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UNITS IN FAMILIES OF TOTALLY COMPLEX ALGEBRAIC NUMBER FIELDS

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Multidimensional continued fraction algorithms associated with $GL_n(\mathbb{Z}_K)$, where $\mathbb{Z}_k$ is the ring of integers of an imaginary quadratic field K, are introduced and applied to find systems of fundamental units in families of totally complex algebraic number fields of degrees four, six, and eight.

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1. Introduction. Let F be an algebraic number field of degree n. There exist exactly n field embeddings of F in $\mathbb{C}$. Let s be the number of embeddings of F whose images lie in $\mathbb{R}$, and let $2t$ be the number of nonreal complex embeddings, so that $n = s+2t$. The pair (s,t) is said to be the *signature* of F. Let $\mathbb{Z}_F$ be the ring of integers of the field F. A *unit* in F is an invertible element of $\mathbb{Z}_F$. The set of units in F forms a multiplicative group which will be denoted by $\mathbb{Z}_F^\times$. In 1840, P. G. Lejeune-Dirichlet determined the structure of the group $\mathbb{Z}_F^\times$. He showed that $\mathbb{Z}_F^\times$ is a finitely generated Abelian group of rank $r = s+t-1$, that is, $\mathbb{Z}_F^\times$ is isomorphic to $\mu_F \times \mathbb{Z}^r$, where μ_F is a finite cyclic group. μ_F is called the *torsion* subgroup of $\mathbb{Z}_F^\times$. Thus, there exist units $\epsilon_1,\dots,\epsilon_r$ such that every element of $\mathbb{Z}_F^\times$ can be written in a unique way as $\zeta\epsilon_1^{n_1}\cdots\epsilon_r^{n_r}$, where $n_i \in \mathbb{Z}$ and ζ is a root of unity in F. Such a set $\{\epsilon_1,\dots,\epsilon_r\}$ is called a *system of fundamental units* of F. Finding a system of fundamental units of F is one of the main computational problems of algebraic number theory (see, e.g., [4, page 217]). Much work has been done to solve this problem for certain classes of algebraic number fields (see, e.g., [11]). In the case of the real quadratic fields, the continued fraction algorithm provides a very efficient method for solving this problem (see, e.g., [11, page 119]). This approach goes back to L. Euler, who applied continued fractions to solve Pell's equation $x^2 - dy^2 = \pm 1$. (If a square-free positive integer $d \equiv 2$ or $3 \bmod 4$ and x, y is an integral solution of this equation, then $x+\sqrt{d}y$ is a unit in the real quadratic field $\mathbb{Q}(\sqrt{d})$. Moreover, any unit in $\mathbb{Q}(\sqrt{d})$ can be obtained this way.) Many attempts have been made to develop a similar algorithm that would find a system of fundamental units in other algebraic number fields. In the case of a cubic field, one of the most successful such algorithms was introduced by Voronoi [16]. A review of the multidimensional continued fraction algorithms and their properties that were known by 1980 can be found in [1].

Let $d > 0$ be a square-free integer. Let $\mathbb{Z}_K$ be the ring of integers of the field $K = \mathbb{Q}(\sqrt{-d})$. The group of units μ_K of K is a finite cyclic group of order 6 if $d = 3$, 4 if $d = 1$, and $\mu_K = \{\pm 1\}$ otherwise. Let $\omega = (1+\sqrt{-d})/2$ if $d \equiv 3 (\bmod 4)$ and $\omega = \sqrt{-d}$

otherwise. Then $\{1,\omega\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_K$. Let F/K be a relative extension of relative degree n, so that the signature of F is $(0,n)$.

In [24], the (multidimensional continued fraction) Algorithm II associated with the discrete group $\mathrm{GL}_n(\mathbb{Z})/\{\pm 1\}$ acting on the symmetric space $\mathcal{P}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ was introduced and applied to the problem of finding a system of fundamental units in an algebraic number field. In the present paper, an analog of Algorithm II associated with the group $\Gamma = \mathrm{GL}_n(\mathbb{Z}_K)/\mu_K$ acting on the Hermitian symmetric space $\mathcal{H} = \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ is applied to the problem of finding a system of fundamental units in the relative extension F/K and in the field F. The space $\mathcal{H}$ can be identified with the set of positive definite Hermitian forms in n complex variables with the leading coefficient one. Denote by $\hat{X}$ the positive definite quadratic form in $2n$ real variables associated with a Hermitian form $X \in \mathcal{H}$. The set $\{\hat{X} : X \in \mathcal{H}\}$ is a totally geodesic submanifold of $\mathcal{P}_{2n}$ of dimension n^2-1 (see, e.g., [2, Chapter II.10]).

Assume that $g \in \mathrm{GL}_n(\mathbb{C})$. Let $ga_i = \lambda_i a_i$, $i = 1,\dots,n$, so that a_i is an eigenvector of g corresponding to its eigenvalue λ_i. For simplicity, assume that all the eigenvalues of g are distinct. Let $P = (a_1,\dots,a_n)$ be the matrix with columns $a_1,\dots,a_n$. The set of points in $\mathcal{H}$ fixed by g will be called the *axis* L_P of g. The axis L_P of g depends only on eigenvectors of g, that is, on P, but not on its eigenvalues (see Section 3). L_P is a totally geodesic submanifold of $\mathcal{H}$ of dimension $n-1$.

In Section 2, the notion of the height of a point in $\mathcal{H}$ is introduced. Let $w = (1,0,\dots,0)^T$ and $W = ww^T$. In what follows, the point W which belongs to the boundary of $\mathcal{H}$ is analogous to the point ∞ in the upper half-space model $H^{n+1} = \{(z,t) : z \in \mathbb{R}^n,\ t > 0\}$ of the $(n+1)$-dimensional hyperbolic space (see [21, 22]). The set $K_n = K(w)$ in $\mathcal{H}$ is defined so that, for every point $X \in \mathcal{H}$, the points in the Γ-orbit of X with the largest height belong to $K(w)$. The images $K_n[g]$ of K_n, $g \in \Gamma$, under the action of Γ form the *K-tessellation* of $\mathcal{H}$. The K-tessellation of $\mathcal{H}$ is Γ-invariant.

If $L_P \cap K_n[g] \neq \emptyset$, $g \in \Gamma$, then the vector $u = g^{-1}w \in \mathbb{Z}_K^n$ is called a *convergent* of L_P. In Section 3, it is shown that if u is a convergent of L_P, then $|\langle a_1,u\rangle \cdots \langle a_n,u\rangle/\det P|$, where $\langle\cdot,\cdot\rangle$ denotes the complex dot product in $\mathbb{C}^n$, is small (Theorem 3.3). Algorithm II, which is introduced in [24], can be applied in $\mathcal{H}$ to find the sets $R(g^{-1}w) = L_P \cap K_n[g] \neq \emptyset$, which form a tessellation of L_P, and the set of convergents of L_P.

It is proved in Section 4 that a system of fundamental units in the relative extension F/K is a system of fundamental units in the field F provided $\mathbb{Z}_{F/K}$ is a free $\mathbb{Z}_K$-module.

The upper half-space $H^3 = \{(z,t) : z \in \mathbb{C},\ t > 0\}$ with the metric $ds^2 = t^{-2}(|dz|^2 + dt^2)$ can be used as a model of the three-dimensional hyperbolic space. $\mathrm{SL}_2(\mathbb{C})$ is the group of orientation-preserving isometries of H^3. In Section 5, for $n = 2$, a bijection ψ of $\mathcal{H}$ and H^3 is introduced, so that ψ is also a bijection between the K-tessellations of $\mathcal{H}$ and H^3. Thus, Algorithm I from [21] in H^3 coincides with Algorithm II from [24] in $\mathcal{H}$ in this case. In Examples 5.3, 5.4, and 5.6, Algorithm I is applied to find fundamental units in some families of number fields with signature $(0,2)$.

If $g \in \Gamma = \mathrm{GL}_n(\mathbb{Z}_K)/\mu_K$, then there are only finitely many sets $R(u)$ which are not congruent modulo the action of Γ. The union of noncongruent sets $R(u)$ forms a fundamental domain of Γ_L in L_P. Assume that the characteristic polynomial $p(x)$ of g is irreducible over K. Let $p(\epsilon) = 0$. In Section 6, the problem of finding a system of fundamental units in F/K is solved for some families of fields $F = \mathbb{Q}(\epsilon)$ with signature $(0,n)$, $n \le 4$, by reducing it, as explained in Section 4, to the problem of finding a set of generators of Γ_L. Here, the families of fields with signature $(0,n)$ are obtained from some families of fields with signature $(n,0)$ by complexification, that is, by replacing a real parameter $t \in \mathbb{Z}$ by a nonreal complex parameter $m \in \mathbb{Z}_K$.

In Example 5.3 (and, for $\delta = -1$, in Example 6.1), the following result is obtained.

