SUMMABILITY OF DOUBLE SEQUENCES BY WEIGHTED MEAN METHODS AND TAUBERIAN CONDITIONS FOR CONVERGENCE IN PRINGSHEIM'S SENSE

FERENC MÓRICZ and U. STADTMÜLLER

After a brief summary of Tauberian conditions for ordinary sequences of numbers, we consider summability of double sequences of real or complex numbers by weighted mean methods which are not necessarily products of related weighted mean methods in one variable. Our goal is to obtain Tauberian conditions under which convergence of a double sequence follows from its summability, where convergence is understood in Pringsheim's sense. In the case of double sequences of real numbers, we present necessary and sufficient Tauberian conditions, which are so-called one-sided conditions. Corollaries allow these Tauberian conditions to be replaced by Schmidt-type slow decrease conditions. For double sequences of complex numbers, we present necessary and sufficient so-called two-sided Tauberian conditions. In particular, these conditions are satisfied if the summable double sequence is slowly oscillating.

1. Introduction. We begin with a brief and concise summary of the corresponding well-known results for single sequences. For basic facts on summability theory, we refer to [4, 9, 12] for ordinary sequences and to [1] for double sequences.

Let $p = (p_k : k = 0, 1, 2, \dots)$ be a fixed sequence of nonnegative numbers with $p_0 > 0$, and set

$$P_m := \sum_{k=0}^{m} p_k, \quad k = 0, 1, \dots. \tag{1.1}$$

Weighted means of a sequence $(s_k : k = 0, 1, \dots)$ of complex numbers are defined by

$$t_m := \frac{1}{P_m} \sum_{k=0}^{m} p_k s_k, \quad m = 0, 1, \dots. \tag{1.2}$$

The sequence (s_k) is said to be summable by the weighted mean method determined by the sequence p, in short, summable $(\overline{N}, p)$, if the sequence (t_m) converges to a finite limit s; in symbols, $s_m \to s(\overline{N}, p)$.

The summability method $(\overline{N}, p)$ is said to be regular, if $s_m \to s$ implies $s_m \to s(\overline{N}, p)$. It is well known that $(\overline{N}, p)$ is regular if and only if $P_m \to \infty$ as $m \to \infty$ (see, e.g., [9, page 16]), which we assume in the following.

We are interested in converse conclusions. In general, $s_m \to s(\overline{N}, p)$ implies $s_m \to s$ only under additional so-called Tauberian conditions. Set

$$a_k := s_k - s_{k-1}, \quad k = 0, 1, \ldots; \ s_{-1} := 0. \tag{1.3}$$

Then, each of the following conditions is Tauberian for the method $(\overline{N}, p)$:

$$\sum_{k=1}^{m} a_k P_{k-1} = o(P_m), \tag{1.4}$$

$$\sum_{k=1}^{\infty} |a_k|^2 \frac{P_{k-1}}{p_{k-1}} < \infty, \quad p_k > 0 \ \forall k > 0, \tag{1.5}$$

$$a_k = o\left(\frac{p_{k-1}}{P_{k-1}}\right), \tag{1.6}$$

since they yield $t_m - s_m \to 0$ as $m \to \infty$. The o-type condition (1.6) can be weakened to an O-type condition

$$a_k = O\left(\frac{p_{k-1}}{P_{k-1}}\right), \tag{1.7}$$

or even to a one-sided condition

$$a_k \le c \frac{p_{k-1}}{P_{k-1}}, \tag{1.8}$$

where c is a positive constant and in the last case we suppose that p also satisfies the condition $p_m / P_m \to 0$ as $m \to \infty$.

A necessary and sufficient Tauberian condition was given in [8], which is implied by either of the conditions (1.4)–(1.8). To present it, we recall the following two definitions. Let $\lambda = (\lambda(m))$, where $\lambda(m) > m$ for all m, be an increasing sequence of natural numbers such that

$$\liminf_{m \to \infty} \frac{P_{\lambda(m)}}{P_m} > 1, \tag{1.9}$$

and denote by Λ_u the set of all such sequences λ. Similarly, let $\mu = (\mu(m))$, where $\mu(m) < m$ for all m, be a nondecreasing sequence of natural numbers such that

$$\liminf_{m \to \infty} \frac{P_m}{P_{\mu(m)}} > 1, \tag{1.10}$$

and denote by Λ_ℓ the set of all such sequences μ.

