follows from our construction that both u_1 and u_2 belong to the *same* factor A_i, so we can write

$$u_1u_2 = zh \quad \text{for some } h\in N,\ z\in X. \tag{3.29}$$

We claim that

$$|(\delta\mu)(g_1,g_2)-(\delta\mu)(u_1,u_2)| \leqslant 10C. \tag{3.30}$$

This estimation is a consequence of the following three inequalities that reflect the three-step transition from g_1,g_2 to u_1,u_2.

Lemma 3.4. (i) *Let* $s_1 = w_1u_1v$, $s_2 = \tau(v)u_2w_2$, *so that* $g_i = s_ih_i$, $i = 1,2$; *then*

$$|(\delta\mu)(g_1,g_2)-(\delta\mu)(s_1,s_2)| \leqslant 4C. \tag{3.31}$$

(ii) *Let* $t_1 = w_1u_1$, $t_2 = u_2w_2$, *so that* $s_1 = t_1v$, $s_2 = \tau(v)t_2$; *then*

$$|(\delta\mu)(s_1,s_2)-(\delta\mu)(t_1,t_2)| \leqslant 2C. \tag{3.32}$$

(iii) $|(\delta\mu)(t_1,t_2)-(\delta\mu)(u_1,u_2)| \leqslant 4C$.

Proof. (i) We have

$$g_1g_2 = s_1s_2([s_2^{-1}h_1s_2]h_2). \tag{3.33}$$

Using Lemma 3.3(i) and (3.11), we obtain the following two inequalities:

$$\begin{gathered} |\mu(g_1g_2)-\mu(s_1s_2)-\mu(h_1)-\mu(h_2)| \leqslant 2C, \\ |(\mu(g_1)+\mu(g_2))-(\mu(s_1)+\mu(s_2)+\mu(h_1)+\mu(h_2))| \leqslant 2C, \end{gathered} \tag{3.34}$$

from which (i) follows.

(ii) We have

$$s_1s_2 = t_1t_2(t_2^{-1}[v\tau(v)]t_2). \tag{3.35}$$

Using Lemma 3.3(i) and (3.11), we obtain

$$|\mu(s_1s_2)-\mu(t_1t_2)-\mu(v\tau(v))| \leqslant C. \tag{3.36}$$

Since $\tau(s_1) = \tau(v)\tau(t_1)$ and $\tau(s_2) = \tau(t_2)v$, Lemma 3.3(ii) combined with the observation that $\mu(\tau(v)) = \mu(v)$ implies that

$$|(\mu(s_1)+\mu(s_2))-(\mu(t_1)+\mu(t_2)+2\mu(v))| \leqslant C. \tag{3.37}$$

But $\mu(v\tau(v)) = f_0(v\tau(v)) = 2\mu(v)$, and (ii) follows.

(iii) We have

$$t_1t_2 = w_1zw_2(w_2^{-1}hw_2). \tag{3.38}$$

Since $\tau(w_1zw_2) = \tau(w_2)\tau(z)\tau(w_1)$, Lemma 3.3(i) and (ii) and (3.11) imply that

$$|\mu(t_1t_2)-(\mu(w_1)+\mu(z)+\mu(w_2)+\mu(h))| \leqslant 2C, \tag{3.39}$$

and therefore

$$\left|\mu(t_1t_2)-(\mu(w_1)+\mu(zh)+\mu(w_2))\right|\leqslant 3C. \tag{3.40}$$

On the other hand, since $\tau(t_1)=\tau(u_1)\tau(w_1)$ and $\tau(t_2)=\tau(w_2)\tau(u_2)$, we have

$$\left|(\mu(t_1)+\mu(t_2))-(\mu(w_1)+\mu(u_1)+\mu(w_2)+\mu(u_2))\right|\leqslant C. \tag{3.41}$$

Since $zh=u_1u_2$, we obtain (iii). □

Next, we calculate $(\delta\eta)(g_1,g_2)$ using Lemma 3.3(iii), (iv), and (v):

$$\begin{aligned}(\delta\eta)(g_1,g_2)&=[\eta(w_1)+\eta(z)+\eta(w_2)]-[\eta(w_1)+\eta(u_1)+\eta(v)]\\&\quad-[\eta(\tau(v))+\eta(u_2)+\eta(w_2)]=\eta(z)-\eta(u_1)-\eta(u_2)\\&=(\delta\eta)(u_1,u_2).\end{aligned} \tag{3.42}$$

From (3.30) and (3.42), we obtain

$$\left|(\delta F)(g_1,g_2)-(\delta F)(u_1,u_2)\right|\leqslant 10C. \tag{3.43}$$

On the other hand, since both u_1 and u_2 belong to the same factor A_i, (3.15) implies that $(\delta F)(u_1,u_2)$ is bounded, and we finally conclude that $(\delta F)(g_1,g_2)$ is bounded, completing the proof of Proposition 3.2.