Theorem 1.1. *Let d be a square-free positive integer and let $K = \mathbb{Q}(\sqrt{-d})$. Let $\{1,\omega\}$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}_K$. Let $p(x) = x^2 - mx + \delta$, where nonreal $m \in \mathbb{Z}_K$, $|m| \ge 4$, and $\delta \in \mathbb{Z}_K^\times$. Assume that either $m^2 - 4\delta$ or $m^2/4 - \delta$ is a square-free ideal in $\mathbb{Z}_K$. Let $p(\epsilon) = 0$ and $F = \mathbb{Q}(\epsilon)$.*

Then $\{1,\omega,\epsilon^{-1},\epsilon^{-1}\omega\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_F$ and $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon\rangle$.

Similar results (Theorems 5.5 and 5.7) are proved in Examples 5.4 and 5.6, where $\Gamma = B_d/\mu_K$ and B_d is the extended Bianchi group (see [19, 20]). Complexification of the family of simplest cubic fields of Shanks [15] leads to the following result obtained in Example 6.2.

Theorem 1.2. *Let d be a square-free positive integer and let $K = \mathbb{Q}(\sqrt{-d})$. Let $f(x) = x^3 - mx^2 - (m+3)x - 1$, where nonreal $m \in \mathbb{Z}_K$, $|m| \ge \sqrt{20}+3$. Assume that m^2+3m+9 is a square-free ideal in $\mathbb{Z}_K$. Let $f(\epsilon) = 0$ and $F = \mathbb{Q}(\epsilon)$.*

Then $\{1,\epsilon,\epsilon^2\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$ and $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon,\epsilon+1\rangle$.

In Example 6.4, the fundamental domain of Γ_L in L_P is found for the family of the simplest quartic fields of Gras [8]. By complexification of this family, in Example 6.5, we prove the following.

Theorem 1.3. *Let d be a square-free positive integer and let $K = \mathbb{Q}(\sqrt{-d})$. Let $f(x) = x^4 - 2mx^3 - 6x^2 + 2mx + 1$, where nonreal $m \in \mathbb{Z}_K$, $\gcd(m,2) = 1$, and $|m| \ge \sqrt{84}$. Assume that m^2+4 is a square-free ideal in $\mathbb{Z}_K$. Let $f(\epsilon) = 0$ and $F = \mathbb{Q}(\epsilon)$.*

Then $\{1,\epsilon,(\epsilon^2-1)/2,\epsilon(\epsilon^2-1)/2\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$, and $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon,(\epsilon-1)/(\epsilon+1),(\epsilon-\epsilon^{-1})/2\rangle$.

Note that the families of algebraic number fields F considered in the theorems above are parameterized by complex parameters $m = a+\omega b \in \mathbb{Z}_K$, $a,b \in \mathbb{Z}$, or by three real parameters a, b, and d.

In [23], Algorithm II is used to find a system of fundamental units in a two-parameter family of complex cubic fields. In [24], it is used to find a system of fundamental units in some families of algebraic number fields F of degree less than or equal to 4, which have at least one real embedding. Thus, the present paper, where Algorithm II is applied only to the totally complex algebraic number fields, can be considered as a complement of [24].

2. Fundamental domains and K-tessellation.

Almost all the definitions in this section and in Section 3 are similar to the corresponding definitions from [24, Sections 2 and 3]. We reproduce them here for completeness.

Let $n \ge 2$ be a positive integer. Let V_n be the vector space of Hermitian $n\times n$ matrices. A complex matrix $X \in V_n$ if and only if $X = X^* = \overline{X}^T$. The real dimension of V_n is $N = n^2$.

The action of $g \in G = \mathrm{GL}(n,\mathbb{C})$ on $X \in V_n$ is given by

$$X \longmapsto X[g] = g^* X g. \tag{2.1}$$

For a subset S of V_n, denote $S[g] = \{X[g] \in V_n : X \in S\}$.

The one-dimensional subspaces of V_n form the real projective space V of dimension $N-1$, so that for any fixed nonzero $X \in V_n$, all the vectors $kX \in V_n$, $0 \neq k \in \mathbb{R}$, represent one point in V. Denote by $\mathcal{H} \subset V$ the set of (positive) definite elements of V and by $\mathcal{B}$ the boundary of $\mathcal{H}$ ($\mathcal{B}$ can be identified with nonnegative elements of V of rank less than n). The group G preserves both $\mathcal{H}$ and $\mathcal{B}$ as does its arithmetic subgroup $\mathrm{GL}(n,\mathbb{Z}_K)$.

The space V_n (and V) can be also identified with the set of Hermitian forms $A[x] = x^* A x$, $A \in V_n$, $x \in \mathbb{C}^n$. With each point $a = (a_1,\dots,a_n)^T \in \mathbb{C}^n$, we associate the matrix $A = aa^* \in \mathcal{B}$ and the Hermitian form

$$A[x] = |\langle a,x\rangle|^2 = |\overline{a}_1 x_1 + \cdots + \overline{a}_n x_n|^2. \tag{2.2}$$

Here, $\langle a,x\rangle = \overline{\langle x,a\rangle} = a^* x$. For $g \in G$, we have $\langle ga,x\rangle = a^* g^* x = \langle a, g^* x\rangle$.

Let $w = (1,0,\dots,0)^T$ and $W = ww^*$. Then $\langle w,x\rangle^2 = x_1^2$ and $W[g] = U = uu^*$, where $u = g^* w$.

Denote by G_∞ and Γ_∞ the stabilizers of w in G and $\Gamma = \mathrm{GL}(n,\mathbb{Z}_F)/\mu_F$, respectively. Then

$$G_\infty = \{g \in G : gw = w\} = \{g \in G : g_1 = w\}, \tag{2.3}$$

where g_1 is the first column of g. Thus, $g \in G_\infty$ if and only if $W[g^*] = W$.

We will say that $A \in V$ is *extremal* if $|A[x]| \geq |A[w]| = |a_{11}|^2$ for any $x \in \mathbb{Z}_K^n$, $x \neq (0,\dots,0)$. Let $\mathcal{A}_n = \{X \in V : X[w] \neq 0\}$. It is clear that $\mathcal{H} \subset \mathcal{A}_n$. For $X \in \mathcal{A}_n$, we will say that $\mathrm{ht}(X) = |\det(X)|^{1/n}/|X[w]|$ is the *height* of X and, for a subset S of V, we define the *height* of S as $\mathrm{ht}(S) = \max \mathrm{ht}(X)$, $X \in S$.

The elements of $\mathcal{A}_n$ will be normalized so that $X[w] = 1$. For a fixed $g \in \Gamma$, the set $\{X \in \mathcal{A}_n : |X[gw]| < 1\}$ is called the *g-strip*. It is clear that the gh-strip coincides with the g-strip for any $h \in \Gamma_\infty$. Since $X[gw] \in \mathbb{R}$ for any $g \in G$, the boundary of the g-strip consists of two planes $X[gw] = \pm 1$. The plane

$$L^+(gw) = L^+(g) = \{X \in \mathcal{A}_n : X[gw] = 1\} \tag{2.4}$$

is the boundary of the g-strip, which cuts $\mathcal{H}$. Let $\mathcal{R}_w$ be the set of all extremal points of V. Denote

$$K_n = K(w) = \mathcal{H} \cap \mathcal{R}_w. \tag{2.5}$$

Note that $K(w) \subset \mathcal{A}_n$ is bounded by the planes $L^+(g)$. If $h \in \Gamma_\infty$, then $X[hw] = X[w]$ and, therefore, $\mathrm{ht}(X[h]) = \mathrm{ht}(X)$. Thus,

$$K_n[h] = K_n, \quad h \in \Gamma_\infty. \tag{2.6}$$

By (2.6), $K_n[hg] = K_n[g]$ for any $g \in \Gamma$ and $h \in \Gamma_\infty$. Thus, the sets $K_n[g]$ are parameterized by the classes $\Gamma_\infty\backslash\Gamma$ or by primitive vectors $u = g^{-1}h^{-1}w = g^{-1}w$, so that $\pm u$

represent the same $K_n[g]$. The sets $K_n[g]$, $g \in \Gamma_\infty\backslash\Gamma$, form a tessellation of $\mathcal{H}$ which will be called the *K-tessellation*. It is clear that the K-tessellation of $\mathcal{H}$ is Γ-invariant.

Let $w_{2n} = (1,0,\dots,0)^T \in \mathcal{P}_{2n}$. Denote by $K(w_{2n})$ the set of w_{2k}-extremal points in $\mathcal{P}_{2k}$. Denote $\mathcal{P}_H = \{\hat{X} : X \in \mathcal{H}\}$ and $K_H = K(w_{2n}) \cap \mathcal{P}_H$. Let $X \in \mathcal{A}_n$. A Hermitian matrix can be reduced to a diagonal form by a unitary transformation. Hence $\det(\hat{X}) = \det^2(X)$ and, therefore, $\mathrm{ht}(\hat{X}) = (\mathrm{ht}(X))^2$, where $\mathrm{ht}(\hat{X})$ is the height of $\hat{X} \in \mathcal{P}_{2n}$ (see [24]). It follows that $X \to \hat{X}$ is a bijection between the K-tessellations of $\mathcal{H}$ and $\mathcal{P}_H$. In Sections 5 and 6, to show that $X \in \mathcal{H}$ is extremal, we will show that $\hat{X}$ is Minkowski-reduced (see, e.g., [6, pages 396–397]).