Now, the following theorem was proved in [8]: for a given sequence (s_m) of real numbers, $s_m \to s(\overline{N}, p)$ implies $s_m \to s$ if and only if

$$\inf_{\lambda \in \Lambda_u} \limsup_{m \to \infty} \frac{1}{P_{\lambda(m)} - P_m} \sum_{k=m+1}^{\lambda(m)} p_k(s_k - s_m) \le 0, \tag{1.11a}$$

$$\inf_{\mu \in \Lambda_\ell} \limsup_{m \to \infty} \frac{1}{P_m - P_{\mu(m)}} \sum_{k=\mu(m)+1}^{m} p_k(s_m - s_k) \le 0. \tag{1.11b}$$

In case of smooth weights, for example, when

$$p_k := (k+1)^\alpha, \quad k = 0,1,\ldots; \ \alpha > -1, \tag{1.12}$$

we may replace Λ_u by $\tilde{\Lambda}_u := \{([\lambda m]) : \lambda > 1\}$ and Λ_ℓ by $\tilde{\Lambda}_\ell := \{([m/\lambda]) : \lambda > 1\}$, where $[a]$ denotes the integer part of $a > 0$. Furthermore, either the condition of slow decrease in the sense of Schmidt [10] (see also [4, pages 124-125])

$$\lim_{\lambda \to 1+} \liminf_{m \to \infty} \min_{m < k \le \lambda m} (s_k - s_m) \ge 0, \tag{1.13}$$

or the condition of slow increase

$$\lim_{\lambda \to 1+} \limsup_{m \to \infty} \max_{m < k \le \lambda m} (s_k - s_m) \le 0, \tag{1.14}$$

implies both (1.11a) and (1.11b) (see [8] for details).

In the same paper, the following theorem for complex sequences was proved: $s_m \to s(\overline{N}, p)$ implies $s_m \to s$ if and only if one of the following Tauberian conditions is satisfied:

$$\inf_{\lambda \in \Lambda_u} \limsup_{m \to \infty} \left| \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda(m)} p_k (s_k - s_m) \right| = 0 \tag{1.15a}$$

or

$$\inf_{\mu \in \Lambda_\ell} \limsup_{m \to \infty} \left| \frac{1}{P_m - P_{\mu(m)}} \sum_{k=\mu(m)+1}^{m} p_k (s_m - s_k) \right| = 0. \tag{1.15b}$$

The following special case is called the condition of slow oscillation (see, e.g., [6]):

$$\inf_{\lambda \in \Lambda_u} \limsup_{m \to \infty} \max_{m < k \le \lambda(m)} |s_k - s_m| = 0. \tag{1.16}$$

In this case (1.15a) (and a fortiori, (1.15b)) is clearly satisfied. The symmetric counterpart involving the class Λ_ℓ can be formulated analogously.

From now on, we will consider double sequences. Let $p = (p_{k,\ell} : k, \ell = 0,1,\ldots)$ be a fixed double sequence of nonnegative numbers with $p_{0,0} > 0$, and set

$$P_{m,n} := \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell}, \quad m,n = 0,1,\ldots. \tag{1.17}$$

The weighted means of a double sequence $(s_{k,\ell} : k, \ell = 0,1,\ldots)$ of complex numbers are defined by

$$t_{m,n} := \frac{1}{P_{m,n}} \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell} s_{k,\ell}, \quad m,n = 0,1,\ldots. \tag{1.18}$$

The sequence $(s_{k,\ell})$ is said to be summable by the weighted mean method determined by the sequence p, in short, summable $(\overline{N}, p)$, if the double sequence $(t_{m,n})$ converges

to a finite number s in Pringsheim's sense; in symbols, $s_{m,n} \to s(\overline{N}, p)$. This type of convergence means that for every $\varepsilon > 0$ there exists a natural number $n_0(\varepsilon)$ such that

$$|t_{m,n} - s| < \varepsilon \quad \forall m, n \geq n_0(\varepsilon). \tag{1.19}$$

