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MULTIVALENT FUNCTIONS AND Q_K SPACES

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We give a criterion for q-valent analytic functions in the unit disk to belong to Q_K, a Möbius-invariant space of functions analytic in the unit disk in the plane for a nondecreasing function $K:[0,\infty)\to[0,\infty)$, and we show by an example that our condition is sharp. As corollaries, classical results on univalent functions, the Bloch space, BMOA, and Q_p spaces are obtained.

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1. Introduction. For analytic univalent function f in the unit disk Δ, Pommerenke [8] proved that $f\in\mathcal{B}$ if and only if $f\in$ BMOA, which easily implies a result of Baernstein II [4] about univalent Bloch functions: *if* $g(z)\neq 0$ *is an analytic univalent function in* Δ, *then* $\log g\in$ BMOA. We know that Pommerenke's result mentioned above was generalized to Q_p spaces for all p, $0<p<\infty$, by Aulaskari et al. (cf. [2, Theorem 6.1]). Their result can be stated as follows.

Theorem 1.1. *Let f be an analytic function in Δ such that*

$$\iint_{|w-w_0|<1} n(w,f)\,dA(w)\le A<\infty, \tag{1.1}$$

for all $w_0\in\mathbb{C}$, where $n(w,f)$ denotes the number of roots of the equation $f(z)=w$ in Δ counted according to their multiplicity and $dA(z)$ is the Euclidean area element on Δ. Then $f\in\mathcal{B}(\mathcal{B}_0)$ if and only if $f\in Q_p(Q_{p,0})$ for all $p\in(0,\infty)$.

Here, Q_p and its subspace $Q_{p,0}$, $0<p<\infty$, denote the spaces of analytic functions f in Δ defined, respectively, as follows (cf. [1, 3]):

$$\begin{aligned} Q_p &= \left\{ f : f \text{ analytic in } \Delta,\ \sup_{a\in\Delta}\iint_\Delta |f'(z)|^2 (g(z,a))^p\,dA(z)<\infty\right\},\\ Q_{p,0} &= \left\{ f\in Q_p : \lim_{|a|\to 1}\iint_\Delta |f'(z)|^2 (g(z,a))^p\,dA(z)=0\right\}, \end{aligned} \tag{1.2}$$

where $g(z,a)=\log 1/|\varphi_a(z)|$ is a Green's function in Δ with pole at $a\in\Delta$, and $\varphi_a(z)=(a-z)/(1-\bar{a}z)$ is a Möbius transformation of Δ.

We know that $Q_1=$ BMOA, the space of all analytic functions of bounded mean oscillation (cf. [5]), and for each $p\in(1,\infty)$, the space Q_p is the *Bloch space* $\mathcal{B}$ (cf. [1]), which is defined as follows:

$$\mathcal{B} = \left\{ f : f \text{ analytic in } \Delta,\ \|f\|_{\mathcal{B}} = \sup_{z\in\Delta} (1-|z|^2)\,|f'(z)| < \infty \right\}. \tag{1.3}$$

Similar to the above we have $Q_{1,0}$ = VMOA, the space of all analytic functions of vanishing mean oscillation (cf. [5]), and $Q_{p,0} = \mathcal{B}_0$ for all $p \in (1,\infty)$, where $\mathcal{B}_0$ denotes the little Bloch space defined by

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z|\to 1} (1-|z|^2)\,|f'(z)| = 0 \right\}. \tag{1.4}$$

In the present paper, we consider a more general space Q_K (see below) and show that all the above-mentioned results are true for space Q_K. Our contribution gives an extended version of Pommerenke's theorem, which is also a slight improvement of all the above results, and the proof presented here is independently developed.

Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. Recall that the space Q_K consists of analytic functions f in Δ for which

$$\|f\|^2_{Q_K} = \sup_{a\in\Delta} \iint_\Delta |f'(z)|^2 K(g(z,a))\,dA(z) < \infty; \tag{1.5}$$

$f \in Q_K$ belongs to the space $Q_{K,0}$ if

$$\iint_\Delta |f'(z)|^2 K(g(z,a))\,dA(z) \longrightarrow 0, \quad |a| \longrightarrow 1. \tag{1.6}$$

Modulo constants, Q_K is a Banach space under the norm defined in (1.5). It is clear that Q_K is Möbius-invariant and a subspace of the Bloch space $\mathcal{B}$ (cf. [6]). For $0 < p < \infty$, $K(t) = t^p$ gives the space Q_p. Choosing $K(t) = 1$, we get the Dirichlet space $\mathcal{D}$.

By [6, Proposition 2.1] we know that if the integral

$$\int_0^{1/e} K\left(\log \frac{1}{\rho}\right) \rho\, d\rho = \int_1^\infty K(t) e^{-2t}\, dt \tag{1.7}$$

is divergent, then the space Q_K is trivial; that is, the space Q_K contains only constant functions. From now on, we assume that the function $K : [0,\infty) \to [0,\infty)$ is right-continuous and nondecreasing and that the integral (1.7) is convergent. Without loss of generality, we can assume that $K(1) > 0$. For a general theory for Q_K spaces, see [6, 11].

2. Main results. A function f analytic in the unit disk is said to be q-valent if the equation $f(z) = w$ has never more than q solutions. Let

$$p(\rho) = \frac{1}{2\pi} \int_0^{2\pi} n(\rho e^{i\phi}, f)\, d\phi. \tag{2.1}$$

If

$$\int_0^R p(\rho)\, d(\rho^2) \le qR^2, \quad R > 0, \tag{2.2}$$

or

$$p(R) \le q, \quad R > 0, \tag{2.3}$$

where q is a positive number, we say that f is areally mean q-valent or circumferentially mean q-valent, respectively (cf. [7, pages 38 and 144]). It is clear that if f is circumferentially mean q-valent, then f is areally mean q-valent.

Note that if (1.1) holds, f will be areally mean q-valent in Δ for some $q > 0$. We know that if f is univalent, then f must be areally and circumferentially mean 1-valent. Thus, it is natural to conjecture that Pommerenke's result and Theorem 1.1 are also true for the areally and circumferentially mean q-valent functions.

We know that the space Q_K can be nontrivial if K is not too big at infinity (see condition (1.7)). For such functions K, the properties of Q_K depend essentially on the behavior of K near the origin. From [6, Theorems 2.3 and 2.5], we know that $Q_K = \mathcal{B}(Q_{K,0} = \mathcal{B}_0)$ if and only if

$$\int_0^1 (1-r^2)^{-2} K\left(\log \frac{1}{r}\right) r\, dr < \infty. \tag{2.4}$$

A natural idea is to look for an integral condition which is weaker than that given by (2.4) such that $f \in \mathcal{B}(\mathcal{B}_0)$ if and only if $f \in Q_K(Q_{K,0})$ for some special f. For the areally mean q-valent case, we present the main result in this paper as follows.

Theorem 2.1. *Let f be an areally mean q-valent function in Δ. If*

$$\int_0^1 \left(\log \frac{1}{1-r}\right)^2 (1-r)^{-1} K\left(\log \frac{1}{r}\right) r\, dr < \infty, \tag{2.5}$$

then

(i) $f \in \mathcal{B}$ *if and only if* $f \in Q_K$*;*
(ii) $f \in \mathcal{B}_0$ *if and only if* $f \in Q_{K,0}$.

Note that (2.4) implies (2.5) since $(\log 1/(1-r))^2 \le 4e^{-2}/(1-r)$ for $0 < r < 1$, but the converse is not true. For example, $K(t) = t$ gives that (2.5) holds but (2.4) fails. By [6, Theorems 2.3 and 2.5], (2.5) is also necessary for Theorem 2.1(i) and (ii) in case f is an areally mean q-valent function in Δ.