3. Axes of elements of G. Let $g \in G$. Let $ga_i = \lambda_i a_i$, $i = 1,\dots,n$, where, for simplicity, we assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Here, a_i is an eigenvector of g corresponding to its eigenvalue λ_i. Assume that $\langle a_i, w\rangle \neq 0$, $i = 1,\dots,n$. Then we can choose a_i so that

$$\langle a_i, w\rangle = 1, \quad i = 1,\dots,n. \tag{3.1}$$

$g \in \Gamma$ is said to be *K-irreducible* if its characteristic polynomial is irreducible over the field K. If $g \in \Gamma$ is K-irreducible, then all its eigenvalues are distinct. Let $\lambda_k \neq \pm 1$. Let $P = (a_1,\dots,a_n)$ be the matrix with columns $a_1,\dots,a_n$, and let $H = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$. Then $g = PHP^{-1}$.

The totally geodesic submanifold L_P of $\mathcal{H}$ fixed by $g = PHP^{-1}$ will be called the *axis* of g. The dimension of L_P is $n-1$. A point $q \in L_P$ can be represented as

$$q = \sum_{k=1}^{n} \mu_k A_k, \quad A_k = a_k a_k^*,\ \mu_k \geq 0,\ \sum_{k=1}^{n} \mu_k = 1. \tag{3.2}$$

It can also be identified with the set of Hermitian forms in $\mathcal{A}_n$

$$q[x] = \sum_{k=1}^{n} \mu_k A_k[x] = \sum_{k=1}^{n} \mu_k \left|\langle x, a_k\rangle\right|^2, \quad \mu_k \geq 0,\ \sum_{k=1}^{n} \mu_k = 1. \tag{3.3}$$

Hence

$$\det q = \mu_1 \cdots \mu_n |\det P|^2. \tag{3.4}$$

It follows from (3.3) that L_P is the axis of $h \in G$ if and only if a_i, $i = 1,\dots,n$, are eigenvectors of h. Hence, the axis of g depends only on its set of eigenvectors, that is, on P, but not on the eigenvalues of g.

Thus, L_P is the simplex with vertices A_k, $k = 1,\dots,n$. All the faces of L_P belong to $\mathcal{B}$. Note that $L_P[g^*] = L_P$.

Denote $K_n(g^{-1}w) = K_n[g]$ and

$$R(g^{-1}w) = K_n[g] \cap L_P \neq \varnothing, \quad g \in \Gamma_\infty\backslash\Gamma. \tag{3.5}$$

The sets $R(u)$, $u = g^{-1}w$, form a tessellation of L_P which is invariant modulo the action of Γ since the K-tessellation of $\mathcal{H}$ is Γ-invariant. We say that this tessellation is *periodic* if there are only a finite number of noncongruent sets $R(u)$ modulo the action of $\text{Stab}(L_P,\Gamma)$. In that case, the union of all noncongruent sets $R(u)$ is a fundamental domain of $\text{Stab}(L_P,\Gamma)$. The number of noncongruent sets $R(u)$ in the tessellation of L_P will be called the *period length*.

Let $N_P(x) = \langle x,a_1\rangle\cdots\langle x,a_n\rangle$, where $\langle x,a_k\rangle = x^* a_k$. Define

$$\nu(L_P) = \inf\left|\frac{N_P(gw)}{\det P}\right|, \tag{3.6}$$

where the infimum is taken over all $g\in\Gamma$. It is clear that $\nu(L_P) = \nu(L_{MP}[h])$ for any $h\in\Gamma$ and $M = \text{diag}(\mu_1,\dots,\mu_n)$, where $\mu_1,\dots,\mu_n\in\mathbb{C}$ and $\mu_1\cdots\mu_n\neq 0$. The projective invariant $\nu(L_P)$ is well known in the geometry of numbers (see, e.g., [3] or [9]).

Let $n = 2$. The approximation constants $\sup\nu(L_P)$ are known for $d = 1,2,3,5,6,7,11,15,19$ (see, e.g., [20, 22], where in the cases of $d = 5,6$, and 15, Γ is the extended Bianchi group). For $d = 1,3$, and 11, more information is available (see [12, 13, 14, 17, 18]). Thus, when $d = 1$, it is proved in [17, 18] that if $\nu(L_P) > 1/2$, then $\nu(L_P) = (4-|m|^{-4})^{-1/2}$ or $14.76^{-1/4}$, where (m,m') is a solution of the Diophantine equation $(m\overline{m'})^2+(m'\overline{m})^2 = |m|^2+|m'|^2$ in nonzero $m,m'\in\mathbb{Z}_K$, the ring of integers of the Gaussian field $K = \mathbb{Q}(i)$.

A point $q_m\in L_P$ is said to be the *summit* of L_P if $|\det(q_m)| = \max|\det(q)|$, the maximum being taken over all $q\in L_P$. It is clear that if $R = L_P\cap K_n(w)\neq\varnothing$, then $q_m\in R$. The following two lemmas are analogous to [24, Lemmas 5 and 6].

Lemma 3.1. *Let L_P be the totally geodesic manifold fixed by $g\in G$ and defined by (3.3), where $ga_i = \lambda_i a_i$. Let $P = (a_1,\dots,a_n)$ be the matrix with columns $a_1,\dots,a_n$. Then*

$$q_m = \frac{1}{n}\sum_{k=1}^{n} A_k \tag{3.7}$$

is the summit of L_P,

$$\begin{aligned}\text{ht}(L_P) &= \frac{1}{n}\left|\frac{\det P}{N_P(w)}\right|^{2/n},\\ \nu(L_P) &= \inf\left(n\,\text{ht}(L_P[g])\right)^{-n/2},\quad g\in\Gamma.\end{aligned} \tag{3.8}$$

Lemma 3.2. *Let L_P be the totally geodesic manifold fixed by $g\in G$ and defined by (3.3), where $ga_i = \lambda_i a_i$. Then*

$$\nu(L_P) = \inf\left(n\,\text{ht}(L_P[g_j])\right)^{-n/2},\quad L_P\cap K_n(g_jw)\neq\varnothing,\quad g_j\in\Gamma. \tag{3.9}$$

Assume that $L_P\cap K_n(gw)\neq\varnothing$, where $g\in\Gamma$. Denote

$$h_n = \inf\left(\text{ht}(X)\right),\quad X\in K_n. \tag{3.10}$$

Since $L_P[g]\cap K_n(w)\neq\varnothing$, by Lemma 3.1,

$$\text{ht}(L_P[g]) = \text{ht}(L_{g^*P}) = \frac{1}{n}\left|\frac{\det P}{N_{g^*P}(w)}\right|^{2/n} > h_n. \tag{3.11}$$

But $N_{g^*P}(x) = \langle x, g^*a_1\rangle \cdots \langle x, g^*a_n\rangle = \langle gx, a_1\rangle \cdots \langle gx, a_n\rangle$. Hence $N_{g^*P}(w) = N_P(gw)$.

A vector $gw \in \mathbb{Z}_K^n$, such that $L_P \cap K_n(gw) \neq \emptyset$, will be called a *convergent* of L_P. We have proved the following.

Theorem 3.3. *If a vector u is a convergent of L_P (i.e., if $L_P \cap K_n(u) \neq \emptyset$), then*

$$|N_P(u)| < C_n^{n/2} |\det P|, \tag{3.12}$$

where $C_n = 1/(nh_n)$. Hence, if L_P cuts infinitely many sets $K_n(u)$, then this inequality has infinitely many solutions in $u \in \mathbb{Z}_K^n$.

A component of the boundary of a set $R(u)$ of codimension one will be called a *face* of $R(u)$.

Algorithm II from [24] can be applied in $\mathcal{H}$. In this case, a simplex $L \subset \mathcal{H}$ has vertices at $A_i \in \mathcal{B}$, where $A_i = a_i a_i^*$, for $i = 1, \dots, n$, and $\det(a_1, \dots, a_n) \neq 0$.

The following result shows that any matrix g in the stabilizer of the simplex L is uniquely determined by the first row of g.