Furthermore, we also consider the so-called bounded convergence (in Pringsheim's sense); in symbols, $b - \lim s_{m,n} = s$, which means the following:

$$s_{m,n} \longrightarrow s, \quad |s_{m,n}| \leq K \quad \forall m, n = 0, 1, \ldots, \tag{1.20}$$

where K is a finite positive number. Similarly, $b - \lim s_{m,n} = s(\overline{N}, p)$ means that $b - \lim t_{m,n} = s$.

It is known (see, e.g., [3]) that

$$b - \lim s_{m,n} = s \Longrightarrow b - \lim s_{m,n} = s(\overline{N}, p) \tag{1.21}$$

if and only if

$$\lim_{m,n \to \infty} \frac{P_{k,n}}{P_{m,n}} = 0 \quad \forall k \tag{1.22}$$

and

$$\lim_{m,n \to \infty} \frac{P_{m,\ell}}{P_{m,n}} = 0 \quad \forall \ell. \tag{1.23}$$

In particular, it follows from (1.22) that $P_{m,n} \to \infty$ as $m, n \to \infty$.

Of particular interest are the weighted means of multiplicative structure, that is, when $p_{k,\ell} = p_k q_\ell$, where (p_k) and (q_ℓ) are sequences of nonnegative real numbers with $p_0, q_0 > 0$. Given a double sequence $(s_{m,n})$, we set

$$a_{m,n} := s_{m,n} - s_{m-1,n} - s_{m,n-1} + s_{m-1,n-1}, \quad m, n = 0, 1, \ldots, \tag{1.24}$$

with the agreement that $s_{m,n} = 0$ if $m < 0$ or $n < 0$.

It is known (see [11]) that if

$$P_m := \sum_{k=0}^{m} p_k \longrightarrow \infty, \quad Q_n := \sum_{\ell=0}^{n} q_\ell \longrightarrow \infty \quad m, n \longrightarrow \infty, \tag{1.25}$$

then the pair of conditions

$$\sup_{n \geq 0} \left| \sum_{\ell=0}^{n} a_{m,\ell} \right| \leq c \frac{p_{m-1}}{P_{m-1}} \quad \forall m, \qquad \sup_{m \geq 0} \left| \sum_{k=0}^{m} a_{k,n} \right| \leq c \frac{q_{n-1}}{Q_{n-1}} \quad \forall n, \tag{1.26}$$

where c is some constant, is a Tauberian condition; that is, under conditions (1.25) and (1.26), $\lim s_{m,n} = s(\overline{N}, p)$ implies $\lim s_{m,n} = s$. We observe that (1.26) can be considered to be a multiplicative version of the Tauberian condition (1.7). However, $a_{m,n} = O(p_{m-1}q_{n-1}/P_{m-1}Q_{n-1})$ (compare (1.7)) is not a Tauberian condition, as it has been shown in [5] for the $(C, 1, 1)$-mean.

2. Main results. In the sequel we will need the following notations. Let m, n, μ, ν be nonnegative integers and set

$$a_\mu(n) := \sum_{\nu=0}^{n} a_{\mu,\nu}, \qquad b_\nu(m) := \sum_{\mu=0}^{m} a_{\mu,\nu}, \tag{2.1}$$

where $a_{\mu,\nu}$ is defined in (1.24). Clearly, we have

$$a_\mu(n) := s_{\mu,n} - s_{\mu-1,n}, \qquad b_\nu(m) := s_{m,\nu} - s_{m,\nu-1}. \tag{2.2}$$

Furthermore, set

$$p_k(n) := \sum_{\nu=0}^{n} p_{k,\nu}, \qquad q_\ell(m) := \sum_{\mu=0}^{m} p_{\mu,\ell}, \quad k,\ell = 0,1,\dots. \tag{2.3}$$

Now, the multivariate version of the Tauberian condition (1.4) reads as follows.

