IJMMS 2004:62, 3333–3338
PII. S0161171204310446
http://ijmms.hindawi.com
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FIXED POINT CHARACTERIZATION OF LEFT AMENABLE LAU ALGEBRAS

R. NASR-ISFAHANI

Received 1 October 2003

The present paper deals with the concept of left amenability for a wide range of Banach algebras known as Lau algebras. It gives a fixed point property characterizing left amenable Lau algebras $\mathcal{A}$ in terms of left Banach $\mathcal{A}$-modules. It also offers an application of this result to some Lau algebras related to a locally compact group G, such as the Eymard-Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$, and the measure algebra $M(G)$. In particular, it presents some equivalent statements which characterize amenability of locally compact groups.

2000 Mathematics Subject Classification: 43A07, 43A10, 43A20, 46H05.

1. Introduction. A *Lau algebra* is a complex Banach algebra $\mathcal{A}$ which is the unique predual of a von-Neumann algebra $\mathcal{M}$ and the identity element of $\mathcal{M}$ is a multiplicative linear functional on $\mathcal{A}$. The subject of this large class of Banach algebras originated with a paper published in 1983 by Lau [3], in which he referred to them as "F-algebras." Later on, in his useful monograph, Pier [11] introduced the name "Lau algebra."

As pointed out in [3], such an $\mathcal{M}$ is not necessarily unique as a dual von-Neumann algebra of $\mathcal{A}$, although it is plainly unique as the dual space of $\mathcal{A}$. We will endow the dual space $\mathcal{A}^*$ of $\mathcal{A}$ with the structure of a fixed von-Neumann algebra whose identity element is a multiplicative linear functional on $\mathcal{A}$.

The main development of the theory of Lau algebras concerns some notions of amenability; however, the theory also deals with some other aspects of analysis on Banach algebras. See [2, 4, 5, 6, 7, 8, 9]. In the pioneering paper [3], Lau introduced and studied a concept of amenability for Lau algebras called left amenability. In the same paper, he obtained several properties characterizing left amenable Lau algebras.

In this paper, we establish a fixed point characterization of left amenable Lau algebras. We also give an application of this result to the following important Lau algebras related to a locally compact group G: the Eymard-Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$, and the measure algebra $M(G)$, as defined in [10].

2. Preliminaries. Let $\mathcal{A}$ be a Lau algebra. By a *left Banach $\mathcal{A}$-module*, we mean a Banach space X equipped with a bounded bilinear map from $\mathcal{A}\times X$ into X, denoted by $(a,\xi)\mapsto a\cdot\xi$ ($a\in\mathcal{A}$, $\xi\in X$) such that

$$a\cdot(b\cdot\xi)=(ab)\cdot\xi \quad (a,b\in\mathcal{A},\ \xi\in X). \tag{2.1}$$

A *right Banach $\mathscr{A}$-module* is defined similarly. A *two-sided Banach $\mathscr{A}$-module* is a left and right Banach $\mathscr{A}$-module such that $(a\cdot\xi)\cdot b = a\cdot(\xi\cdot b)$ for all $a,b\in\mathscr{A}$ and $\xi\in X$.

The dual space X^* of a left (resp., right) Banach $\mathscr{A}$-module X becomes a right (resp., left) Banach $\mathscr{A}$-module with

$$\langle \xi^*\cdot a,\xi\rangle = \langle \xi^*,a\cdot\xi\rangle \quad (\text{resp., } \langle a\cdot\xi^*,\xi\rangle = \langle \xi^*,\xi\cdot a\rangle) \tag{2.2}$$

for all $\xi\in X$, $\xi^*\in X^*$, and $a\in\mathscr{A}$.

Now, let X be a left Banach $\mathscr{A}$-module, and denote by $\mathscr{B}(X^{**})$ the Banach space of bounded linear operators on the second dual space X^{**} of X. By *weak* operator topology* on $\mathscr{B}(X^{**})$, we mean the locally convex topology of $\mathscr{B}(X^{**})$ determined by the family $\{q(\xi^{**},\xi^*) : \xi^{**}\in X^{**},\ \xi^*\in X^*\}$ of seminorms on $\mathscr{B}(X^{**})$, where

$$q(\xi^{**},\xi^*)(T) = |\langle T\xi^{**},\xi^*\rangle| \quad \forall T\in\mathscr{B}(X^{**}). \tag{2.3}$$