Proposition 3.4. *Let $f(x)$ be an irreducible polynomial over $\mathbb{Z}_K$ of degree n with coefficients in $\mathbb{Z}_K$. Let E^* be the companion matrix of $f(x)$. Let L_P be the axis of E. Let $g = (g_{ij}) \in G_L$, the torsion-free subgroup of the stabilizer of L_P in G. Then*

$$g = g_{11} I + g_{12} E + \cdots + g_{1n} E^{n-1}. \tag{3.13}$$

Proof. The first row of E^{i-1} is the standard unit vector $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. □

4. Fundamental units in $\mathbb{Z}_F$ and $\mathbb{Z}_{F/K}$. Let $\mathbb{Z}_F$ be the ring of integers of the field F. Assume that $\mathbb{Z}_F$ is a free $\mathbb{Z}_K$-module. Let $\{1, \alpha_2, \dots, \alpha_n\}$ be a $\mathbb{Z}_K$-basis of $\mathbb{Z}_F$. Then the $\mathbb{Z}$-basis of $\mathbb{Z}_F$ is $\{1, \omega, \alpha_2, \omega\alpha_2, \dots, \alpha_n, \omega\alpha_n\}$. Let $a_1 = (1, \alpha_2, \dots, \alpha_n)^T$. Let $\gamma \in \mathbb{Z}_F$. Then $\gamma\alpha_j = \sum m_{jk}\alpha_k$ or $\gamma a_1 = M_\gamma a_1$, where $\alpha_1 = 1$, $m_{jk} \in \mathbb{Z}_K$, and $M_\gamma = (m_{jk})$ is a square matrix of order n. Let σ_i be the n distinct embeddings of F/K in $\mathbb{C}$. Let $a_k = \sigma_k(a_1)$ and $\gamma_k = \sigma_k(\gamma)$, where $\gamma_1 = \gamma$. Then $\gamma_k a_k = M_\gamma a_k$ for $k = 1, \dots, n$. Thus, a_k is an eigenvector of M_γ corresponding to its eigenvalue γ_k. It is clear that the map $\gamma \mapsto M_\gamma$ is an isomorphism of the ring of integers $\mathbb{Z}_{F/K}$ and the commutative ring of $\mathbb{Z}_K$-integral square matrices of order n with the common axis L_P. The relative norm of γ equals $\det(M_\gamma)$ so that γ is a unit in $\mathbb{Z}_{F/K}$ if and only if $M_\gamma \in \mathrm{GL}_n(\mathbb{Z}_K)$. The torsion-free subgroup Γ_L of the stabilizer of L_P is isomorphic to $\mathbb{Z}_{F/K}^\times/\mu_{F/K}$. Thus, the problem of finding a system of fundamental units of F/K is equivalent to the problem of finding a set of generators of Γ_L. The analog of the (multidimensional continued fraction) Algorithm II introduced in [24] can be used to solve the latter problem. In Section 6, a set of generators of Γ_L, and, therefore, a system of fundamental units, is found in some families of relative extensions F/K of relative degree $n \leq 4$ and in the fields F.

Let $\hat{a}_1 = (1, \omega, \alpha_2, \omega\alpha_2, \dots, \alpha_n, \omega\alpha_n)$. Let $m_{jk} = b_{jk} + \omega c_{jk}$, where $b_{jk}, c_{jk} \in \mathbb{Z}$. Then $\gamma\alpha_j = \sum b_{jk}\alpha_k + \sum c_{jk}\omega\alpha_k$. Let $d_1 = (d+1)/4$. If $d \equiv 3 \pmod 4$, then $\gamma\omega\alpha_j = \sum(-d_1 c_{jk})\alpha_k + \sum(b_{jk} + c_{jk})\omega\alpha_k$, and $\gamma\omega\alpha_j = \sum(-dc_{jk})\alpha_k + \sum b_{jk}\omega\alpha_k$ otherwise. Denote $\widehat{M}_\gamma = (\widehat{m}_{jk})$, where

$$\widehat{m}_{jk} = \begin{cases} \begin{bmatrix} b_{jk} & c_{jk} \\ -d_1 c_{jk} & b_{jk} + c_{jk} \end{bmatrix} & \text{if } d \equiv 3 \pmod 4, \\ \begin{bmatrix} b_{jk} & c_{jk} \\ -dc_{jk} & b_{jk} \end{bmatrix} & \text{otherwise.} \end{cases} \tag{4.1}$$

Then $\gamma_k \hat{a}_k = \widehat{M}_\gamma \hat{a}_k$, where $\hat{a}_k = \sigma_k(\hat{a}_1)$, for $k = 1, \dots, n$, and $\widehat{(\overline{a_k})}$ is also an eigenvector of $\widehat{M}_\gamma$ corresponding to its eigenvalue $\overline{\gamma}_k$. Let $\hat{L}_P$ be the axis of $\widehat{M}_\gamma$ in $\mathcal{P}_{2n}$. It is clear that the map $\gamma \mapsto \widehat{M}_\gamma$ is an isomorphism of the ring of integers $\mathbb{Z}_F$ and the commutative ring of $\mathbb{Z}$-integral square matrices of order $2n$ with the common axis $\hat{L}_P$. The norm of γ equals $\det(\widehat{M}_\gamma)$ so that γ is a unit in $\mathbb{Z}_F$ if and only if $\widehat{M}_\gamma \in \mathrm{GL}_{2n}(\mathbb{Z})$. The torsion-free subgroup $\hat{\Gamma}_L$ of the stabilizer of $\hat{L}_P$ is isomorphic to $\mathbb{Z}_F^\times/\mu_F$. Thus, the problem of finding a system of fundamental units of F is equivalent to the problem of finding a set of generators of $\hat{\Gamma}_L$. Note that $\det(\widehat{M}_\gamma) = |\det(M_\gamma)|^2$ since $N(\gamma) = |N_{F/K}(\gamma)|^2$.

We have proved the following.

Lemma 4.1. *Let $d > 0$ be a square-free integer. Let $\mathbb{Z}_K$ be the ring of integers of the field $K = \mathbb{Q}(\sqrt{-d})$. Let F be an extension of K. Let $\mathbb{Z}_F$ be the ring of integers of the field F. Assume that $\mathbb{Z}_F$ is a free $\mathbb{Z}_K$-module. Then a system of fundamental units of the relative extension F/K is a system of fundamental units of F.*

5. 2×2 Hermitian matrices. In this section, we consider a model of the three-dimensional hyperbolic space which is similar to the Klein model of the hyperbolic plane used in Example 1 from [23] or [24].

When $n = 2$, the space V consists of all Hermitian 2×2 matrices

$$A = \begin{bmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_4 \end{bmatrix}, \tag{5.1}$$

where $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The formula

$$\rho(g)A = g^* A g = A[g], \tag{5.2}$$

where $g \in \mathrm{PSL}(2, \mathbb{C})$, $A \in V$, defines a representation ρ of the group $\mathrm{PSL}(2, \mathbb{C})$ in the space V. All the transformations $\rho(g)$ as well as the complex conjugation $A \mapsto \overline{A}$ preserve the form $\Delta(A) = \det(A) = x_1x_4 - x_2^2 - x_3^2$. The space $\mathcal{H}$ of (positive) definite matrices in V, considered with the action of the group $\rho(\mathrm{PSL}(2, \mathbb{C}))$ extended by the complex conjugation, is isomorphic to the three-dimensional hyperbolic space.

The action of $g = \left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{C})$ on $(z,t) \in H^3$ is given by

$$g(z,t) = \left(\frac{(\alpha z+\beta)(\overline{\gamma z+\delta})+\alpha\overline{\gamma}t^2}{|\gamma z+\delta|^2+|\gamma|^2t^2}, \frac{t}{|\gamma z+\delta|^2+|\gamma|^2t^2}\right) \tag{5.3}$$

(see, e.g., [7, page 569]). Thus, the height of $g(z,t)$ is $t(|\gamma z+\delta|^2+|\gamma|^2t^2)^{-1}$.

Lemma 5.1 [7, page 409]. *Define $\psi : \mathcal{H} \mapsto H^3$ by*

$$\psi(A) = \left(\frac{x_2+ix_3}{x_1}, \frac{\sqrt{\Delta(A)}}{|x_1|}\right), \quad A = \begin{bmatrix} x_1 & x_2+ix_3 \\ x_2-ix_3 & x_4 \end{bmatrix} \in \mathcal{H}. \tag{5.4}$$

Then $\psi(A[g]) = g\psi(A)$. Hence ψ induces a bijection of $\mathcal{H}$ and H^3, which commutes with the action of $\mathrm{PSL}(2,\mathbb{C})$.

The height of $\psi(A)$ is $\sqrt{\Delta(A)}/|x_1|$.

Let $g \in \Gamma = \mathrm{PGL}(2,\mathbb{Z}_K)$. Denote

$$K(\infty) = \{(z,t) \in H^3 : |g_{21}z+g_{22}|^2+|g_{12}|^2t^2 \ge 1,\ g = (g_{ij}) \in \Gamma\}. \tag{5.5}$$

We have the following.

Theorem 5.2. *Let ψ be the bijection of $\mathcal{H}$ and H^3 defined in Lemma 5.1. Then $\psi(K_2) = K(\infty)$ and, therefore, ψ is a bijection between the K-tessellations of $\mathcal{H}$ and H^3. Hence ψ is a bijection between the tessellations of the axis of $g \in \mathrm{SL}(2,\mathbb{C})$ in H^3 and the axis of g in $\mathcal{H}$. Thus, Algorithm I from* [22] *in H^3 coincides with Algorithm II from* [24] *in $\mathcal{H}$ in this case.*

Proof. The height of $\psi(A) \in H^3$ equals $\mathrm{ht}(A)$ in $\mathcal{H}$. Hence $\psi(K_2) = K(\infty)$. □

Let F be a field with signature $(0,2)$, which has an imaginary quadratic subfield K, so that $\mathbb{Z}_F$, the ring of integers of F, is a free $\mathbb{Z}_K$-module. Lemma 4.1 and Theorem 5.2 imply that to find a fundamental unit in F, one can apply either Algorithm I in H^3 or Algorithm II in $\mathcal{H}$. But, in general, it is easier to apply Algorithm II in $\mathcal{H}$ than Algorithm I in H^3, since to find the point of intersection of the axis L_P of $g \in \Gamma$ with the boundary of $K(w)$ in $\mathcal{H}$, one has to solve a system of linear equations. On the other hand, to solve this problem in H^3, we have to find the point of intersection of a semicircle with a hemisphere. However, the application of Algorithm I in H^3 in Examples 5.3, 5.4, and 5.6 is quite simple. In the next section, we apply Algorithm II in $\mathcal{H}$ to find a system of fundamental units in some families of fields with signature $(0,n)$, $n \le 4$. The period length in any of Examples 5.3, 5.4, 5.6, 6.1, 6.2, 6.4, 6.5 is one.