THEOREM 2.1. *If $s_{m,n} \to s(\overline{N},p)$ and one of the following Tauberian conditions is satisfied,*

$$\sum_{\mu=1}^{n} P_{\mu-1,n} \max_{\ell \leq n} |a_\mu(\ell)| + \sum_{\nu=1}^{n} P_{m,\nu-1} |b_\nu(m)| = o(P_{m,n}) \tag{2.4a}$$

or

$$\sum_{\mu=1}^{n} P_{\mu-1,n} |a_\mu(n)| + \sum_{\nu=1}^{n} P_{m,\nu-1} \max_{k \leq m} |b_\nu(k)| = o(P_{m,n}), \tag{2.4b}$$

then $s_{m,n} \to s$.

The multivariate version of Tauberian condition (1.5) is more involved.

THEOREM 2.2. *Assume that (1.22) holds. If $b - \lim s_{m,n} = s(\overline{N},p)$ and one of the following Tauberian conditions is satisfied: for every $\varepsilon > 0$ there exist natural numbers μ_0, ν_0 such that*

$$\sum_{\mu=\mu_0}^{\infty} \frac{P_{\mu-1,n}}{p_{\mu-1}(n)} \max_{\ell \leq n} |a_\mu(\ell)|^2 < \varepsilon \quad \text{for large } n,$$
$$\sum_{\nu=\nu_0}^{\infty} \frac{P_{m,\nu-1}}{q_{\nu-1}(m)} |b_\nu(m)|^2 < \varepsilon \quad \text{for large } m \tag{2.5a}$$

or

$$\sum_{\mu=\mu_0}^{\infty} \frac{P_{\mu-1,n}}{p_{\mu-1}(n)} |a_\mu(n)|^2 < \varepsilon \quad \text{for large } n,$$
$$\sum_{\nu=\nu_0}^{\infty} \frac{P_{m,\nu-1}}{q_{\nu-1}(m)} \max_{k \leq m} |b_\nu(k)|^2 < \varepsilon \quad \text{for large } m, \tag{2.5b}$$

then $b - \lim s_{m,n} = s$.

Now, we turn to our main result which provides a necessary and sufficient Tauberian condition to deduce the conclusion $s_{m,n} \to s$ from $s_{m,n} \to s(\overline{N}, p)$. A result of this type was already discussed in [7] for the $(C,1,1)$-mean and for more general weighted means in [11], however, the assumptions there can be simplified and the proof in the present paper is direct.

To formulate Theorem 2.5, we introduce the notation

$$\triangle(m,n;k,\ell) := \sum_{m<\mu\leq k}\sum_{n<\nu\leq\ell} p_{\mu,\nu} = P_{k,\ell} - P_{k,n} - P_{m,\ell} + P_{m,n}, \tag{2.6}$$

where $k > m$ and $\ell > n$. We consider a pair of nondecreasing sequences $(\lambda_1(m) > m, \lambda_2(n) > n : m,n = 1,2,\dots)$ of natural numbers such that

$$\liminf_{m,n\to\infty} \frac{P_{\lambda_1(m),\lambda_2(n)}}{P_{m,n}} > 1, \tag{2.7}$$

$$\limsup_{m,n\to\infty} \frac{P_{\lambda_1(m),\lambda_2(n)}}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} < \infty, \tag{2.8}$$

and denote by Λ_u the set of all such pairs of sequences.

Furthermore, we consider a pair of nondecreasing sequences $(\mu_1(m) < m, \mu_2(n) < n)$ of natural numbers such that

$$\liminf_{m,n\to\infty} \frac{P_{m,n}}{P_{\mu_1(m),\mu_2(n)}} > 1,$$

$$\limsup_{m,n\to\infty} \frac{P_{m,n}}{\triangle(\mu_1(m),\mu_2(n);m,n)} < \infty, \tag{2.9}$$

and denote by Λ_ℓ the set of all such pairs of sequences.

