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COMPLETE CONVERGENCE FOR ARRAYS OF MINIMAL ORDER STATISTICS

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For arrays of independent Pareto random variables, this paper establishes complete convergence for weighted partial sums for the smaller order statistics within each row. This result improves on past strong laws. Moreover, it shows that we can obtain a finite nonzero limit for our normalized partial sums under complete convergence even though the first moment of our order statistics is infinite.

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Let $\{X_{nj},\ 1 \le j \le m_n,\ n \ge 1\}$ be independently distributed random variables with density $f_{X_{nj}}(x) = p_n x^{-p_n-1} I\ (x \ge 1)$, where $p_n > 0$. Let $X_{n(k)}$ be the kth smallest order statistic from each row of our array. Thus the density of $X_{n(k)}$ is

$$f_{X_{n(k)}}(x) = \frac{p_n \cdot m_n!}{(k-1)!(m_n-k)!} x^{-p_n(m_n-k+1)-1}\left(1-x^{-p_n}\right)^{k-1} I \quad (x \ge 1). \tag{1}$$

We will establish laws of large numbers of the form

$$\sum_{N=k}^{\infty} c_N P\left\{\left|\frac{\sum_{n=k}^{N} a_n X_{n(k)}}{b_N} - L\right| > \epsilon\right\} < \infty \tag{2}$$

for all $\epsilon > 0$, where L is not zero even though $EX_{n(k)} = \infty$ and of course $\sum_{N=k}^{\infty} c_N = \infty$. In order to have $EX_{n(k)} = \infty$, we need $p_n \to 0$ whenever $m_n \to \infty$. For results on fixed sample sizes, see [2]. Strangely the theorems involving the larger order statistics proved to be much simpler to prove than the corresponding theorems involving the smaller order statistics, see [1].

This type of strong law is part of the fair games problem. The idea is to balance sums of random variables with a sequence of constants. The random variables $a_n X_{n(k)}$ can be considered as winnings from a game and the sequence of constants $\{b_N,\ N \ge 1\}$ as the cumulative entrance fee. We try to make the limit of the ratio of these two approaches a nonzero constant. This can never happen when $a_n = 1$. That is why we must examine weighted sums of random variables. Having a nonzero limit makes the game equable to both the house and the gambler. Otherwise, one of them would have an unfair advantage.

We use the partition

$$\begin{aligned} b_N^{-1}\sum_{n=k}^{N} a_n X_{n(k)} &= b_N^{-1}\sum_{n=k}^{N} a_n\left[X_{n(k)} I\left(X_{n(k)} \le d_n\right) - E X_{n(k)} I\left(X_{n(k)} \le d_n\right)\right] \\ &\quad + b_N^{-1}\sum_{n=k}^{N} a_n X_{n(k)} I\left(X_{n(k)} > d_n\right) \\ &\quad + b_N^{-1}\sum_{n=k}^{N} a_n E X_{n(k)} I\left(X_{n(k)} \le d_n\right), \end{aligned} \tag{3}$$

where $d_n = b_n/a_n$, in order to prove Theorem 2. Most of our proof will be devoted to showing that the middle term of (3) vanishes. In order to achieve this, we will use a result due to [4]. Their theorem is as follows.

Theorem 1. *Let* $\{(Y_{Nn},\ 1 \le n \le n_N),\ N \ge 1\}$ *be an array of rowwise independent random variables and* $\{c_N,\ N \ge 1\}$ *a sequence of positive constants such that* $\sum_{N=1}^{\infty} c_N = \infty$*. Suppose that for all* $\epsilon > 0$ *and some* $\delta > 0$*,*

(i) $\sum_{N=1}^{\infty} c_N \sum_{n=1}^{n_N} P\{|Y_{Nn}| > \epsilon\} < \infty$,

(ii) *there exists* $J \ge 2$ *such that* $\sum_{N=1}^{\infty} c_N \left(\sum_{n=1}^{n_N} E Y_{Nn}^2 I(|Y_{Nn}| \le \delta)\right)^J < \infty$,

(iii) $\sum_{n=1}^{n_N} E Y_{Nn} I(|Y_{Nn}| \le \delta) \to 0$ *as* $N \to \infty$*.*

Then $\sum_{N=1}^{\infty} c_N P\{|\sum_{n=1}^{n_N} Y_{Nn}| > \epsilon\} < \infty$ *for all* $\epsilon > 0$*.*

It should be noted that if $p_n(m_n - k + 1) > 1$, then $EX_{n(k)}$ exists, hence classical strong laws exist. When $p_n(m_n - k + 1) < 1$ in past papers, see [2], it was shown that not even an exact weak law can hold. So our concern was to establish a result when $p_n(m_n - k + 1) = 1$. This work started out as an attempt to extend the results that can be found in [3]. That paper established exact strong laws for weighted sums for the smallest order statistics from a Pareto distribution. The first result in that paper obtained a strong law for the first order statistic in each row, no matter how slow or fast our sample size grew. Unfortunately, that result could not be extended to the mode of convergence here. However, the main theorem in that paper can be extended.