The discriminant of F is $d_K^2|d_{F/K}|^2$, where d_K is the discriminant of K and $d_{F/K}$ is the discriminant of the extension F/K (see, e.g., [5, page 209]). In all the examples below, we assume that $\mathbb{Z}_F$ has a free basis over $\mathbb{Z}_K$. In the case when F/K is a quadratic extension, (i.e., $F = K(\sqrt{\Delta})$), as in Examples 5.3, 5.4, and 5.6, such a basis exists if and only if $\mathfrak{D}_{F/K}/\sqrt{\Delta}$ is a principal ideal (of $\mathbb{Z}_F$) generated by an element of K (see, e.g., [5, page 222]). Here, $\mathfrak{D}_{F/K}$ is the relative different.

Example 5.3. Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & -\delta \\ 1 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.6}$$

where $m = a + \omega b \in \mathbb{Z}_K$, $\delta \in \mathbb{Z}_K^\times$. Then reflection S fixes the unit hemisphere ϕ_1 with equation $|z|^2 + t^2 = 1$ in the hyperbolic space H^3. The axes of reflections $U' = UW$ and $S' = WS$ are perpendicular to the axis A_g of $g = US = U'S'$ in H^3. Let M and M_1 be the points of intersection of A_g with axes of S' and U', respectively. Let R_0 be the arc MM_1 on A_g. Since the axis of U' is the vertical line in H^3 through the point $m/2 \in \mathbb{C}$, $R_0 \subset A_g \cap K(\infty)$ if and only if $M \in K(\infty)$. For $|m|$ fixed, it can be easily seen that the height of M is smallest when A_g and the axis of S' lie in the same vertical plane in H^3. It is clear that the part of the unit hemisphere ϕ_1 which lies above $|z| \le 1/2$ belongs to $K(\infty)$ for any d. It follows that $M \in K(\infty)$ and $R_0 \subset A_g \cap K(\infty)$ if $|m| \ge 4$. Thus, g is a generator of the torsion-free subgroup of the stabilizer of A_g in $\mathrm{PGL}(2,\mathbb{Z}_F)$ and, by Theorem 5.2, of Γ_L, provided $|m| \ge 4$. The characteristic polynomial of g is $p(x) = x^2 - mx + \delta$ with discriminant $d(p) = m^2 - 4\delta$. Let $p(\epsilon) = 0$. Let $F = \mathbb{Q}(\epsilon)$. If either the ideal $d(p) = m^2 - 4\delta$ or $d(p)/4$ is square-free in $\mathbb{Z}_K$, then $\{1,\epsilon\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$, and $\{1,\omega,\epsilon^{-1},\epsilon^{-1}\omega\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_F$. By Lemma 4.1, $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon\rangle$. Note that $\hat{a} = (1,\omega,\epsilon^{-1},\epsilon^{-1}\omega)^T$ is an eigenvector of $\hat{g}$ corresponding to its eigenvalue ϵ. We have proved Theorem 1.1.

Example 5.4. Let $d = 5$ or 6. Let B_d be the extended Bianchi group (see [19]). Let $c = 1 + \sqrt{-5}$ for $d = 5$ and $c = \sqrt{-6}$ for $d = 6$. In that case, the floor of an isometric fundamental domain of B_d in H^3 lies in the hemisphere ϕ_1, which is the unit hemisphere defined in Example 5.3, and hemisphere ϕ_2 with center $c/2$ and radius $1/\sqrt{2}$ (see [19, page 308]). Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \qquad S_6 = \begin{bmatrix} c & 2 \\ 2 & -c \end{bmatrix}, \qquad S_5 = \begin{bmatrix} c & \bar{c} \\ 2 & -c \end{bmatrix}, \qquad W = \begin{bmatrix} -1 & c \\ 0 & 1 \end{bmatrix}, \tag{5.7}$$

where nonreal $m = a + b\sqrt{-d} \in \mathbb{Z}_K$. The axes of reflections $U' = UW$ and $S_d' = WS_d$ are perpendicular to the axis A_g of $g = US_d = U'S_d'$. As in Example 5.3, it can be shown that g is a generator of the torsion-free subgroup of the stabilizer of A_g in B_d, provided $|m| \ge \sqrt{6}$.

The characteristic polynomial of g is $p(x) = x^2 - 2mx + 2$ with discriminant $d(p) = 4(m^2 - 2)$. Let $p(\alpha) = 0$ and $F = K(\alpha)$. $g \notin \mathrm{GL}(\mathbb{Z}_K)$ since $\det g = 2$, but $(1/2)g^2 \in \mathrm{SL}(\mathbb{Z}_K)$. Hence $\alpha^2/2 = m\alpha - 1 \in \mathbb{Z}_F^\times$. (Similarly, the case of $g' = US_d' = U'S_d$ can be considered. In this case, $\alpha^2/2 = m\alpha + 1 \in \mathbb{Z}_F^\times$.) If either $d = 5$ and $(a-b)$ is odd, or $d = 6$ and a is even, then $N_{F/K}((\alpha+c)/2) \in \mathbb{Z}_K$. If $d(p)/8 = m^2/2 - 1$ is a square-free ideal in $\mathbb{Z}_K$, then $\{(\alpha+c)/2, 1\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$, $\{(\alpha+c)/2, \omega(\alpha+c)/2, 1, \omega\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_F$, and $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon\rangle$, where $\epsilon = \alpha^2/2 = m\alpha - 1$. We have proved the following.

Theorem 5.5. *Let $d = 5$ or 6. Let $c = 1 + \sqrt{-5}$ for $d = 5$ and $c = \sqrt{-6}$ for $d = 6$. Let $\{1,\omega\}$ be the standard $\mathbb{Z}$-basis of $\mathbb{Z}_K$, where $K = \mathbb{Q}(\sqrt{-d})$. Let α be a root of $p(x) = x^2 - 2mx + 2\delta$, where nonreal $m = a + b\sqrt{-d} \in \mathbb{Z}_K$, $|m| \ge \sqrt{6}$, and $\delta = \pm 1$. Let $F = K(\alpha)$.*

Assume that either $d = 5$ *and* $(a-b)$ *is odd, or* $d = 6$ *and* a *is even. If* $m^2/2-\delta$ *is a square-free integer in* $\mathbb{Z}_K$*, then* $\{(\alpha+c)/2, \omega(\alpha+c)/2, 1, \omega\}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}_F$*, and* $\mathbb{Z}_F^\times/\mu_F = \langle\epsilon\rangle$*, where* $\epsilon = \alpha^2/2 = m\alpha - 1$*.*

Note that $\hat{a} = ((\alpha+c)/2, \omega(\alpha+c)/2, 1, \omega)^T$ is an eigenvector of $\hat{g}$ corresponding to its eigenvalue α.

Example 5.6. Let $d = 15$. Let $K = \mathbb{Q}(\sqrt{-15})$. The floor of an isometric fundamental domain of B_{15} in H^3 lies in ϕ_1, which is the unit hemisphere defined in Example 5.3, and the hemisphere ϕ_2 with center $\omega/2$ and radius $1/\sqrt{2}$ (see [22, page 2313]). Let

$$U = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} -1-\overline{\omega} & 1+\omega \\ 1+\omega & 1+\overline{\omega} \end{bmatrix}, \qquad W = \begin{bmatrix} -1 & \omega \\ 0 & 1 \end{bmatrix}, \tag{5.8}$$

where $m = a+\omega b \in \mathbb{Z}_K$. Let A_g be the axis of $g = US = U'S'$ in H^3, where $U' = UW$ and $S' = WS$. As above, it can be shown that the arc $R = K(\infty)\cap A_g$ is a fundamental domain of the torsion-free subgroup of the stabilizer of A_g in B_{15} on A_g, and that g is a generator of this subgroup, provided $|m| \geq 4$.