We remind the reader that the limit infimum of a double sequence of real numbers is defined by

$$\liminf_{m,n\to\infty} b_{m,n} := \lim_{N\to\infty} \inf_{m,n\geq N} b_{m,n}, \tag{2.10}$$

where the right-hand side may be $-\infty$ or ∞. Obviously,

$$\limsup_{m,n\to\infty} b_{m,n} = -\liminf_{m,n\to\infty}(-b_{m,n}). \tag{2.11}$$

EXAMPLE 2.3. Put $p_{m,n} = m+n$, then we have

$$P_{m,n} = \left(\frac{1}{2}\right)m(m+1)n + \left(\frac{1}{2}\right)n(n+1)m \sim \left(\frac{1}{2}\right)nm(n+m), \tag{2.12}$$

and a typical example of a sequence from Λ_u is $\lambda_1(m) = [\lambda m]$, $\lambda_2(n) = [\mu n]$ with λ and $\mu > 1$ leading to

$$\triangle(m,n;\lambda_1(m),\lambda_2(n)) \sim \left(\frac{1}{2}\right)nm(m(\lambda^2\mu-\lambda^2-\mu+1)+n(\mu^2\lambda-\mu^2-\lambda+1)). \tag{2.13}$$

We cannot expect that (2.8) is always satisfied, since for example with $p_{k,\ell} = e^{-k\ell}$ we have $P_{m,n} = n+m+o(1)$. However, in this case (1.22) is not satisfied.

LEMMA 2.4. *Under (1.22), the sets Λ_u and Λ_ℓ are not empty.*

PROOF. Given m_0, n_0, we can find by (1.22) integers $m \geq m_0$, $n \geq n_0$ large enough such that $P_{m,n} \geq 3P_{m_0,n}$ and $P_{m,n} \geq 3P_{m,n_0}$ which implies that $\triangle(m_0, n_0, m, n) \geq (1/3)P_{m,n}$. □

The next result contains a multivariate version of the Tauberian conditions in (1.11).

THEOREM 2.5. *Assume that the sets Λ_u and Λ_ℓ are nonempty and $s_{m,n}$ is a double sequence of real numbers. Then, $s_{m,n} \to s(\overline{N}, p)$ implies $s_{m,n} \to s$ if and only if both of the following conditions are satisfied:*

$$\inf_{\Lambda_u} \limsup_{m,n \to \infty} \frac{1}{\triangle(m, n : \lambda_1(m), \lambda_2(n))} \sum_{m < k \leq \lambda_1(m)} \sum_{n < \ell \leq \lambda_2(n)} p_{k,\ell}(s_{k,\ell} - s_{m,n}) \leq 0, \quad (2.14a)$$

$$\inf_{\Lambda_\ell} \limsup_{m,n \to \infty} \frac{1}{\triangle(\mu_1(m), \mu_2(n); m, n)} \sum_{\mu_1(m) < k \leq m} \sum_{\mu_2(n) < \ell \leq n} p_{k,\ell}(s_{m,n} - s_{k,\ell}) \leq 0. \quad (2.14b)$$

REMARKS 2.6. (i) The conditions in (2.14) are obviously satisfied if

$$\inf_{\Lambda_u} \limsup_{m,n \to \infty} \max_{\substack{m < k \leq \lambda_1(m) \\ n < \ell \leq \lambda_2(n)}} (s_{k,\ell} - s_{m,n}) \leq 0, \quad (2.15a)$$

$$\inf_{\Lambda_\ell} \limsup_{m,n \to \infty} \max_{\substack{\mu_1(m) < k \leq m \\ \mu_2(n) < \ell \leq n}} (s_{m,n} - s_{k,\ell}) \leq 0. \quad (2.15b)$$

(ii) In the special (but important) case of multiplicative weights $p_{k,\ell} = p_k q_\ell$ with sequences p, q as before, conditions (2.7) and (2.8) are satisfied if the sequences $(\lambda_1(m))$ and $(\lambda_2(n))$ of natural numbers are chosen such that

$$\liminf_{m \to \infty} \frac{P_{\lambda_1(m)}}{P_m} > 1, \qquad \liminf_{n \to \infty} \frac{Q_{\lambda_2(n)}}{Q_n} > 1. \quad (2.16)$$