In the light of the following example it is impossible to drop the assumption of areally mean q-valence of the functions f in Theorem 2.1. Indeed, choose $K_1(t) = t^{2\alpha-1}$ and

$$f_1(z) = \sum_{j=1}^{\infty} 2^{-j(1-\alpha)} z^{2^j}, \quad \frac{1}{2} < \alpha < 1. \tag{2.6}$$

It is easy to see that $f_1 \in \mathcal{B}$ and (2.5) holds for K_1. Since f_1 has a gap series representation, f_1 is not an areally mean q-valent in Δ. The following argument shows that $f \notin Q_{K_1}$.

For $r \in [3/4,1)$, we find k so that $1/2 \le 2^k(1-r) < 1$. Using the inequality $\log r \ge 2(r-1)$, $1/2 < r < 1$, we see that

$$\begin{aligned}\int_0^{2\pi} |f_1'(re^{i\theta})|^2 d\theta &= 2\pi \sum_{j=1}^{\infty} 2^{j2\alpha} r^{2^{j+1}-2} \\ &\ge 2\pi(1-r)^{-2\alpha} \sum_{j=1}^{\infty} (2^j(1-r))^{2\alpha} \exp(-2^{j+2}(1-r)) \\ &\ge 2^{-2\alpha+1}\pi(1-r)^{-2\alpha} \sum_{j=1}^{\infty} 2^{(j-k)(2\alpha)} \exp(-2^{j-k+2}) \\ &\ge 2^{-2\alpha+1}\pi(1-r)^{-2\alpha} \sum_{j=0}^{\infty} (2^{j2\alpha} \exp(-2^{j+2})) \\ &= C(\alpha)(1-r)^{-2\alpha}.\end{aligned} \tag{2.7}$$

Hence

$$\begin{aligned}&\sup_{a\in\Delta} \iint_\Delta |f_1'(z)|^2 K_1(g(z,a))\,dA(z) \\ &\quad\ge \iint_\Delta |f_1'(z)|^2 K_1\left(\log\frac{1}{|z|}\right) dA(z) \\ &\quad= \int_0^1 K\left(\log\frac{1}{r}\right) r\,dr \int_0^{2\pi} |f_1'(re^{i\theta})|^2 d\theta \\ &\quad\ge C(\alpha)\int_{3/4}^1 (1-r)^{-2\alpha}\left(\log\frac{1}{r}\right)^{2\alpha-1} r\,dr.\end{aligned} \tag{2.8}$$

Since the last integral is divergent, we conclude that $f_1 \notin Q_K$.

Theorem 2.2. *Let f be a circumferentially mean q-valent and nonvanishing function in Δ. If (2.5) holds, then* $\log f \in Q_K$.

It is clear that the integral in (2.5) is convergent for $K(t) = t^p$, $p > 0$. Thus, we have the following result which extends Theorem 1.1.

Corollary 2.3. *Let f be an areally mean q-valent function in Δ, $0 < p < \infty$. Then*
(i) $f \in \mathcal{B}$ *if and only if* $f \in Q_p$;
(ii) $f \in \mathcal{B}_0$ *if and only if* $f \in Q_{p,0}$.

3. Proofs. In the proofs of Theorems 2.1 and 2.2, we need two lemmas, the first one can be considered as a generalization of a result of Pommerenke (cf. [9, page 174]).

Lemma 3.1. *Let f be areally mean q-valent in Δ. Then*

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \le \frac{4q\pi(M(\sqrt{r},f))^2}{1-r}, \quad \frac{1}{2} < r < 1, \tag{3.1}$$

where $M(r,f) = \sup_{|z|=r}|f(z)|$, $0 < r < 1$.