The characteristic polynomial of g is $p(x) = x^2 - m(1+\omega)x + 3$ with discriminant $d(p) = (1+\omega)^2(m^2+\omega)$. Let $p(\alpha) = 0$. Let $F = \mathbb{Q}(\alpha)$. $\alpha \notin \mathbb{Z}_F^\times$ since $\det(g) = 3$ and $g \notin \mathrm{GL}(\mathbb{Z}_K)$. But $(1/3)g^2 \in \mathrm{SL}(\mathbb{Z}_K)$. Hence $\epsilon = \alpha^2/3 = m\alpha(1+\omega)/3 - 1 \in \mathbb{Z}_K^\times$. Let $\beta = (\alpha - 1 - \overline{\omega})/(1+\omega)$. $N_{F/K}(\beta) = -1-2b+2(a+b)/\overline{\omega} \in \mathbb{Z}_F$ if and only if $(a+b) \in 2\mathbb{Z}$. Assume that $(a+b)$ is even. Then $\{\beta, 1\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$, and $\{\beta, \beta\omega, 1, \omega\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_F$, provided $m^2+\omega$ is a square-free ideal in $\mathbb{Z}_K$. We have obtained the following.

Theorem 5.7. *Let* $K = \mathbb{Q}(\sqrt{-15})$*. Let* $m = a+\omega b \in \mathbb{Z}_K$*, where* $(a+b) \in 2\mathbb{Z}$*. Let* $p(x) = x^2 - m(1+\omega)x + 3$ *and* $p(\alpha) = 0$*. Assume that* $|m| \geq 4$ *and* $m^2+\omega$ *is a square-free ideal in* $\mathbb{Z}_K$*. Let* $F = \mathbb{Q}(\alpha)$*. Let* $\beta = (\alpha-1-\overline{\omega})/(1+\omega)$*. Then* $\{\beta, \beta\omega, 1, \omega\}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}_L$ *and* $\mathbb{Z}_L^\times/\mu_L = \langle\epsilon\rangle$*, where* $\epsilon = \alpha^2/3 = m\alpha(1+\omega)/3 - 1$*.*

Note that $(\beta, \beta\omega, 1, \omega)^T$ is an eigenvector of $\hat{g}$ corresponding to its eigenvalue α.

6. Complexification of families of totally real cyclic fields. In this section, systems of fundamental units are found in some families of totally complex fields of degrees 4, 6, and 8, which are cyclic extensions of imaginary quadratic fields. These families are obtained by replacing the real parameter $t \in \mathbb{Z}$ in Examples 1 and 2 from [24] and Example 6.4 by a nonreal complex parameter $m \in \mathbb{Z}_K$.

Example 6.1. Let $f(x) = x^2 - mx - 1$, where $m \in \mathbb{Z}_K$. Let $f(\epsilon) = 0$. If $m \in \mathbb{Z}$, then we obtain the family of real quadratic fields $\mathbb{Q}(\epsilon)$ considered in [24, Example 1]. Assume that $m \notin \mathbb{Z}$ and that either m^2+4 or $m/4+1$ is a square-free ideal in $\mathbb{Z}_F$. Then $\{1, \epsilon\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_{F/K}$, where $F = K(\epsilon)$. The family of fields considered here is a particular case of the family of fields from Example 1.

Let nonreal $m = a+ib = a_1+\omega b_1 = \epsilon - 1/\epsilon$, $\epsilon = u+iv$, where $a, b, u, v \in \mathbb{R}$ and $a_1, b_1 \in \mathbb{Z}$. Let $\eta = \eta_0\sqrt{d} = v/(u^2+v^2+1)$ and $c = |m|^2+4$. Then $c/b = (1+4\eta^2)/\eta$.

Hence η_0 is a root of the polynomial $r(x) = 4b_1dx^2 - c_1x + b_1$, where $c_1 = 2c$ if $d \equiv 3 \pmod 4$ and $c_1 = c$ otherwise. The discriminant of $r(x)$ is $d_r = c_1^2 - 16db_1^2 = |d(f)|^2$, where $d(f)$ is the discriminant of f.

Let E^* be the companion matrix of $f(x)$. Let L_P be the axis of E. Let Γ_L be the torsion-free subgroup of the stabilizer of L_P in Γ. Let $E^*a_i = \epsilon_i a_i$ and $A_i = a_i a_i^*$, where $\epsilon_0 = \epsilon$, $a_i = (1, \epsilon_i)$, $i = 0, 1$. Then $q(\mu_0, \mu_1) = \mu_0 A_0 + \mu_1 A_1$, $\mu_i > 0$, $\mu_0 + \mu_1 = 1$ is an equation of L_P. Let F_1 be the intersection of L_P and $L^+(E)$. Then $F_1 = q(|\epsilon|^2, 1)$ in the projective coordinates. Let

$$h = \begin{bmatrix} 1 & -\overline{m} \\ 0 & 1 \end{bmatrix}, \qquad F_0 = \begin{bmatrix} 1 & 2i\eta \\ -2i\eta & 1 \end{bmatrix}. \tag{6.1}$$

Assume that $|\epsilon| < 1$. Then $|\eta| < |\epsilon| < (|m|^2 - 4)^{-1/2}$ since $|m| = |\epsilon + 1/\epsilon| \le |\epsilon| + 1/|\epsilon|$.

Thus, if $|m| \ge \sqrt{20}$, then $F_o = F_1[h]$ is Minkowski-reduced. Hence, F_1 and $F_2 = F_1[E]$ are extremal. Thus, the interval $R = [F_1, F_2] = L_P \cap K(w)$ is a fundamental domain of Γ_L on L_P. It follows that $\Gamma_L = \langle E \rangle$ and, therefore, $\mathbb{Z}_F^\times / \mu_F = \langle \epsilon \rangle$.

A point $X = (x_{ij}) \in \mathcal{P}_n$ is said to be rational over a field M if all $x_{ij} \in M$. A subset S of $\mathcal{P}_n$ is rational over M if the set of rational points of S is dense in S. If the summit of the axis L_P of $g \in \mathrm{GL}_n(\mathbb{Z})$ is rational over a field M, then L_P is rational over M (see [24, Section 4]). By (3.7), the summit of L_P is

$$q_m = \begin{bmatrix} 1 & \dfrac{\overline{m}}{2} \\ \dfrac{m}{2} & \dfrac{|m|^2 + |d(f)|}{4} \end{bmatrix}. \tag{6.2}$$

Hence, $\widehat{L_P} = \{\hat{X} : X \in L_P\} \subset \mathcal{P}_4$ is rational over the real quadratic field $\mathbb{Q}(|d(f)|) = \mathbb{Q}(\eta_0)$.

Example 6.2. Let $m \in \mathbb{Z}_K$. Let $\Gamma = \mathrm{GL}(3, \mathbb{Z}_K)$. Here, we consider complexification of the simplest cubic fields (see [15] and [24, Example 2]). These are the relative cyclic fields of relative discriminant $d_{F/K} = (m^2 + 3m + 9)^2$. Assume that $m \notin \mathbb{Z}$. The sextic field $F = K(\epsilon_1)$ is generated by a root ϵ_1 of $f(x) = x^3 - mx^2 - (m+3)x - 1$. Assume that $m^2 + 3m + 9$ is a square-free ideal in $\mathbb{Z}_K$. Then $\{1, \epsilon_1, \epsilon_1^2\}$ is a $\mathbb{Z}_K$-basis of $\mathbb{Z}_F$ and both units ϵ_1 and $\epsilon_2 = \sigma(\epsilon_1) = -1/(1+\epsilon_1)$ are the roots of this polynomial.

Let nonreal $m = a + ib = a_1 + \omega b_1$, $\epsilon_1 = u + iv$, where $a_1, b_1 \in \mathbb{Z}$, $a, b, u, v \in \mathbb{R}$. Let $\eta = \eta_0\sqrt{d} = v/(u^2 + v^2 + u + 1)$. Since $b/\eta - a - 3 = |\epsilon_1|^2 + |\epsilon_2|^2 + |\epsilon_3|^2$, η does not depend on a chosen root of $f(x)$. Denote $c = |m|^2 + 3a + 9 \in (1/2)\mathbb{Z}$, so that $c - b \in \mathbb{Z}$. Then η_0 is the real root of the polynomial $r(x) = c_1dx^3 - 9b_1dx^2 + c_1x - b_1$, where $c_1 = 2c$ if $d \equiv 3 \pmod 4$, and $c_1 = c$ otherwise. The discriminant of $r(x)$ is $d_r = -4d(c_1^2 - 27db_1^2)^2$.

Let E^* be the companion matrix of $f(x)$ and let $E_1 = E + I$. Let L_P be the axis of E. Let Γ_L be the torsion-free subgroup of the stabilizer of L_P in Γ. Let $E^*a_i = \epsilon_i a_i$ and $A_i = a_i a_i^*$, where $a_i = (1, \epsilon_i, \epsilon_i^2)$, $i = 1, 2, 3$. Then $q(\mu_1, \mu_2, \mu_3) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$, $\mu_i > 0$, $\mu_1 + \mu_2 + \mu_3 = 1$, is an equation of L_P.