Then, condition (2.7) is obviously satisfied. In addition, we have

$$\frac{P_{\lambda_1(m),\lambda_2(n)}}{\triangle(m, n; \lambda_1(m), \lambda_2(n))} = \left\{ \left(1 - \frac{P_m}{P_{\lambda_1(m)}}\right)\left(1 - \frac{Q_n}{Q_{\lambda_2(n)}}\right) \right\}^{-1}, \quad (2.17)$$

whence it follows that

$$\begin{aligned} \limsup_{m,n \to \infty} &\frac{P_{\lambda_1(m),\lambda_2(n)}}{\triangle(m, n; \lambda_1(m), \lambda_2(n))} \\ &= \left\{ \left(1 - \limsup_{m \to \infty} \frac{P_m}{P_{\lambda_1(m)}}\right)\left(1 - \limsup_{n \to \infty} \frac{Q_n}{Q_{\lambda_2(n)}}\right) \right\}^{-1} < \infty, \end{aligned} \quad (2.18)$$

which means that this time (2.8) is automatically satisfied.

(iii) In the special case when

$$\lim_{m \to \infty} \frac{P_{m+1}}{P_m} = 1, \qquad \lim_{n \to \infty} \frac{Q_{n+1}}{Q_n} = 1, \quad (2.19)$$

we may replace (2.14a) by

$$\inf_{\rho>1} \limsup_{m,n\to\infty} \frac{1}{(P_{m(\rho)}-P_m)(Q_{n(\rho)}-Q_n)} \sum_{m<k\le m(\rho)} \sum_{n<\ell\le n(\rho)} p_k q_\ell (s_{k,\ell}-s_{m,n}) \le 0, \quad (2.20)$$

where

$$m(\rho) := \min\left\{k>m : \frac{P_k}{P_m} \ge \rho\right\}, \tag{2.21}$$

and $n(\rho)$ is defined analogously based on (Q_n). A similar reformulation of condition (2.14b) reads as follows:

$$\inf_{0<\sigma<1} \limsup_{m,n\to\infty} \frac{1}{(P_m-P_{m(\sigma)})(Q_n-Q_{n(\sigma)})} \sum_{m(\sigma)<k\le m} \sum_{n(\sigma)<\ell\le n} p_k q_\ell (s_{m,n}-s_{k,\ell}) \le 0,$$
$$\tag{2.22}$$

where

$$m(\sigma) := \max\left\{k<m : \frac{P_k}{P_m} \le \sigma\right\} \tag{2.23}$$

and $n(\sigma)$ is defined analogously based on (Q_n).

(iv) It is clear that (2.20) and (2.22) are implied by the following conditions of slow increase:

$$\lim_{\rho\to 1+} \limsup_{m,n\to\infty} \max_{\substack{m<k\le m(\rho) \\ n<\ell\le n(\rho)}} (s_{k,\ell}-s_{m,n}) \le 0, \tag{2.24a}$$

$$\lim_{\sigma\to 1-} \limsup_{m,n\to\infty} \max_{\substack{m(\sigma)<k\le m \\ n(\sigma)<\ell\le n}} (s_{m,n}-s_{k,\ell}) \le 0, \tag{2.24b}$$

respectively. Now, one can verify that the last two conditions are equivalent. Furthermore, all particular cases discussed in [11] can be deduced from (2.24a).

Finally, the counterpart of Theorem 2.5 when $(s_{m,n})$ is a double sequence of complex numbers reads as follows.