3.2. Exactness in the term $PX(A_1 *_N A_2)$. The exactness of (2.11) in $PX(A_1 *_N A_2)$ is based on the following fact.

Lemma 3.5. *Let G be an arbitrary group and N its normal subgroup. If a pseudocharacter $f\in PX(G)$ has zero restriction to N, then it satisfies $f(gh)=f(g)$ for all $h\in N$, $g\in G$. In other words, the natural sequence*

$$0\longrightarrow PX(G/N)\longrightarrow PX(G)\longrightarrow PX(N) \tag{3.44}$$

is exact.

Proof. Suppose that $f(gh)\neq f(g)$ for some $h\in N$, $g\in G$, and let $a=f(gh)-f(g)$. Then

$$\left|f((gh)^n)-f(g^n)\right|=n|a|\longrightarrow\infty\quad\text{when } n\longrightarrow\infty. \tag{3.45}$$

On the other hand, $(gh)^n$ can be written as g^nh' for $h'\in N$, so

$$\left|f((gh)^n)-f(g^n)\right|=\left|f(g^nh')-f(g^n)-f(h')\right| \tag{3.46}$$

is bounded independent of n. □

If $f\in\operatorname{Ker}\beta$, then $f|_N=0$. Lemma 3.5 implies that f factors through the group homomorphism

$$A_1 *_N A_2\longrightarrow(A_1 *_N A_2)/N\cong(A_1/N)*(A_2/N), \tag{3.47}$$

immediately implying that $f \in \operatorname{Im}\alpha$ and proving the inclusion $\operatorname{Ker}\beta \subset \operatorname{Im}\alpha$. The opposite inclusion is obvious.

3.3. Remarks. The general construction of a quasicharacter $F \in QX(A_1 *_N A_2)$ lifting given pseudocharacters $f_i \in PX(A_i)$ $(i = 1,2)$ with the same restrictions to N essentially simplifies in the following two particular cases:

(1) $S = \varnothing$, that is, when the quotients A_1/N and A_2/N do not have elements of order two;

(2) $T = \varnothing$, that is, when these quotients are groups of exponent two.

In the first case, with the above choice of the coset representative systems, F can be extended "by linearity":

$$F(x_1 \cdots x_n h) = \sum_{i=1}^{n} F(x_i) + f_0(h). \tag{3.48}$$

4. The case of an arbitrary subgroup. If we do not assume that the amalgamated subgroup H is normal in both factors A_1 and A_2, then two difficulties arise. First of all, when we switch representatives of cosets modulo H with elements of H in order to write a product of two words in the canonical form, the representative of a coset will change. Secondly, there is no natural candidate for the term on the extreme left. We restrict our attention to special classes of quasicharacters and pseudocharacters which we call *strongly H-spherical* and *H-spherical*, respectively. We would like to point out that the only explicitly known quasicharacters on amalgamated products (see Example 4.4) are strongly H-spherical. Below is a brief analysis of what restrictions should be imposed on pseudocharacters.

Let H be a subgroup of a group G. The first conjecture, which naturally arises after preliminary considerations, is to look at the following class of pseudocharacters:

$$\{f \in PX(G) \mid f(xh) = f(x) + f(h)\ \forall x \in G,\ h \in H\}. \tag{4.1}$$

However, it has to be discarded as the following observation shows.

Lemma 4.1. *Let $f \in PX(G)$ and suppose that*

$$f(xb) = f(x) + f(b) \tag{4.2}$$

holds for all $x \in G$ and all b in a certain subset $B \subset G$. Then (4.2) also holds for all $x \in G$ and all b in the normal subgroup $N \subset G$ generated by B. Moreover, if $f(xb) = f(x)$ for all $x \in G$ and $b \in B$, then the same is true for all $x \in G$ and all $b \in N$.