We denote by $\mathscr{P}(\mathscr{A},X^{**})$ the closure of the set $\{\Lambda_a : a\in P_1(\mathscr{A})\}$ in the weak* operator topology of $\mathscr{B}(X^{**})$. Here, $P_1(\mathscr{A})$ is the set of all elements $a\in\mathscr{A}$ with norm one that induces positive functionals on the dual von-Neumann algebra $\mathscr{A}^*$, and $\Lambda_a\in\mathscr{B}(X^{**})$ is the operator of left action of $a\in P_1(\mathscr{A})$ on X^{**}; that is,

$$\Lambda_a(\xi^{**}) = a\cdot\xi^{**} \quad \forall a\in P_1(\mathscr{A}),\ \xi^{**}\in X^{**}. \tag{2.4}$$

We remark that $P_1(\mathscr{A})$ with the multiplication of $\mathscr{A}$ is a semigroup. This can be readily checked by using the interesting equality

$$P_1(\mathscr{A}) = \{a\in\mathscr{A} : \|a\| = \langle u,a\rangle = 1\}, \tag{2.5}$$

where u denotes the identity element of $\mathscr{A}^*$; the latter equality follows at once from the fact that a bounded linear functional ϕ on a C^*-algebra with identity is positive if and only if $\|\phi\|$ is equal to the value of ϕ at the identity [12, Propositions 1.5.1 and 1.5.2]. In particular, the set $\{\Lambda_a : a\in P_1(\mathscr{A})\}$ is a subsemigroup of the semigroup $\mathscr{B}(X^{**})$ with the ordinary multiplication of linear operators, and as easily verified, so is its closure $\mathscr{P}(\mathscr{A},X^{**})$ in the weak* operator topology of $\mathscr{B}(X^{**})$.

The second dual $\mathscr{A}^{**}$ of $\mathscr{A}$ is a Lau algebra with the first Arens product defined by the equations

$$\langle F\odot H,f\rangle = \langle F,Hf\rangle, \qquad \langle Hf,a\rangle = \langle H,fa\rangle, \qquad \langle fa,b\rangle = \langle f,ab\rangle \tag{2.6}$$

for all $F,H\in\mathscr{A}^{**}$, $f\in\mathscr{A}^*$, and $a,b\in\mathscr{A}$; see [3, Proposition 3.2] (for $R=\mathscr{A}^*$ in the notation of [3]). Considering $\mathscr{A}$ as a left Banach $\mathscr{A}$-module, we have the following lemma

Lemma 2.1. *Let $\mathscr{A}$ be a Lau algebra. Then, the mapping Φ defined by $\Phi(N)(F) = N\odot F$ for all $N\in P_1(\mathscr{A}^{**})$ and $F\in\mathscr{A}^{**}$ is a semigroup homomorphism of $P_1(\mathscr{A}^{**})$ onto $\mathscr{P}(\mathscr{A},\mathscr{A}^{**})$.*

Proof. In view of [4, Lemma 2.1], the set of states in the predual of a von-Neumann algebra is weak* dense in the set of states in its dual space. In particular, $P_1(\mathscr{A})$ is weak* dense in $P_1(\mathscr{A}^{**})$. So, the bounded linear operator $\Phi(N)$ on $\mathscr{A}^{**}$ lies in $\mathscr{P}(\mathscr{A},\mathscr{A}^{**})$ for all $N \in P_1(\mathcal{A}^{**})$; indeed, there is a net (b_β) in $P_1(\mathcal{A})$ converging to N in the weak* topology of $\mathcal{A}^{**}$, and hence for every $f \in \mathcal{A}$ and $F \in \mathcal{A}^{**}$,

$$\begin{aligned} \langle b_\beta \cdot F, f\rangle &= \langle b_\beta \odot F, f\rangle = \langle b_\beta, Ff\rangle \\ &\longrightarrow \langle N, Ff\rangle = \langle N \odot F, f\rangle, \end{aligned} \tag{2.7}$$

which implies that the net $(\Lambda_{b_\beta}) \subseteq \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ converges to $\Phi(N)$ in the weak* operator topology of $\mathcal{B}(\mathcal{A}^{**})$. This shows that Φ is well defined. Since Φ is obviously a semigroup homomorphism, it remains to prove that Φ is onto. To that end, let $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$. Choose a net (a_γ) in $P_1(\mathcal{A})$ such that $\Lambda_{a_\gamma} \to \Lambda$ in the weak* operator topology, and let N be a weak* cluster point of (a_γ) in $P_1(\mathcal{A}^{**})$. Then, for each $f \in \mathcal{A}^*$ and $F \in \mathcal{A}^{**}$ we conclude from (2.4) and (2.6) that

$$\begin{aligned} \langle \Lambda F, f\rangle &= \lim_\delta \langle \Lambda_{a_\delta} F, f\rangle = \lim_\delta \langle a_\delta \cdot F, f\rangle \\ &= \lim_\delta \langle a_\delta \odot F, f\rangle = \lim_\delta \langle a_\delta, Ff\rangle \\ &= \langle N, Ff\rangle = \langle N \odot F, f\rangle, \end{aligned} \tag{2.8}$$

where (a_δ) is a subnet of (a_γ) converging to N in the weak* topology, that is, $\Lambda = \Phi(N)$ as required. □