Before we establish our results, we need a few comments. As for notation, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. Also, the constant C will denote a generic real number that is not necessarily the same in each appearance.

Theorem 2. *If* $m_n = [\![\lg n]\!]$, $p_n(m_n - k + 1) = 1$, $\alpha + k > 0$, *and* $a > 1$, *then for all* $\epsilon > 0$,

$$\sum_{N=k}^{\infty} \frac{1}{N(\lg_2 N)^a} P\left\{\left|\frac{\sum_{n=k}^{N}\left((\lg n)^{\alpha}/n\right) X_{n(k)}}{(\lg N)^{\alpha+k+1}} - \frac{\gamma_k}{\alpha+k+1}\right| > \epsilon\right\} < \infty, \tag{4}$$

where

$$\gamma_k = \frac{1}{(k-1)!}\left[1+\sum_{j=1}^{k-1}\frac{\binom{k-1}{j}(-1)^{j+1}}{je^j}-\sum_{j=1}^{k-1}\frac{1}{j}\right]. \tag{5}$$

Proof. We set $a_n = (\lg n)^\alpha/n$, $b_n = (\lg n)^{\alpha+k+1}$, $d_n = b_n/a_n = n(\lg n)^{k+1}$, and $c_N = 1/(N(\lg_2 N)^a)$. From [3] we see that the last term in (3) converges to $\gamma_k/(\alpha+k+1)$.

As for the first term in (3), we use Chebyshev's inequality. Setting

$$W_{nk} = X_{n(k)}I(X_{n(k)} \le d_n) - EX_{n(k)}I(X_{n(k)} \le d_n), \tag{6}$$

we have, since $\alpha+k>0$,

$$\begin{aligned}
&\sum_{N=k}^{\infty} c_N P\left\{\left|\sum_{n=k}^{N} a_n W_{nk}\right| > \epsilon b_N\right\} \\
&\quad < C\sum_{N=k}^{\infty}\frac{c_N}{b_N^2}\sum_{n=k}^{N} a_n^2 EX_{n(k)}^2 I(X_{n(k)} \le d_n) \\
&\quad < C\sum_{N=k}^{\infty}\frac{c_N}{b_N^2}\sum_{n=k}^{N} a_n^2 \int_1^{d_n}\frac{m_n\cdots(m_n-k+2)}{(k-1)!}dx \\
&\quad < C\sum_{N=k}^{\infty}\frac{c_N}{b_N^2}\sum_{n=k}^{N} a_n^2 m_n^{k-1} d_n \\
&\quad < C\sum_{N=k}^{\infty}\frac{c_N}{(\lg N)^{2\alpha+2k+2}}\sum_{n=k}^{N}\frac{(\lg n)^{2\alpha}}{n^2}(\lg n)^{k-1} n(\lg n)^{k+1} \\
&\quad = C\sum_{N=k}^{\infty}\frac{c_N}{(\lg N)^{2\alpha+2k+2}}\sum_{n=k}^{N}\frac{(\lg n)^{2\alpha+2k}}{n} \\
&\quad < C\sum_{N=k}^{\infty}\frac{c_N}{(\lg N)^{2\alpha+2k+2}}(\lg N)^{2\alpha+2k+1} \\
&\quad = C\sum_{N=k}^{\infty}\frac{c_N}{\lg N} \\
&\quad = C\sum_{N=k}^{\infty}\frac{1}{N\lg N(\lg_2 N)^a} \\
&\quad < \infty.
\end{aligned} \tag{7}$$

As for the middle term of (3), we will use Theorem 1 with

$$Y_{Nn} = \frac{a_n X_{n(k)} I(X_{n(k)} > d_n)}{b_N}. \tag{8}$$

Note that in proving (i) we can, without any loss of generality, let $0<\epsilon<1$. Observe that there exist $0<\gamma_1<\gamma_2<1$ such that