Proof. It suffices to show that

$$H = \{b \in G \mid f(xb) = f(x) + f(b)\ \forall x \in G\} \tag{4.3}$$

is a normal subgroup of G. If $b_1, b_2 \in H$, then

$$f(x(b_1 b_2)) = f((xb_1)b_2) = f(x) + f(b_1) + f(b_2) = f(x) + f(b_1 b_2) \tag{4.4}$$

for all $x \in G$, so $b_1b_2 \in H$. Similarly, for $b \in H$, we have

$$f(x) = f((xb)b^{-1}) = f(xb) - f(b) = f(xb) + f(b^{-1}) \tag{4.5}$$

which means that

$$f(yb^{-1}) = f(y) + f(b^{-1}) \tag{4.6}$$

for all $y \in G$, that is, $b^{-1} \in H$. Finally, for fixed $b \in H$, $g \in G$, and an arbitrary $x \in G$, we have

$$f(x(gbg^{-1})) = f((g^{-1}xg)b) = f(g^{-1}xg) + f(b) = f(x) + f(gbg^{-1}), \tag{4.7}$$

proving that $gbg^{-1} \in H$, hence our first assertion. The argument for the second assertion is similar: one shows that

$$\{b \in G \mid f(xb) = f(x)\ \forall x \in G\} \tag{4.8}$$

is a normal subgroup of G. □

Corollary 4.2. *Suppose that G has the property that every nontrivial normal subgroup has finite index. If the center of G is trivial, then, given a nonzero pseudocharacter $f \in PX(G)$ and an element $a \in G$, $a \neq e$, there exists $x \in G$ such that $f(xa) \neq f(x) + f(a)$.*

Proof. Assume the contrary. Then, according to Lemma 4.1, the equality $f(xb) = f(x) + f(b)$ holds for all $x \in G$ and all b in $N :=$ the normal subgroup of G generated by a. In particular, the restriction $f|_N$ is a character of N. Moreover, since f is constant on conjugacy classes in G, it vanishes on the commutator subgroup $[G,N]$. Since the center of G is trivial, $[G,N] \neq \{e\}$ and therefore has finite index in G. But then $f|_{[G,N]} = 0$ implies $f = 0$ as required. □

Further analysis leads to the following definition.

Definition 4.3. Let H be a subgroup of a group G. A quasicharacter $F \in QX(G)$ is *strongly H-spherical* if

(i) $F(h_1gh_2) = F(h_1) + F(g) + F(h_2)$ for all $h_1, h_2 \in H$ and $g \in G$;
(ii) $F(g^{-1}) = -F(g)$ for all $g \in G$.

A pseudocharacter $f \in PX(G)$ is *H-spherical* if there exists a strongly H-spherical quasicharacter F such that the difference $f - F$ is bounded.

Example 4.4. We briefly recall the construction of quasicharacters used to prove the result in [7] (a similar construction with a more geometric flavor was discovered independently in [5]) as these are essentially the only known quasicharacters on amalgamated products. A product $u_1 \cdots u_n$ in the amalgamated product $G = A_1 *_H A_2$ is called *reduced* if the following holds:

(i) every u_i belongs to either A_1 or A_2;
(ii) u_i and u_{i+1} belong to different factors A_j, $j = 1,2$;

(iii) if $n > 1$, then none of u_i belongs to H;
(iv) if $n = 1$, then $u_1 \neq 1$.

Grigorchuk's construction is based on the fact that if $u_1 \cdots u_n = v_1 \cdots v_m$ are two reduced products in G, then $n = m$ and, for every $i = 1,\dots,n$, the elements u_i and v_i belong to the same double coset modulo H, which is a simple consequence of the structure theorem for reduced words in amalgamated products. Two words u and v are called generally equal if there exist reduced products $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_m$ such that $n = m$ and, for every $i = 1,\dots,n$, the elements u_i and v_i belong to the same double coset modulo H. A reduced word $w = w_1 \cdots w_k$ is said to *generally occur* in a reduced word $u = u_1 \cdots u_n$ if there is a subword $u_i \cdots u_{i+k-1}$ of u which is generally equal to $w_1 \cdots w_k$. We define $\#_w(u)$ as the number of general occurrences of w in u, and, for any $g \in G$, we let

$$F_w(g) = \#_w(u) - \#_{w^{-1}}(u), \tag{4.9}$$

where u is any reduced word representing g. It turns out that if w is a reduced word, then the function F_w is a quasicharacter of G. In case $|H \backslash A_1 / H| \geqslant 3$ and $[A_2 : H] \geqslant 2$, it is possible to exhibit an infinite sequence of reduced words $\{w_n\}$ such that the quasicharacters $\{F_{w_n}\}$ are linearly independent, whence the infinite dimensionality of the second bounded cohomology group. It is immediate from (4.9) that Grigorchuk's quasicharacters are strongly H-spherical.