Denote $E_2 = EE_1^{-1}$. Let F_1 be the intersection of L_P, $L^+(E)$, and $L^+(E_2)$, and let G_1 be the intersection of L_P, $L^+(E)$, and $L^+(E_1)$. Let $m = 3n+k$, where $n, k = k_1 + \omega k_2 \in \mathbb{Z}_K$, $|k_1| \le 1$ and $|k_2| \le 1$. Denote

$$h = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1-\overline{m} \\ 0 & 0 & 1 \end{bmatrix}, \qquad F_0 = F_1[h] = \begin{bmatrix} 1 & -\alpha & \overline{\alpha} \\ -\overline{\alpha} & 1 & \alpha \\ \alpha & \overline{\alpha} & 1 \end{bmatrix}, \tag{6.3}$$

where $\alpha = 2\eta/(\eta - i)$, and

$$h_1 = \begin{bmatrix} 1 & 0 & -1-\overline{n} \\ 0 & 1 & -2\overline{n} \\ 0 & 0 & 1 \end{bmatrix}, \qquad G_0 = \begin{bmatrix} 1 & \gamma_{13} & -i\overline{k}\eta \\ \overline{\gamma}_{13} & 1 & \dfrac{\overline{k}(1+i\eta)}{2} \\ ik\eta & \dfrac{k(1-i\eta)}{2} & \gamma_{33} \end{bmatrix}, \tag{6.4}$$

where

$$\gamma_{13} = -\frac{1}{2} - \frac{3}{2}\eta i, \qquad \gamma_{33} = \frac{c}{6}\frac{(1-3\eta^2)^2}{1+9\eta^2} + \frac{|k|^2}{3}. \tag{6.5}$$

Assume that $|\epsilon| < 1/2$, where $\epsilon = \epsilon_1$. Then $|\eta| < |\epsilon| < ((|m|-3)^2 - 4)^{-1/2}$ since $|m| = |\epsilon - 1/(\epsilon+1) + (\epsilon+1)/\epsilon| \le 1 + |\epsilon| + 1/|\epsilon| + 1/(1-|\epsilon|) \le 3 + |\epsilon| + 1/|\epsilon|$. Note that $\eta \to 0$ and, therefore, $\alpha \to 0$ and $\gamma_{13} \to -1/2$, as $|m| \to \infty$.

Thus, if $|m| \ge \sqrt{20} + 3$, then $F_0 = F_1[h]$ and $G_0 = G_1[h_1]$ with

$$\det(F_0) = \left(1 - \frac{4\eta^2}{1+\eta^2}\right)^3, \qquad \det(G_0) = \frac{1}{8}(|t|^2 + 3a + 9)\frac{(1-3\eta^2)^3}{1+9\eta^2} \tag{6.6}$$

are Minkowski-reduced and, therefore, F_i, G_i, $i = 1,2,3$, are extremal. Hence, $R = L_P \cap K(w)$ is the hexagon with vertices at $F_1, F_2 = F_1[E]$, $F_3 = F_1[E_2]$, $G_1, G_2 = G_1[E_1]$, $G_3 = G_1[E]$. The sides of R are identified as follows: $E : F_1G_1 \to F_2G_3$; $E_1 : F_3G_1 \to F_2G_2$; $E_2 : F_1G_2 \to F_3G_3$. Thus, R is a fundamental domain of $\Gamma_L = \langle E, E_1 \rangle$ and, therefore, $\mathbb{Z}_F^{\times}/\mu_F = \langle \epsilon, \epsilon+1 \rangle$. Theorem 1.2 is proved.

Note that $\widehat{L_P} = \{\widehat{X} : X \in L_P\} \subset \mathcal{P}_6$ is rational over the real cubic field $\mathbb{Q}(\eta_0)$. Also, note that $F_1 = q(|\epsilon_1+1|^2, |\epsilon_1(\epsilon_1+1)|^2, |\epsilon_1|^2)$ and $G_1 = q(1, |\epsilon_1+1|^2, |\epsilon_1|^2)$ in the projective coordinates, and if $F_1 = q(\mu_1, \mu_2, \mu_3)$, then $F_2 = q(\mu_2, \mu_3, \mu_1)$ and $F_3 = q(\mu_3, \mu_1, \mu_2)$. The same relations hold for G_1, G_2, and G_3. For the summit q_m of L_P, we have

$$q_m = \frac{1}{3}\sum A_i = \frac{1}{3}\sum F_i = \frac{1}{3}\sum G_i. \tag{6.7}$$

Remark 6.3. The properties of the vertices F_i and G_i of the fundamental domain R of Γ_L mentioned above, in the case of the simplest cubic fields, can be explained as follows.

Let $m = t \in \mathbb{Z}$. Then $v = 0$ and F is the simplest cubic field. Let $\mathrm{Gal}(F) = \langle \sigma \rangle$. Since

$$r = t^2 + 3t + 9 = \frac{(u^2+u+1)^3}{u^2(u+1)^2}, \tag{6.8}$$

where $u = \epsilon_1$, the divisor $\rho = u^2 + u + 1$ is ramified in F. Thus, $\sigma(\rho)$ and $\sigma^2(\rho)$ both are divisible by ρ, and therefore $t_r = \text{trace}(\rho) = \rho + \sigma(\rho) + \sigma^2(\rho)$ is divisible by ρ. But, $t_r \in \mathbb{Z}$, hence t_r is divisible by r. It is easy to verify that $t_r = r$. Let $\mu_i = \sigma^i(\rho)/r$, $i = 0,1,2$. Then

$$\mu_i > 0, \quad \sum \mu_i = 1. \tag{6.9}$$

The point $F_2 = \sum \mu_i A_i$ belongs to L_P and it is integral since any entry of F_2 has a form $\sum \sigma^i(\rho\alpha)/r \in \mathbb{Z}$. Since $u, u+1 \in \mathbb{Z}_L^\times$ and $\text{trace}(\rho^2/\epsilon^2) = 2r$, where $\epsilon = u, u+1$, or $u(u+1)$, if we choose $\mu_i = \sigma^i(\rho^2/\epsilon^2)/(2r)$, then (6.9) holds, and we obtain one of the points G_k. Note that (6.9) for F_i can be written in the form

$$u^2 + (u+1)^2 + (u^2+u)^2 = (u^2+u+1)^2 \tag{6.10}$$

and, for G_i, in the form

$$u^2 + (u+1)^2 + 1 = 2(u^2+u+1). \tag{6.11}$$

Example 6.4. Let t be an odd integer. Let $f(x) = x^4 - 2tx^3 - 6x^2 + 2tx + 1$. (Out of the four possible cases enumerated in [10, page 315], here we consider only Case 2.) Let $f(\epsilon) = 0$ and $\epsilon_1 = (\epsilon - 1)/(\epsilon + 1)$. Then $f(\epsilon_1) = f(-1/\epsilon) = f(-1/\epsilon_1) = 0$. The discriminant of $f(x)$ is $d(f) = 4^4(t^2+4)^3$. Let $\theta = (\epsilon - \epsilon^{-1})/2$. Then $\theta^2 - t\theta - 1 = 0$, $\theta = (t \pm \sqrt{t^2+4})/2$, and $N = \mathbb{Q}(\sqrt{d})$, $d = t^2+4$, is a quadratic subfield of the cyclic quartic field $F = \mathbb{Q}(\epsilon)$. If t^2+4 is square-free, then $\{1,\epsilon\}$ is a $\mathbb{Z}_N$-basis of $\mathbb{Z}_{F/N}$. Hence $\{1,\epsilon,\theta,\theta\epsilon,\}$ or $\{1,\epsilon,(\epsilon^2-1)/2,(\epsilon^3-\epsilon)/2\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_F$, provided θ is a fundamental unit of N. Let

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \tag{6.12}$$

Then $a_0 = (1,\epsilon,(\epsilon^2-1)/2,(\epsilon^3-\epsilon)/2)^T = \tau(1,\epsilon,\epsilon^2,\epsilon^3)^T$. Let $(E')^T$ be the companion matrix of $f(x)$ and let $E^T = \tau(E')^T\tau^{-1}$. Then $E^T a_i = \epsilon_i a_i$, where $a_i = (1,\epsilon_i,(\epsilon_i^2-1)/2,(\epsilon_i^3-\epsilon_i)/2)$, $\epsilon_0 = \epsilon$, $\epsilon_2 = -1/\epsilon$, $\epsilon_3 = -1/\epsilon_1$. Denote $E_1 = (E-I)(E+I)^{-1}$ and $E_2 = (E-E^{-1})/2$, where I is the identity matrix. Let L_P be the axis of E in $\mathcal{P}_4$. Let

$$E_- = \frac{1}{\sqrt{2}}(E-I), \qquad E_+ = \frac{1}{\sqrt{2}}(E+I). \tag{6.13}$$

Then $E_1 = E_-E_+^{-1}$ and $E = E_-E_+E_2^{-1}$. Let $\Delta_L = \langle E, E_1, E_2\rangle$ and $\Gamma_L' = \langle E_2, E_-, E_+\rangle$. Then Δ_L is a subgroup of index two in Γ_L'. Thus, if we show that Γ_L' equals the extension of the torsion-free subgroup Γ_L of the stabilizer of L_P in Γ by E_-, then $\Gamma_L = \Delta_L$.