Proof. If $1/2 < r < 1$, we obtain

$$\begin{aligned}\iint_{|z|<\sqrt{r}} |f'(z)|^2 dA(z) &= \int_0^{\sqrt{r}} \rho \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 d\theta\, d\rho \\ &\geq \frac{1}{4}(1-r)\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta.\end{aligned} \tag{3.2}$$

Since f is areally mean q-valent, we deduce that

$$\begin{aligned}\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta &\leq \frac{4}{1-r}\iint_{|z|<\sqrt{r}} |f'(z)|^2 dA(z) \\ &\leq \frac{4}{1-r}\iint_{|w|<M(\sqrt{r},f)} n(w,f)\,dA(w) \\ &\leq \frac{4q\pi(M(\sqrt{r},f))^2}{1-r},\end{aligned} \tag{3.3}$$

which proves Lemma 3.1. □

Lemma 3.2. *Let K be defined as in Section 1. Then*
(i) $Q_{K,0} \subset \mathcal{B}_0$;
(ii) *an analytic function f belongs to $\mathcal{B}_0$ if and only if there exists an $r \in (0,1)$ such that*

$$\lim_{|a|\to 1} \iint_{\Delta(a,r)} |f'(z)|^2 K(g(z,a))\,dA(z) = 0, \tag{3.4}$$

where $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$.

Proof. See [6, Thereom 2.4]. □

Now we turn to give the proofs of our main theorems.

Proof of Theorem 2.1. We first prove (i). Since $Q_K \subset \mathcal{B}$, it suffices to prove that if a Bloch function f is areally mean q-valent in Δ, then $f \in Q_K$. We use the change of variable $w = \varphi_a(z)$ to deduce that

$$\begin{aligned}&\iint_{\Delta\setminus\Delta(a,1/2)} |f'(z)|^2 K(g(z,a))\,dA(z) \\ &\quad= \iint_{\Delta\setminus\Delta(a,1/2)} |(f(z)-f(a))'|^2 K\left(\log\frac{1}{|\varphi_a(z)|}\right) dA(z) \\ &\quad= \iint_{1/2<|w|<1} |(f\circ\varphi_a(w)-f(a))'|^2 K\left(\log\frac{1}{|w|}\right) dA(w) \\ &\quad= \int_{1/2}^1 K\left(\log\frac{1}{r}\right) r \int_0^{2\pi} |(f\circ\varphi_a(re^{i\theta})-f(a))'|^2 d\theta\, dr.\end{aligned} \tag{3.5}$$

It is known that if $g \in \mathcal{B}$, then

$$|g(z)-g(0)| \leq \frac{1}{2}\|g\|_{\mathcal{B}} \log \frac{1+|z|}{1-|z|}. \tag{3.6}$$

Choosing $g = f \circ \varphi_a - f(a)$ and observing that $\|g\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$, we obtain

$$M(r, f \circ \varphi_a - f(a)) \leq \frac{1}{2}\|f\|_{\mathcal{B}} \log \frac{1+r}{1-r}. \tag{3.7}$$

It follows from (3.5) and Lemma 3.1 that

$$\begin{aligned}
&\iint_{\Delta \setminus \Delta(a,1/2)} |f'(z)|^2 K(g(z,a))\, dA(z) \\
&\quad = \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) r \int_0^{2\pi} |(f \circ \varphi_a(re^{i\theta}) - f(a))'|^2 d\theta\, dr \\
&\quad \leq 4q\pi \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r\, dr \\
&\quad \leq q\pi C \|f\|_{\mathcal{B}}^2 \int_{1/2}^{1} K\left(\log \frac{1}{r}\right)\left(\log \frac{1}{1-r}\right)^2 (1-r)^{-1} r\, dr.
\end{aligned} \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}
&\iint_{\Delta(a,1/2)} |f'(z)|^2 K(g(z,a))\, dA(z) \\
&\quad \leq \|f\|_{\mathcal{B}}^2 \iint_{\Delta(a,1/2)} (1-|z|^2)^{-2} K(g(z,a))\, dA(z) \\
&\quad = \|f\|_{\mathcal{B}}^2 \iint_{\Delta(0,1/2)} (1-|w|^2)^{-2} K\left(\log \frac{1}{|w|}\right) dA(w) \\
&\quad \leq 4\pi \|f\|_{\mathcal{B}}^2 \int_0^{1/2} K\left(\log \frac{1}{r}\right) r\, dr.
\end{aligned} \tag{3.9}$$

Combining the upper bounds given by (3.8), (3.9), and (2.5), we see that $f \in Q_K$, which proves part (i) of Theorem 2.1.