Notice that if F is a strongly H-spherical quasicharacter, then the restriction of F to H is a character of H; in particular, $F(1) = 0$. Also, if H_1 and H_2 are subgroups of a group G and F is a strongly H_i-spherical quasicharacter for $i = 1,2$, then F is a strongly H-spherical quasicharacter, where H is the subgroup of G generated by H_1 and H_2. We denote the space of H-spherical pseudocharacters of G by $PX(G)_H$.

In the sequel, we will need the following observation.

Lemma 4.5. *If F is a strongly H-spherical quasicharacter and $g_1 g_2 \in H$ for some $g_1, g_2 \in G$, then $F(g_1 g_2) = F(g_1) + F(g_2)$.*

Proof. Our claim follows from

$$F(g_2) = F(g_1^{-1}) + F(g_1 g_2) = -F(g_1) + F(g_1 g_2). \tag{4.10}$$

□

The canonical (surjective) homomorphism

$$\theta : A_1 * A_2 \longrightarrow A_1 *_H A_2 \tag{4.11}$$

gives rise to the following embedding of the spaces of pseudocharacters:

$$\iota : PX(A_1 *_H A_2) \hookrightarrow PX(A_1 * A_2), \tag{4.12}$$

which allows us to identify the former with a subspace of the latter. We denote the kernel of the linear map

$$\beta : PX(A_1 * A_2) \longrightarrow PX(A_1) \oplus PX(A_2) \tag{4.13}$$

by $PX_0(A_1 * A_2)$. The following analog of Theorem 2.1 holds for H-spherical pseudocharacters in the case when the amalgamated subgroup H is arbitrary.

Theorem 4.6. *Let H be an arbitrary subgroup of A_1 and A_2, let $\theta : A_1 * A_2 \to A_1 *_H A_2$ be the canonical homomorphism, let $\mathcal{H}$ be the subgroup of $A_1 * A_2$ generated by $H * H$ and* $\operatorname{Ker}\theta$, *and let $PX_{0,\operatorname{Ker}\theta}(A_1 * A_2)_{\mathcal{H}}$ be the subspace of $PX_0(A_1 * A_2)_{\mathcal{H}}$ consisting of pseudocharacters with trivial restriction to* $\operatorname{Ker}\theta$. *Then the sequence of vector spaces*

$$\begin{aligned} 0 &\longrightarrow PX_{0,\operatorname{Ker}\theta}(A_1 * A_2)_{\mathcal{H}} \longrightarrow PX(A_1 *_H A_2)_H \\ &\xrightarrow{\beta} PX(A_1)_H \oplus PX(A_2)_H \xrightarrow{\gamma} PX(H) \end{aligned} \tag{4.14}$$

is exact.

5. Proof of Theorem 4.6

5.1. Exactness in the term $PX(A_1)_H \oplus PX(A_2)_H$. To prove the exactness of (4.14) in the term $PX(A_1)_H \oplus PX(A_2)_H$, we need to show that given H-spherical pseudocharacters $f_i \in PX(A_i)_H$, $i = 1,2$, satisfying $f_1|_H = f_2|_H$, there exists an H-spherical pseudocharacter $f \in PX(A_1 *_H A_2)_H$ such that $f|_{A_i} = f_i$. Let $F_i \in QX(A_i)$, $i = 1,2$, be strongly H-spherical quasicharacters with the property that the differences $F_i - f_i$ are bounded; also, let $C = \max\{C_{F_1}, C_{F_2}\}$. An analog of Lemma 3.1 shows that for the existence of f, it suffices to construct a strongly H-spherical quasicharacter $F \in QX(A_1 *_H A_2)$ with the property that the differences $F|_{A_i} - F_i$ are bounded for $i = 1,2$.