Let $(\mu_0,\mu_1,\mu_2,\mu_3)$ be the coordinates of the point $\sum_{k=1}^4 \mu_k A_k$, where $A_k = a_k a_k^T$, in L_P. Let $\delta = u^2+1$, where $u = \epsilon$. Let i be defined modulo 4. Define

$$B_i = \frac{1}{\delta^3}(\beta_i^2, \beta_{i+1}^2, \beta_{i+2}^2, \beta_{i+3}^2), \tag{6.14}$$

where

$$\beta_0 = (u-1)(u+1), \qquad \beta_1 = \sqrt{2}u(u+1),$$
$$\beta_2 = u(u-1)(u+1), \qquad \beta_3 = \sqrt{2}u(u-1)(u+1). \tag{6.15}$$

Define

$$C_i = \frac{1}{\delta^3}\left(\gamma_i^2, \gamma_{i+1}^2, \gamma_{i+2}^2, \gamma_{i+3}^2\right), \tag{6.16}$$

where

$$\gamma_0 = 2u, \qquad \gamma_1 = \frac{1}{\sqrt{2}}(u-1)(u+1)^2,$$
$$\gamma_2 = 2u^2, \qquad \gamma_3 = \frac{1}{\sqrt{2}}u(u-1)^2(u+1). \tag{6.17}$$

Let

$$h_B = \begin{bmatrix} 1 & 0 & 0 & -\overline{m} \\ 0 & 1 & -\overline{m} & -2 \\ 0 & 0 & 1 & -2\overline{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad h_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2\overline{m} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{6.18}$$

where $m = t$. Then $B_0[h_B] = C_0[h_C] = I$. Thus, all the points B_i, C_i are integral and, therefore, extremal, with $\det(B_i) = \det(C_i) = 1$. The point B_0 is the intersection of L_P, $L^+(E)$, $L^+(E_2^{-1})$, and $L^+(EE_2^{-1})$, and $B_1 = B_0[EE_-^{-1}], B_2 = B_0[E]$, $B_3 = B_0[EE_+^{-1}]$, $C_0 = B_0[E_2^{-1}]$, $C_1 = B_0[E_+]$, $C_2 = B_0[EE_2^{-1}]$, and $C_3 = B_0[E_-]$.

The polytope $R = L_P \cap K(w)$ is bounded by two quadrangles lying in $L^+(E_2^{\pm 1})$ and eight triangles lying in $L^+(g^{\pm 1})$, $g = E_+, E_-, E_+E_2^{-1}, E_-E_2^{-1}$. It has 8 vertices, 16 edges, and 10 faces. Note that at any vertex B_i of R, four faces of R meet, but at any C_i, only three do. The projections of R into a plane which is “parallel” to its quadrangular faces are shown in Figure 6.1. The faces of R are identified as follows: $E_2 : C_0B_1C_2B_3 \to B_0C_1B_2C_3$; $E_+ : B_0C_0B_3 \to C_1B_1B_2$; $E_- : B_0C_0B_1 \to C_3B_3B_2$; $E_2E_+^{-1} : B_1B_2C_2 \to B_0C_3B_3$; $E_2E_-^{-1} : B_2B_3C_2 \to C_1B_0B_1$. Thus, R is a fundamental domain of Γ_L' in L_P, $\Gamma_L' = \langle E_2, E_-, E_+\rangle$, and $\Gamma_L = \langle E, E_1, E_2\rangle$. Hence, $\mathbb{Z}_F^\times/\{\pm 1\} = \langle \epsilon, (\epsilon-1)/(\epsilon+1), (\epsilon-\epsilon^{-1})/2\rangle$.

Example 6.5. Here, we consider complexification of the cyclic quartic fields from Example 6.4. Let $f(x) = x^4 - 2mx^3 - 6x^2 + 2mx + 1$, where $m = a + ib = a_1 + \omega b_1 \in \mathbb{Z}_K$, $\gcd(m, 2) = 1$, $a_1, b_1 \in \mathbb{Z}$, $a, b, u, v \in \mathbb{R}$, and $b \neq 0$. Let $f(\epsilon) = 0$. Then $F = \mathbb{Q}(\epsilon)$ is a totally complex field of degree eight. Let $\epsilon = u + iv$ and $\eta = \eta_0\sqrt{d} = v/(u^2 + v^2 + 1)$. Denote $c = 2|m|^2 + 8 \in \mathbb{Z}$. Then η_0 is a real root of the polynomial $r(x) = b_1(16d^2x^4 + 24dx^2 + 1) - c_1(4dx^3 + x)$, where $c_1 = 2c$ if $d \equiv 3 \pmod 4$, and $c_1 = c$ otherwise. The discriminant of $r(x)$ is $d_r = 256d^3(64db_1^2 - c_1^2)^3$. Define E, E_1, E_2, E_-, E_+, h_B, and h_C as in Example 6.4.

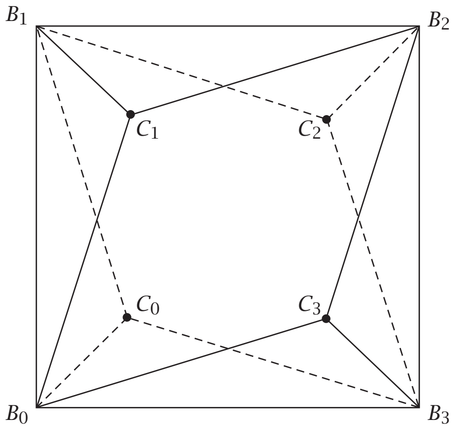

FIGURE 6.1 Fundamental domain of $\langle E_2, E_-, E_+\rangle$.

The point

$$C_0 = \frac{1}{(1+4\eta^2)(1+|\epsilon|^2)^3}\left(4|\epsilon|^2, \frac{1}{2}|1+\epsilon|^4|1-\epsilon|^2, 4|\epsilon|^4, \frac{1}{2}|1-\epsilon|^4|1+\epsilon|^2\right) \tag{6.19}$$

is the intersection of L_P, $L^+(E)$, $L^+(E_+)$, and $L^+(EE_2)$. Let $C_0[h_C] = (c_{ij}) = (\overline{c}_{ji})$. Then $c_{ii} = 1$, $i = 1,2,3,4$,

$$\begin{gathered} c_{12} = -c_{34} = -2i\eta, \qquad c_{14} = c_{23} = -4i\frac{\eta}{1+4\eta^2}, \\ c_{13} = -c_{24} = c_{12}c_{14} = -8\frac{\eta^2}{1+4\eta^2}. \end{gathered} \tag{6.20}$$

The point

$$B_0 = \frac{1}{(1+4\eta^2)(1+|\epsilon|^2)^3}\left(\left|\epsilon^2-1\right|^2, 2|\epsilon|^2|1+\epsilon|^2, |\epsilon|^2\left|\epsilon^2-1\right|^2, 2|\epsilon|^2|1-\epsilon|^2\right) \tag{6.21}$$

is the intersection of L_P, $L^+(E)$, $L^+(E_+)$, and $L^+(E_2^{-1})$. Let $B_0[h_B] = (b_{ij}) = (\overline{b}_{ji})$. Then $b_{ii} = 1$, $i = 1,2,3,4$,

$$\begin{gathered} b_{12} = -b_{34} = -2i\eta, \qquad b_{14} = b_{23} = 4i\frac{\eta}{1+4\eta^2}, \\ b_{13} = -b_{24} = b_{12}b_{14} = 8\frac{\eta^2}{1+4\eta^2}. \end{gathered} \tag{6.22}$$

Let $\alpha = 2\epsilon/(1-\epsilon^2)$. Then $|m| = 2|\alpha - 1/\alpha| \le 2(|\alpha| + 1/|\alpha|)$. Hence, if $|\alpha| < 1$, then $|\alpha| < 2(|m|^2 - 16)^{-1/2}$ and $|\epsilon| + 1/|\epsilon| > (|m|^2 - 16)^{1/2}$. Thus, if $|\epsilon| < 1$, then $|\eta| < |\epsilon| < (|m|^2 - 20)^{-1/2}$. It follows that $B_0[h_B] \to I$ as $|m| \to \infty$. If $|m| \ge \sqrt{84}$, then $|\eta| < |\epsilon| \le 1/8$, $B_0[h_B]$ and $C_0[h_C]$ are Minkowski-reduced, and, therefore, B_k, C_k, $k = 1,\dots,4$, which are defined as in Example 6.4, are extremal. Note that $\det(B_k) = \det(C_k) = (1-4\eta^2)^6/(1+4\eta^2)^4$. Thus, the polytope $R = L_P \cap K(w)$ is the same as in Example 6.4, R is a fundamental domain of Γ'_L in L_P, $\Gamma'_L = \langle E_2, E_-, E_+\rangle$, and $\Gamma_L = \langle E, E_1, E_2\rangle$. Hence, $\mathbb{Z}_F^{\times}/\mu_F = \langle \epsilon, (\epsilon-1)/(\epsilon+1), (\epsilon-\epsilon^{-1})/2\rangle$. Theorem 1.3 is proved.

Note that $\widehat{L_P} = \{\hat{X} : X \in L_P\} \subset \mathcal{P}_8$ is rational over the real quadric field $\mathbb{Q}(\eta_0)$.

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