To prove (ii), we assume that f is an areally mean q-valent function in Δ which is also in $\mathcal{B}_0$. By Lemma 3.2(i), it suffices to prove that $f \in Q_{K,0}$. By Lemma 3.2(ii), there exists an r_0, $1/2 < r_0 < 1$, such that

$$\lim_{|a| \to 1} \iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))\, dA(z) = 0. \tag{3.10}$$

Now we show that

$$\lim_{|a| \to 1} \iint_{\Delta \setminus \Delta(a,r_0)} |f'(z)|^2 K(g(z,a))\, dA(z) = 0. \tag{3.11}$$

By the proof of part (i) and assumption (2.5), we see that

$$\begin{aligned}
&\iint_{\Delta\setminus\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))\,dA(z)\\
&\quad= \int_{r_0}^1 K\left(\log\frac{1}{r}\right) r \int_0^{2\pi} |(f\circ\varphi_a(re^{i\theta})-f(a))'|^2\,d\theta\,dr\\
&\quad\le 4q\pi \int_{r_0}^1 K\left(\log\frac{1}{r}\right)(M(\sqrt{r},f\circ\varphi_a-f(a)))^2(1-r)^{-1} r\,dr\\
&\quad\le q\pi\|f\|_{\mathcal{B}}^2 \int_{r_0}^1 K\left(\log\frac{1}{r}\right)\left(\log\frac{1+r}{1-r}\right)^2(1-r)^{-1} r\,dr < \infty
\end{aligned}\tag{3.12}$$

for all $a\in\Delta$. Thus, for any given $\varepsilon>0$, there exists an r_1, $r_0<r_1<1$, such that

$$\int_{r_1}^1 K\left(\log\frac{1}{r}\right)(M(\sqrt{r},f\circ\varphi_a-f(a)))^2(1-r)^{-1} r\,dr < \varepsilon \tag{3.13}$$

for all $a\in\Delta$. Hence, what we need to prove is that

$$\lim_{|a|\to 1}\int_{r_0}^{r_1} K\left(\log\frac{1}{r}\right)(M(\sqrt{r},f\circ\varphi_a-f(a)))^2(1-r)^{-1} r\,dr = 0. \tag{3.14}$$

In fact, we have

$$\begin{aligned}
&\int_{r_0}^{r_1} K\left(\log\frac{1}{r}\right)(M(\sqrt{r},f\circ\varphi_a-f(a)))^2(1-r)^{-1} r\,dr\\
&\qquad\le C(r_0,r_1)K\left(\log\frac{1}{r_0}\right)(M(r_2,f\circ\varphi_a-f(a)))^2,
\end{aligned}\tag{3.15}$$

where $r_2=\sqrt{r_1}$ and $C(r_0,r_1)$ is a constant depending on r_0 and r_1. Define $f_t(z)=f(tz)$ for $0<t<1$ and then

$$\begin{aligned}
&(M(r_2,f\circ\varphi_a-f(a)))^2\\
&\quad\le 2\left(\frac{1}{4}\|f-f_t\|_{\mathcal{B}}^2\left(\log\frac{1+r_2}{1-r_2}\right)^2+(M(r_2,f_t\circ\varphi_a-f_t(a)))^2\right).
\end{aligned}\tag{3.16}$$