Let X_i be an arbitrary system of representatives of left cosets $\neq H$ in A_i/H, $i = 1,2$, and let $X = X_1 \cup X_2$. Similarly to Section 3, we introduce a function F on X whose restriction to X_i is F_i:

$$F(x) = F_i(x) \quad \text{if } x \in X_i, \tag{5.1}$$

and let W be the set of all words of the form $x_1 \cdots x_n$, where $x_i \in X$ and for every $i = 1,\dots,n-1$, the elements x_i and x_{i+1} belong to different parts X_1 or X_2 of X (by convention, the empty word is included in W and corresponds to $n = 0$). Then any element $g \in G := A_1 *_H A_2$ admits a unique canonical presentation of the form

$$g = x_1 \cdots x_n h \tag{5.2}$$

for some $h \in H$ and some word $x_1 \cdots x_n \in W$.

Since the restrictions of f_i to H coincide, the difference $F_1|_H - F_2|_H$ is bounded. However, the restrictions $F_i|_H$, $i = 1,2$, are the characters of H, hence

$$F_1|_H - F_2|_H = 0 \tag{5.3}$$

and we let F_0 denote the common restriction of F_1 and F_2 to H:

$$F_0 := F_1|_H = F_2|_H \in X(H). \tag{5.4}$$

We now extend F to a function on $A_1 *_H A_2$ using the canonical form (5.2):

$$F(g) = F(x_1) + \cdots + F(x_n) + F_0(h). \tag{5.5}$$

To complete the proof of exactness of (4.14) in $PX(A_1)_H \oplus PX(A_2)_H$, it suffices to establish the following.

Proposition 5.1. *The function F defined by (5.5) is a strongly H-spherical quasicharacter of $A_1 *_H A_2$ such that the differences $F|_{A_i} - F_i$ are bounded for $i = 1,2$.*

The property that the differences $F|_{A_i} - F_i$ are bounded for $i = 1,2$ follows immediately from (5.5) (moreover, $F|_{A_i} = F_i$).

Next, we are going to show that F is a quasicharacter of $A_1 *_H A_2$. When we switch a representative of a coset modulo H and an element of H, both of them will change. Since it is necessary to keep track of all these changes, we introduce the following notation: given elements $x \in X_i$ and $h \in H$, there exist elements $x^{\langle h\rangle} \in X_i$ and $h^{\langle x\rangle} \in H$ such that

$$hx = x^{\langle h\rangle} h^{\langle x\rangle}. \tag{5.6}$$

To simplify notation, we will write $h^{\langle x_1,x_2\rangle}$ instead of $(h^{\langle x_1\rangle})^{\langle x_2\rangle}$ and similarly for $x^{\langle h_1,h_2\rangle}$. From (5.6), we derive that

$$F(x) + F(h) = F(x^{\langle h\rangle}) + F(h^{\langle x\rangle}) \tag{5.7}$$

which is a crucial equality in our argument. One of the main consequences of this equality is the following fact which follows from (5.7) by induction on m.

Lemma 5.2. *Let $y_1,\dots,y_m \in X$ and $h \in H$. Then*

$$\begin{aligned} &F\left(y_1^{\langle h\rangle}\right) + F\left(y_2^{\langle h^{\langle y_1\rangle}\rangle}\right) + F\left(y_3^{\langle h^{\langle y_1,y_2\rangle}\rangle}\right) + \cdots + F\left(y_m^{\langle h^{\langle y_1,\dots,y_{m-1}\rangle}\rangle}\right) + F\left(h^{\langle y_1,\dots,y_m\rangle}\right) \\ &\quad = F(y_1) + \cdots + F(y_m) + F(h). \end{aligned} \tag{5.8}$$

Given two elements $g_1, g_2 \in G$, we fix their canonical presentations

$$g_1 = x_1 \cdots x_m h_1, \qquad g_2 = y_1 \cdots y_n h_2, \tag{5.9}$$

where $x_1 \cdots x_m$, $y_1 \cdots y_n \in W$ and $h_1, h_2 \in H$. Suppose first that x_m and y_1 belong to different factors A_i, $i = 1,2$. Then the canonical presentation of $g_1 g_2$ is

$$g_1 g_2 = x_1 \cdots x_m y_1^{\langle h_1\rangle} y_2^{\langle h_1^{\langle y_1\rangle}\rangle} y_3^{\langle h_1^{\langle y_1,y_2\rangle}\rangle} \cdots y_n^{\langle h_1^{\langle y_1,\dots,y_{n-1}\rangle}\rangle} \left(h_1^{\langle y_1,\dots,y_n\rangle} h_2\right) \tag{5.10}$$

and

$$\begin{aligned} |(\delta F)(g_1,g_2)| \leqslant \Bigg| &\left[F\left(y_1^{\langle h_1\rangle}\right) + F\left(y_2^{\langle h_1^{\langle y_1\rangle}\rangle}\right) + \cdots + F\left(y_n^{\langle h_1^{\langle y_1,\dots,y_{n-1}\rangle}\rangle}\right) + F\left(h_1^{\langle y_1,\dots,y_n\rangle}\right)\right] \\ &- \left[F(y_1) + \cdots + F(y_n) + F(h_1)\right]\Bigg| + C. \end{aligned} \tag{5.11}$$