3. The main result. Let $\mathcal{A}$ be a Lau algebra. A *derivation* from $\mathcal{A}$ into the dual space X^* of a two-sided Banach $\mathcal{A}$-module X is a linear map D such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad \forall a, b \in \mathcal{A}. \tag{3.1}$$

The Lau algebra $\mathcal{A}$ is called *left amenable* if for each two-sided Banach $\mathcal{A}$-module X with $a \cdot \xi = \xi$ $(a \in P_1(\mathcal{A}),\ \xi \in X)$, every bounded derivation $D : \mathcal{A} \to X^*$ is inner; that is, there exists $\xi^* \in X^*$ with

$$D(a) = a \cdot \xi^* - \xi^* \cdot a \quad \forall a \in \mathcal{A}. \tag{3.2}$$

We now give the following fixed point characterization of left amenable Lau algebras. First, we recall that an element $M \in P_1(\mathcal{A}^{**})$ is called a *topological left invariant mean* on $\mathcal{A}^*$ if $a \odot M = M$ for all $a \in P_1(\mathcal{A})$.

Theorem 3.1. *Let $\mathcal{A}$ be an arbitrary Lau algebra. Then, the following conditions are equivalent.*

(a) *$\mathcal{A}$ is left amenable.*

(b) *There exists $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$.*

(c) *For each left Banach $\mathcal{A}$-module X, there exists $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ such that $\Lambda_a \Lambda = \Lambda$ for all $a \in P_1(\mathcal{A})$.*

Proof. (a)⇒(c). Suppose that $\mathcal{A}$ is left amenable. Appealing to [3, Theorem 4.6], there exists a net (a_γ) in $P_1(\mathcal{A})$ such that

$$\|a a_\gamma - a_\gamma\| \longrightarrow 0 \quad \forall a \in P_1(\mathcal{A}). \tag{3.3}$$

Now, recall that the operator algebra $\mathcal{B}(X^{**})$ can be identified with the dual space $(X^{**}\hat{\otimes}X^*)^*$ of the projective tensor product $X^{**}\hat{\otimes}X^*$ in a natural way; see, for example, [1, Corollary VIII.2.2]. In particular, the weak* operator topology of $\mathcal{B}(X^{**})$ coincides with the weak* topology of $(X^{**}\otimes X^*)^*$ on bounded subsets of $\mathcal{B}(X^{**})$, and therefore $\mathcal{P}(\mathcal{A},X^{**})$ is compact in the weak* operator topology of $\mathcal{B}(X^{**})$; indeed, as readily checked, $\|\Lambda\|\le K$ for all $\Lambda\in\mathcal{P}(\mathcal{A},X^{**})$, where K is a constant satisfying

$$\|b\cdot\xi\|\le K\|b\|\|\xi\| \quad \forall b\in\mathcal{A},\ \xi\in X. \tag{3.4}$$

Since (Λ_{a_γ}) is contained in $\mathcal{P}(\mathcal{A},X^{**})$, we may find $\Lambda\in\mathcal{P}(\mathcal{A},X^{**})$ with $\|\Lambda\|\le K$ and a subnet (a_δ) of (a_γ) such that $\Lambda_{a_\delta}\to\Lambda$ in the weak* operator topology of $\mathcal{B}(X^{**})$. For each $a\in P_1(\mathcal{A})$, we therefore have $\Lambda_a\Lambda_{a_\delta}\to\Lambda_a\Lambda$ in the weak* operator topology. Also, by (3.3) and (3.4) we have

$$||\Lambda_a\Lambda_{a_\delta}-\Lambda_{a_\delta}||\le K||aa_\delta-a_\delta||\longrightarrow 0. \tag{3.5}$$

It follows that $\Lambda_a\Lambda=\Lambda$ for all $a\in P_1(\mathcal{A})$.

(c)⇒(b). The implication is clear.