Since $f\in\mathcal{B}_0$, $\|f-f_t\|_{\mathcal{B}}\to 0$, $t\to 1$. Also,

$$\max_{|z|\le r_2}|f_t\circ\varphi_a(z)-f_t(a)| \le \frac{1-|a|^2}{(1-r_2)^2}\max_{|w|\le t}|f'(w)|, \tag{3.17}$$

which implies that

$$\lim_{|a|\to 1} M(r_2,f_t\circ\varphi_a-f_t(a)) = 0. \tag{3.18}$$

Thus we have (3.14). Hence

$$\lim_{|a|\to 1}\iint_{\Delta}|f'(z)|^2K(g(z,a))\,dA(z)=0, \tag{3.19}$$

which shows that $f\in Q_{K,0}$. The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Assume that f is a nonvanishing circumferentially mean q-valent function in Δ. According to [7, Theorem 5.1], we have $\log f\in\mathcal{B}$. From [7, Lemma 5.2] and the argument in the beginning of the proof of [7, Theorem 5.1], we see that we can define a single-valued branch of $f(z)^{1/q}$ which is circumferentially mean 1-valent in Δ and such that on each circle $\{|w|=R\}$ there exists a point which is not assumed by $f(z)^{1/q}$. It follows that

$$\begin{gathered}\int_{-\infty}^{\infty}n\left(\log\rho+i\phi,\frac{1}{q}\log f\right)d\phi=\int_0^{2\pi}n(\rho e^{i\phi},f^{1/q})\,d\phi\le 2\pi,\\ \iint_{|w|<R}n(w,\log f)\,dA(w)\le 4\pi Rq,\end{gathered} \tag{3.20}$$

which means that $\log f$ is areally mean q_1-valued in Δ for some $q_1>0$. It follows from Theorem 2.1 that $\log f\in Q_K$. □

4. Further discussion.

In [10] we studied the conditions for analytic univalent Bloch function f to belong to Q_K spaces. The log-order of the function $K(r)$ is defined as

$$\rho=\overline{\lim_{r\to\infty}}\frac{\log^+\log^+K(r)}{\log r}, \tag{4.1}$$

where $\log^+x=\max\{\log x,0\}$, and if $0<\rho<\infty$, the log-type of the function $K(r)$ is defined as

$$\sigma=\overline{\lim_{r\to\infty}}\frac{\log^+K(r)}{r^{\rho}}. \tag{4.2}$$

Theorem 4.1. *Let f be an analytic univalent function in Δ and let $K:[0,\infty)\to[0,\infty)$ satisfy that $K(t)=O((t\log 1/t)^p)$ as $t\to 0$ for some $p>0$. If the log-order ρ and the log-type σ of K satisfy one of the conditions*

(i) $0\le\rho<1$,

(ii) $\rho=1$ *and* $\sigma<2$,

then $f\in\mathcal{B}$ if and only if $f\in Q_K$.

We note that Theorem 4.1 can be viewed as a consequence of Theorem 2.1. In fact, conditions (i) and (ii) of Theorem 4.1 show that the space Q_K is not trivial. That is, the integral (1.7) is convergent in this case. Suppose that $K(t)=O((t\log 1/t)^p)$, $t\to 0$. There exist an $r_0\in(1/2,1)$ and a constant $C>0$ such that both $\log 1/r\le 2(1-r)$ and

$$K\left(\log\frac{1}{r}\right)\le C\left(\log\frac{1}{r}\log\left(\log\frac{1}{r}\right)^{-1}\right)^p \tag{4.3}$$