THEOREM 2.7. *Assume that the sets Λ_u and Λ_ℓ are nonempty. Then, $s_{m,n} \to s(\overline{N},p)$ implies $s_{m,n} \to s$ if and only if one of the following conditions is satisfied:*

$$\inf_{\Lambda_u} \limsup_{m,n\to\infty} \left| \frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \sum_{m<k\le\lambda_1(m)} \sum_{n<\ell\le\lambda_2(n)} p_{k,\ell}(s_{k,\ell}-s_{m,n}) \right| = 0 \tag{2.25a}$$

or

$$\inf_{\Lambda_\ell} \limsup_{m,n \to \infty} \left| \frac{1}{\triangle(\mu_1(m), \mu_2(n); m, n)} \sum_{\mu_1(m) < k \le m} \sum_{\mu_2(n) < \ell \le n} p_{k,\ell}(s_{m,n} - s_{k,\ell}) \right| = 0.$$

(2.26a)

REMARK 2.8. The following special case may be called the condition of slow oscillation. If

$$\inf_{\Lambda_u} \limsup_{m,n \to \infty} \max_{\substack{m < k \le \lambda_1(m) \\ n < \ell \le \lambda_2(n)}} |s_{k,\ell} - s_{m,n}| = 0,$$

(2.27)

then condition (2.25a) is obviously satisfied. The symmetric counterpart of (2.27) from which (2.26a) follows can be formulated analogously.

Originally, these conditions were considered in the case of multiplicative weights in [2, 11].

3. Proofs

PROOF OF THEOREM 2.1. It hinges on the following decomposition. By (2.1), we may write

$$
\begin{aligned}
&\left| P_{m,n}(s_{m,n} - t_{m,n}) \right| \\
&= \left| \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell}(s_{m,n} - s_{k,\ell}) \right| \\
&= \left| \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell} \left\{ \sum_{\mu=k+1}^{m} \sum_{\nu=0}^{\ell} a_{\mu,\nu} + \sum_{\mu=0}^{m} \sum_{\nu=\ell+1}^{n} a_{\mu,\nu} \right\} \right| \\
&= \left| \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell} \left\{ \sum_{\mu=k+1}^{m} a_\mu(\ell) + \sum_{\nu=\ell+1}^{n} b_\nu(m) \right\} \right| \\
&\le \sum_{\mu=1}^{m} P_{\mu-1,n} \max_{1 \le \ell \le n} |a_\mu(\ell)| + \sum_{\nu=1}^{n} P_{m,\nu-1} |b_\nu(m)|.
\end{aligned}
$$

(3.1)

Thus, (2.4a) implies that $s_{m,n} - t_{m,n} \to 0$ as $m,n \to \infty$.

In the case of condition (2.4b), the above reasoning works if we use the decomposition

$$\sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell}(s_{m,n} - s_{k,\ell}) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} p_{k,\ell} \left\{ \sum_{\mu=k+1}^{m} \sum_{\nu=0}^{n} a_{\mu,\nu} + \sum_{\mu=0}^{k} \sum_{\nu=\ell+1}^{n} a_{\mu,\nu} \right\}.$$

(3.2) □

PROOF OF THEOREM 2.2. Assume that condition (2.5a) is satisfied. By the Cauchy inequality and (2.3), we have

$$
\begin{aligned}
&\sum_{\mu=1}^{m} P_{\mu-1,n} \max_{1 \le \ell \le n} |a_\mu(\ell)| \\
&\le \left\{ \sum_{\mu=1}^{m} \frac{P_{\mu-1,n}}{p_{\mu-1}(n)} \max_{1 \le \ell \le n} |a_\mu(\ell)|^2 \sum_{\mu=1}^{n} P_{\mu-1,n} p_{\mu-1}(n) \right\}^{1/2} \le c P_{m,n},
\end{aligned}
$$

(3.3)

where c is a constant. Similarly, we have

$$\sum_{\nu=1}^{n} P_{m,\nu-1} |b_\nu(m)| \le cP_{m,n}. \tag{3.4}$$

Taking into account (3.1) and the boundedness of the double sequence $(t_{m,n})$, from the inequalities above we conclude that $(s_{m,n})$ is also bounded, that is,

$$|s_{m,n}| \le K \quad \forall m,n = 0,1,\dots \tag{3.5}$$

with some constant K. By (2.2), it follows that

$$|a_\mu(\ell)| \le 2K, \quad |b_\nu(k)| \le 2K \quad \forall \mu,\nu,k,\ell = 0,1,\dots. \tag{3.6}$$