Lemma 5.2 implies that, in this case, $|(\delta F)(g_1,g_2)| \leqslant C$.

In the general case, there might be some cancelation in the middle in the product $g_1 g_2$, and we indicate several steps to write the canonical form of $g_1 g_2$ in a convenient way. First, we write it in the form (which is not a canonical form in general)

$$g_1g_2 = x_1\cdots x_m y_1^{\langle h_1\rangle} y_2^{\langle h_1^{\langle y_1\rangle}\rangle} y_3^{\langle h_1^{\langle y_1,y_2\rangle}\rangle}\cdots y_n^{\langle h_1^{\langle y_1,\dots,y_{n-1}\rangle}\rangle}\left(h_1^{\langle y_1,\dots,y_n\rangle}h_2\right) \tag{5.12}$$

and let

$$\begin{gathered} z_1 = y_1^{\langle h_1\rangle},\ z_2 = y_2^{\langle h_1^{\langle y_1\rangle}\rangle},\dots,\ z_n = y_n^{\langle h_1^{\langle y_1,\dots,y_{n-1}\rangle}\rangle} \in X, \\ h_0 = h_1^{\langle y_1,\dots,y_n\rangle}h_2 \in H, \end{gathered} \tag{5.13}$$

then (5.12) becomes

$$g_1g_2 = x_1\cdots x_m z_1\cdots z_n h_0. \tag{5.14}$$

It remains to consider the case when x_m and z_1 belong to the same factor A_i; then

$$x_m z_1 = u_1 a_1, \tag{5.15}$$

where $u_1 \in X$ or $u_1 = e$ and $a_1 \in H$. If $u_1 \in X$, then

$$g_1g_2 = x_1\cdots x_{m-1}u_1 z_2^{\langle a_1\rangle} z_3^{\langle a_1^{\langle z_2\rangle}\rangle} z_4^{\langle a_1^{\langle z_2,z_3\rangle}\rangle}\cdots z_n^{\langle a_1^{\langle z_2,\dots,z_{n-1}\rangle}\rangle}\left(a_1^{\langle z_2,\dots,z_n\rangle}h_0\right) \tag{5.16}$$

is the canonical form of g_1g_2. If $u_1 = e$, then

$$g_1g_2 = x_1\cdots x_{m-1}a_1 z_2\cdots z_n h_0 = x_1\cdots x_{m-1} z_2^{\langle a_1\rangle} a_1^{\langle z_2\rangle} z_3\cdots z_n h_0. \tag{5.17}$$

Notice that we do not transfer a_1 all the way to the right. Since x_{m-1} and z_2 must belong to the same X_i, we next write

$$x_{m-1} z_2^{\langle a_1\rangle} a_1^{\langle z_2\rangle} = u_2 a_2, \tag{5.18}$$

where $u_2 \in X$ or $u_2 = e$ and $a_2 \in H$. We continue this process until we find a positive integer k such that

$$x_{m-j+1} z_j^{\langle a_{j-1}\rangle} a_{j-1}^{\langle z_j\rangle} = a_j \in H \quad \text{for } 2 \leqslant j \leqslant k-1, \tag{5.19}$$

but

$$x_{m-k+1} z_k^{\langle a_{k-1}\rangle} a_{k-1}^{\langle z_k\rangle} = u_k a_k, \tag{5.20}$$

where $u_k \in X$ and $a_k \in H$. Then the canonical form of g_1g_2 is

$$g_1g_2 = x_1\cdots x_{m-k}u_k z_{k+1}^{\langle a_k\rangle} z_{k+2}^{\langle a_k^{\langle z_{k+1}\rangle}\rangle}\cdots z_n^{\langle a_k^{\langle z_{k+1},\dots,z_{n-1}\rangle}\rangle}\left(a_k^{\langle z_{k+1},\dots,z_n\rangle}h_0\right). \tag{5.21}$$

Before we can estimate $|(\delta F)(g_1,g_2)|$, we need the following fact.