hold for $r_0 < r < 1$. Thus

$$\begin{aligned}
&\int_0^1 \left(\log\frac{1}{1-r}\right)^2 (1-r)^{-1} K\left(\log\frac{1}{r}\right) r\,dr \\
&\quad = \int_0^{r_0} + \int_{r_0}^1 \left(\log\frac{1}{1-r}\right)^2 (1-r)^{-1} K\left(\log\frac{1}{r}\right) r\,dr \\
&\quad \le \left(\log\frac{1}{1-r_0}\right)^2 (1-r_0)^{-1} \int_0^{r_0} K\left(\log\frac{1}{r}\right) r\,dr \\
&\qquad + C\int_{r_0}^1 \left(\log\frac{1}{1-r}\right)^2 (1-r)^{-1} \left(\log\frac{1}{r}\log\left(\log\frac{1}{r}\right)^{-1}\right)^p r\,dr \qquad (4.4)\\
&\quad \le C_1 + C_2\int_{r_0}^1 \left(\log\frac{1}{1-r}\right)^{2+p} (1-r)^{p-1} r\,dr \\
&\quad \le C_1 + C_2\int_{R_0}^\infty e^{-ps} s^{2+p}\,ds \\
&\quad \le C_1 + C_2 p^{-3-p}\Gamma(3+p) < \infty.
\end{aligned}$$

For a general analytic function f, we have the following theorem.

Theorem 4.2. *Suppose that (2.5) holds. If*

$$\sup_{a\in\Delta} \iint_{|z|<r} \left|(f\circ\varphi_a(z))'\right|^2 dA(z) = O\left(\left(\log\frac{1}{1-r}\right)^2\right), \qquad (4.5)$$

then

(i) $f \in \mathcal{B}$ *if and only if* $f \in Q_K$;

(ii) $f \in \mathcal{B}_0$ *if and only if* $f \in Q_{K,0}$.

Proof. We know that

$$\begin{aligned}
\int_0^{2\pi} \left|(f\circ\varphi_a(re^{i\theta}))'\right|^2 d\theta &\le \frac{4}{1-r}\iint_{|z|<\sqrt{r}} \left|(f\circ\varphi_a(z))'\right|^2 dA(z) \\
&\le \frac{1}{1-r} O\left(\left(\log\frac{1}{1-\sqrt{r}}\right)^2\right) \qquad (4.6)\\
&\le \frac{C}{1-r}\left(\log\frac{1}{1-r}\right)^2.
\end{aligned}$$

The proof can be completed by an argument similar to that used in the proof of Theorem 2.1. □

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References

[1] R. Aulaskari and P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex Analysis and Its Applications (Hong Kong, 1993), Pitman Res. Notes Math. Ser., vol. 305, Longman Scientific & Technical, Harlow, 1994, pp. 136–146.

[2] R. Aulaskari, P. Lappan, J. Xiao, and R. Zhao, *On α-Bloch spaces and multipliers of Dirichlet spaces*, J. Math. Anal. Appl. **209** (1997), no. 1, 103–121.

[3] R. Aulaskari, J. Xiao, and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), no. 2, 101–121.

[4] A. Baernstein II, *Univalence and bounded mean oscillation*, Michigan Math. J. **23** (1976), no. 3, 217–223.

[5] ______, *Analytic functions of bounded mean oscillation*, Aspects of Contemporary Complex Analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979), Academic Press, London, 1980, pp. 3–36.

[6] M. Essén and H. Wulan, *On analytic and meromorphic functions and spaces of Q_K-type*, Illinois J. Math. **46** (2002), no. 4, 1233–1258.

[7] W. K. Hayman, *Multivalent Functions*, 2nd ed., Cambridge Tracts in Mathematics, vol. 110, Cambridge University Press, Cambridge, 1994.

[8] Ch. Pommerenke, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv. **52** (1977), no. 4, 591–602 (German).

[9] ______, *Boundary Behaviour of Conformal Maps*, Grundlehren der mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[10] H. Wulan, *On univalent Bloch functions*, Acta Math. Sci. Ser. B Engl. Ed. **21** (2001), no. 1, 37–40.

[11] H. Wulan and P. Wu, *Characterizations of Q_T spaces*, J. Math. Anal. Appl. **254** (2001), no. 2, 484–497.

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