$$\begin{aligned}
&\sum_{N=k}^{\infty} c_N \sum_{n=k}^{N} P\{|Y_{Nn}|>\epsilon\} \\
&\quad= \sum_{N=k}^{\infty} c_N \sum_{n=k}^{N} P\left\{X_{n(k)}>\max\left\{d_n,\frac{\epsilon b_N}{a_n}\right\}\right\} \\
&\quad< \sum_{N=k}^{\infty} c_N \left[\sum_{n=k}^{\gamma_2 N} P\left\{X_{nk}>\frac{\epsilon b_N}{a_n}\right\}+\sum_{n=\gamma_1 N}^{N} P\{X_{nk}>d_n\}\right] \\
&\quad< \sum_{N=k}^{\infty} c_N \left[\sum_{n=k}^{\gamma_2 N} m_n^{k-1}\int_{\epsilon b_N/a_n}^{\infty} x^{-2}dx+\sum_{n=\gamma_1 N}^{N} m_n^{k-1}\int_{d_n}^{\infty} x^{-2}dx\right] \\
&\quad< C\sum_{N=k}^{\infty} c_N \left[\sum_{n=k}^{\gamma_2 N}\frac{a_n m_n^{k-1}}{b_N}+\sum_{n=\gamma_1 N}^{\infty}\frac{m_n^{k-1}}{d_n}\right] \\
&\quad< C\sum_{N=k}^{\infty} c_N \left[\frac{\sum_{n=k}^{\gamma_2 N}((\lg n)^{\alpha+k-1}/n)}{(\lg N)^{\alpha+k+1}}+\sum_{n=\gamma_1 N}^{\infty}\frac{1}{n(\lg n)^2}\right] \\
&\quad< C\sum_{N=k}^{\infty} c_N \left[\frac{(\lg N)^{\alpha+k}}{(\lg N)^{\alpha+k+1}}+\frac{1}{\lg(\gamma_1 N)}\right] \\
&\quad< C\sum_{N=k}^{\infty}\frac{c_N}{\lg N}=C\sum_{N=k}^{\infty}\frac{1}{N\lg N(\lg_2 N)^{a}}<\infty
\end{aligned} \tag{9}$$

since $a>1$.

We next establish (ii) with $J=2$ and $\delta=1$. Thus

$$\begin{aligned}
&\sum_{N=k}^{\infty} c_N \left(\sum_{n=k}^{N}\frac{a_n^2}{b_N^2}EX_{n(k)}^2 I\left(d_n<X_{n(k)}\le\frac{b_N}{a_n}\right)\right)^2 \\
&\quad< \sum_{N=k}^{\infty} c_N \left(\sum_{n=k}^{N}\frac{a_n^2 m_n^{k-1}}{b_N^2}\int_1^{b_N/a_n} dx\right)^2 \\
&\quad< \sum_{N=k}^{\infty} c_N \left(\sum_{n=k}^{N}\frac{a_n m_n^{k-1}}{b_N}\right)^2 \\
&\quad< C\sum_{N=k}^{\infty} c_N \left(\frac{\sum_{n=k}^{N}((\lg n)^{\alpha+k-1}/n)}{(\lg N)^{\alpha+k+1}}\right)^2 \\
&\quad< C\sum_{N=k}^{\infty} c_N \left(\frac{(\lg N)^{\alpha+k}}{(\lg N)^{\alpha+k+1}}\right)^2 \\
&\quad= C\sum_{N=k}^{\infty}\frac{c_N}{(\lg N)^2} \\
&\quad= C\sum_{N=k}^{\infty}\frac{1}{N(\lg N)^2(\lg_2 N)^{a}}<\infty.
\end{aligned} \tag{10}$$

Finally, we show that (iii) holds, where once again $\delta = 1$. Hence

$$\begin{aligned}
&\sum_{n=k}^{N} EY_{Nn} I(|Y_{Nn}| \le 1) \\
&\quad = \sum_{n=k}^{N} \frac{a_n}{b_N} EX_{n(k)} I\left(d_n < X_{n(k)} \le \frac{b_N}{a_n}\right) \\
&\quad < \frac{1}{b_N} \sum_{n=k}^{N} a_n m_n^{k-1} \int_{d_n}^{b_N/a_n} x^{-1} dx \\
&\quad = \frac{1}{b_N} \sum_{n=k}^{N} a_n m_n^{k-1} [\lg(b_N) - \lg(a_n) - \lg(d_n)] \\
&\quad < \frac{C}{(\lg N)^{\alpha+k+1}} \sum_{n=k}^{N} \frac{(\lg n)^{\alpha+k-1}}{n} [(\alpha+k+1)\lg_2 N + \lg n - \alpha \lg_2 n - \lg n + (k+1)\lg_2 n] \\
&\quad = \frac{C}{(\lg N)^{\alpha+k+1}} \sum_{n=k}^{N} \frac{(\lg n)^{\alpha+k-1}}{n} [(\alpha+k+1)\lg_2 N + (k-\alpha+1)\lg_2 n] \\
&\quad < \frac{C \lg_2 N}{(\lg N)^{\alpha+k+1}} \sum_{n=k}^{N} \frac{(\lg n)^{\alpha+k-1}}{n} \\
&\quad < \frac{C \lg_2 N}{(\lg N)^{\alpha+k+1}} (\lg N)^{\alpha+k} \\
&\quad = \frac{C \lg_2 N}{\lg N} \longrightarrow 0
\end{aligned} \tag{11}$$

as $N \to \infty$. Therefore, via Theorem 1, the second term of (3) converges to zero, which completes the proof. □

It should be pointed out how delicate these proofs are. Just observe how we needed the lower bound of d_n in order to have $\sum_{n=k}^{N} EY_{Nn} I(|Y_{Nn}| \le 1) \to 0$. If we were not able to cancel the $\lg n$ from the $\lg(a_n)$ term via the $\lg(d_n)$ term, this proof would fail.

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