Lemma 5.3. *In the current notation,*

$$\left|F(u_k)+F(a_k)-\left[F(x_{m-k+1})+\cdots+F(x_m)+F(z_1)+\cdots+F(z_k)\right]\right| \leqslant C. \tag{5.22}$$

Proof. Since F_i, $i=1,2$, are strongly H-spherical quasicharacters, then, using (5.7), we conclude that

$$\begin{aligned} F(u_k)+F(a_k) &= F_i(u_k)+F_i(a_k) = F_i(u_k a_k) \\ &= F_i\left(x_{m-k+1} z_k^{\langle a_{k-1}\rangle} a_{k-1}^{\langle z_k\rangle}\right) \\ &= F_i\left(x_{m-k+1} z_k^{\langle a_{k-1}\rangle}\right) + F\left(a_{k-1}^{\langle z_k\rangle}\right) \\ &= F_i(a_{k-1})+F_i(x_{m-k+1})+F_i(z_k) \\ &\quad + \left[F_i\left(x_{m-k+1} z_k^{\langle a_{k-1}\rangle}\right) - F_i(x_{m-k+1}) - F_i\left(z_k^{\langle a_{k-1}\rangle}\right)\right]. \end{aligned} \tag{5.23}$$

For $2 \leqslant j \leqslant k-1$, $x_{m-j+1} z_j^{\langle a_{j-1}\rangle} \in H$ by (5.19), so Lemma 4.5 and (5.7) imply that

$$\begin{aligned} F_i(a_j) &= F_i\left(x_{m-j+1} z_j^{\langle a_{j-1}\rangle} a_{j-1}^{\langle z_j\rangle}\right) \\ &= F_i\left(x_{m-j+1} z_j^{\langle a_{j-1}\rangle}\right) + F_i\left(a_{j-1}^{\langle z_j\rangle}\right) \\ &= F_i(x_{m-j+1}) + F_i\left(z_j^{\langle a_{j-1}\rangle}\right) + F_i\left(a_{j-1}^{\langle z_j\rangle}\right) \\ &= F_i(x_{m-j+1}) + F_i(z_j) + F_i(a_{j-1}). \end{aligned} \tag{5.24}$$

Finally,

$$F_i(a_1) = F_i(x_m z_1) = F_i(x_m) + F_i(z_1) \tag{5.25}$$

by Lemma 4.5. □

We obtain the following inequalities which show that $F \in QX(A_1 *_H A_2)$:

$$\begin{aligned} \left|(\delta F)(g_1,g_2)\right| &\leqslant \big|\left[F(x_1)+\cdots+F(x_{m-k})+F(u_k)+F(a_k)\right. \\ &\qquad \left. +F(z_{k+1})+\cdots+F(z_n)+F(h_0)\right] \\ &\qquad -\left[F(x_1)+\cdots+F(x_m)+F(h_1)\right] \\ &\qquad -\left[F(y_1)+\cdots+F(y_n)+F(h_2)\right]\big| + C \quad \text{(Lemma 5.2)} \\ &\leqslant \big|\left[F(z_1)+\cdots+F(z_n)+F(h_0)\right] \\ &\qquad -\left[F(y_1)+\cdots+F(y_n)+F(h_2)+F(h_1)\right]\big| + 2C \quad \text{(Lemma 5.3)} \\ &\leqslant 3C \quad \text{(Lemma 5.2)}. \end{aligned} \tag{5.26}$$

To finish the proof of Proposition 5.1, it remains to show that F satisfies Definition 4.3(i) and (ii).

To prove (i), we write an arbitrary $g \in G$ in the canonical form $g = x_1 \cdots x_n h$; then, for any $h_1, h_2 \in H$, the canonical form of $h_1 g h_2$ is

$$x_1^{\langle h_1\rangle} x_2^{\langle h_1^{\langle x_1\rangle}\rangle} \cdots x_n^{\langle h_1^{\langle x_1,\dots,x_{n-1}\rangle}\rangle}\left(h_1^{\langle x_1,\dots,x_n\rangle} h h_2\right). \tag{5.27}$$

Since the restriction of F to H is a character of H, we obtain

$$F\left(h_1^{\langle x_1,\dots,x_n\rangle} h h_2\right) = F\left(h_1^{\langle x_1,\dots,x_n\rangle}\right) + F(h) + F(h_2) \tag{5.28}$$

and Lemma 5.2 implies that

$$\begin{aligned} F(h_1gh_2) &= F(x_1)+F(x_2)+\cdots+F(x_n)+F(h_1)+F(h)+F(h_2) \\ &= F(h_1)+F(g)+F(h_2) \end{aligned} \tag{5.29}$$

as required.