(b)⇒(a). Suppose that (b) holds and choose an element Λ of $\mathcal{P}(\mathcal{A},\mathcal{A}^{**})$ such that $\Lambda_a\Lambda=\Lambda$ for all $a\in P_1(\mathcal{A})$. Using Lemma 2.1, there exists an element M of $P_1(\mathcal{A}^{**})$ such that

$$\Lambda F=M\odot F \quad \forall F\in\mathcal{A}^{**}. \tag{3.6}$$

We therefore have $a\odot(M\odot F)=M\odot F$ for all $F\in\mathcal{A}^{**}$. It follows that for each $N\in P_1(\mathcal{A}^{**})$, $M\odot N$ is a topological left invariant mean on $\mathcal{A}^*$. Now, (a) follows from the fact that left amenability of $\mathcal{A}$ is equivalent to the existence of a topological left invariant mean on $\mathcal{A}^*$; see [3, Theorem 4.1]. □

As an application of this result, we present the following descriptions of amenable locally compact groups. First, we recall that a locally compact group G is called *amenable* if there is a *left invariant mean* on the dual $L^\infty(G)$ of $L^1(G)$; that is a functional $m\in P_1(L^\infty(G)^*)$ such that $m({}_xg)=m(g)$ for all $g\in L^\infty(G)$ and $x\in G$, where $({}_xg)(y)=g(xy)$ for all $y\in G$. Examples of amenable groups include all commutative groups and all compact groups; refer to Pier [10] for details.

Corollary 3.2. *Let G be a locally compact group with left Haar measure λ. Each of the following properties characterizes the amenability of G.*

(i) *There exists $\Lambda\in\mathcal{P}(L^1(G),L^\infty(G)^*)$ such that $\Lambda_f\Lambda=\Lambda$ for all $f\in L^1(G)$ with $f\ge 0$ and $\int_G f\,d\lambda=1$.*

(ii) *For each left Banach $L^1(G)$-module X, there exists $\Lambda\in\mathcal{P}(L^1(G),X^{**})$ such that $\Lambda_f\Lambda=\Lambda$ for all $f\in L^1(G)$ with $f\ge 0$ and $\int_G f\,d\lambda=1$.*

Proof. The identity element of $L^\infty(G)$ is the constant function 1_G. So, the result follows from Theorem 3.1 and the fact that the amenability of G is equivalent to the left amenability of $L^1(G)$; see [3, Theorem 4.1] and [10, Theorem 4.19]. □

Corollary 3.3. *A locally compact group G is amenable if and only if any of the following properties holds.*

(i) *There exists $\Lambda \in \mathcal{P}(M(G), M(G)^{**})$ such that $\Lambda_\mu \Lambda = \Lambda$ for all $\mu \in M(G)$ with $\mu \geq 0$ and $\mu(G) = 1$.*

(ii) *For each left Banach $M(G)$-module X, there exists $\Lambda \in \mathcal{P}(M(G), X^{**})$ such that $\Lambda_\mu \Lambda = \Lambda$ for all $\mu \in M(G)$ with $\mu \geq 0$ and $\mu(G) = 1$.*

Proof. The Lau algebra $M(G)$ is left amenable if and only if G is amenable; see [3, Corollary 4.3]. So, the result follows from Theorem 3.1 together with that the identity element of the dual W^* algebras of $M(G)$ is the functional defined by $u(\mu) = \mu(G)$ for all $\mu \in M(G)$. □

We round up this paper by giving another consequence of our main result. Recall that $A(G)$ is spanned by functions with compact support in $P(G)$, and $B(G)$ is spanned by $P(G)$, where $P(G)$ denotes the set of all continuous positive definite functions on G; see Pier [10] for details.

Corollary 3.4. *Let G be a locally compact group with identity e. Then, for each left Banach $A(G)$-module X, there exists $\Lambda \in \mathcal{P}(A(G), X^{**})$ such that $\Lambda_\varphi \Lambda = \Lambda$ for all $\varphi \in A(G) \cap P(G)$ with $\varphi(e) = 1$. A similar result holds for $B(G)$.*

Proof. Recall from [3] that the identity element of the dual W^* algebras of $A(G)$ is the functional defined by $u(\varphi) = \varphi(e)$ for all $\varphi \in A(G)$. Since $A(G)$ is a commutative Lau algebra, the result follows from Theorem 3.1 together with the fact that any commutative Lau algebra is left amenable; see [3, Example 1]. The proof of the second assertion is similar. □

Acknowledgment. The author would like to thank the referees of this paper for their valuable suggestions.

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R. Nasr-Isfahani: Department of Mathematics, Isfahan University of Technology, Isfahan, Iran
E-mail address: isfahani@cc.iut.ac.ir