To prove (ii), we first suppose that the canonical form of $g \in G$ is $x_1\cdots x_n$, that is, there is no H-component. Then the canonical form of g^{-1} is $x_n^{-1}\cdots x_1^{-1}$ and

$$F(g^{-1}) = F_i(x_n^{-1})+\cdots+F_i(x_1^{-1}) = -F_i(x_n)-\cdots-F_i(x_1) = -F(g). \tag{5.30}$$

In the general case, write $g = g_0h$, where the canonical form of g_0 has no H-component. Since we have already shown that F satisfies (i) of strongly H-spherical quasicharacters, we use (5.30) to obtain

$$F(g^{-1}) = F(h^{-1}g_0^{-1}) = F(h^{-1})+F(g_0^{-1}) = -F(h)-F(g_0) = -F(g) \tag{5.31}$$

as required.

5.2. Exactness in the term $PX(A_1 *_H A_2)_H$. Given $f \in PX_0(A_1 *_H A_2)_H = \operatorname{Ker}\beta$, we let F denote the corresponding strongly H-spherical quasicharacter of $A_1 *_H A_2$. We claim that $\tilde{F} := F\circ\theta$ is a strongly $\mathcal{H}$-spherical quasicharacter of $A_1 * A_2$. Indeed, the boundedness of $\delta\tilde{F}$ follows from that of δF and, moreover, $\tilde{F}$ is both strongly $(H*H)$-spherical (since $\theta(H*H) = H$) and strongly $\operatorname{Ker}\theta$-spherical, hence strongly $\mathcal{H}$-spherical. There exists a pseudocharacter $\tilde{f} \in PX(A_1 * A_2)$ such that the difference $\tilde{F}-\tilde{f}$ is bounded. Thus, $\tilde{f}$ is an $\mathcal{H}$-spherical pseudocharacter of $A_1 * A_2$ and, clearly,

$$\tilde{f}|_{A_1} = \tilde{f}|_{A_2} = \tilde{f}|_{\operatorname{Ker}\theta} = 0. \tag{5.32}$$

This shows the inclusion $\operatorname{Ker}\beta \subset PX_{0,\operatorname{Ker}\theta}(A_1 * A_2)_{\mathcal{H}}$.

For the opposite inclusion, we consider $\tilde{f} \in PX_{0,\operatorname{Ker}\theta}(A_1 * A_2)_{\mathcal{H}}$ and let $\tilde{F}$ be the corresponding strongly $\mathcal{H}$-spherical quasicharacter. Then

$$\tilde{F}(gx) = \tilde{F}(g)+\tilde{F}(x) \quad \forall g \in A_1 * A_2,\ x \in \operatorname{Ker}\theta. \tag{5.33}$$

Since $\tilde{f}|_{\operatorname{Ker}\theta} = 0$, the restriction $\tilde{F}|_{\operatorname{Ker}\theta}$ is bounded. But $\operatorname{Ker}\theta \subset \mathcal{H}$ and $\tilde{F}$ is a character of $\mathcal{H}$. We conclude that $\tilde{F}|_{\operatorname{Ker}\theta} = 0$, and thus

$$\tilde{F}(gx) = \tilde{F}(g) \quad \forall g \in A_1 * A_2,\ x \in \operatorname{Ker}\theta. \tag{5.34}$$

Therefore there exists a function F on $A_1 *_H A_2$ such that $\tilde{F} = F\circ\theta$. It is immediate that F is a strongly H-spherical quasicharacter of $A_1 *_H A_2$ with bounded restrictions to A_1 and A_2. Hence we can construct an H-spherical pseudocharacter f of $A_1 *_H A_2$ whose restrictions to A_1 and A_2 are trivial. Therefore

$$\operatorname{Ker}\beta \supset PX_{0,\operatorname{Ker}\theta}(A_1 * A_2)_{\mathcal{H}} \tag{5.35}$$

and the proof of Theorem 4.6 is complete.

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