Now, we can proceed as follows. For any $\varepsilon > 0$, choose natural numbers μ_0, ν_0 according to the assumptions in Theorem 2.2 and follow the estimations above to obtain

$$\left\{ \sum_{\mu=1}^{\mu_0-1} + \sum_{\mu=\mu_0}^{m} \right\} P_{\mu-1,n} \max_{1 \le \ell \le n} |a_\mu(\ell)|$$

$$\le (\mu_0-1) P_{\mu_0-2,n} 2K + \left\{ \sum_{\mu=\mu_0}^{\infty} \frac{P_{\mu-1,n}}{p_\mu(n)} \max_{1 \le \ell \le n} |a_\mu(\ell)|^2 \sum_{\mu=1}^{m} P_{\mu-1,n} p_{\mu-1}(n) \right\}^{1/2} \tag{3.7}$$

$$\le P_{m,n} \left\{ (\mu_0-1) 2K \frac{P_{\mu_0-2,n}}{P_{m,n}} + \varepsilon^{1/2} \right\} \le 2P_{m,n}\varepsilon^{1/2},$$

provided that m, n are large enough (observe (1.22)). Similarly, we obtain

$$\left\{ \sum_{\nu=1}^{\nu_0-1} + \sum_{\nu=\nu_0}^{n} \right\} P_{m,\nu-1} |b_\nu(m)| \le 2P_{m,n}\varepsilon^{1/2}, \tag{3.8}$$

provided again that m, n are large enough. Combining (3.1), (3.7), and (3.8) yields

$$|s_{m,n} - t_{m,n}| \le 4\varepsilon^{1/2}, \tag{3.9}$$

which completes the proof since ε was arbitrary. The proof is analogous if (2.5b) is satisfied. $\square$

For the proof of Theorem 2.5, we need the following auxiliary result which is interesting in itself.

LEMMA 3.1. *If* $s_{m,n} \to s(\overline{N}, p)$, *then for any pair of sequences* $(\lambda_1(m)), (\lambda_2(n)) \in \Lambda_u$,

$$\lim_{m,n\to\infty} \frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \sum_{m<k\le\lambda_1(m)} \sum_{n<\ell\le\lambda_2(n)} p_{k,\ell} s_{k,\ell} = s, \tag{3.10}$$

and for any pair of sequences $(\mu_1(m)), (\mu_2(n)) \in \Lambda_\ell$,

$$\lim_{m,n\to\infty} \frac{1}{\triangle(\mu_1(m),\mu_2(n);m,n)} \sum_{\mu_1(m)<k\leq m} \sum_{\mu_2(n)<\ell\leq n} p_{k,\ell} s_{k,\ell} = s. \tag{3.11}$$

PROOF. By definition, we may write that

$$\sum_{m<k\leq\lambda_1(m)} \sum_{n<\ell\leq\lambda_2(n)} p_{k,\ell} s_{k,\ell}$$
$$= P_{\lambda_1(m),\lambda_2(n)} t_{\lambda_1(m),\lambda_2(n)} - P_{\lambda_1(m),n} t_{\lambda_1(m),n} - P_{m,\lambda_2(n)} t_{m,\lambda_2(n)} + P_{m,n} t_{m,n}. \tag{3.12}$$

Hence, a simple rearrangement gives

$$\frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \sum_{m<k\leq\lambda_1(m)} \sum_{n<\ell\leq\lambda_2(n)} p_{k,\ell} s_{k,\ell}$$
$$= t_{\lambda_1(m),\lambda_2(n)} + \frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \Big\{ \big(P_{\lambda_1(m),n} - P_{m,n}\big) \big(t_{\lambda_1(m),\lambda_2(n)} - t_{\lambda_1(m),n}\big)$$
$$+ \big(P_{m,\lambda_2(n)} - P_{m,n}\big) \big(t_{\lambda_1(m),\lambda_2(n)} - t_{m,\lambda_2(n)}\big)$$
$$+ P_{m,n} \big(t_{\lambda_1(m),\lambda_2(n)} - t_{\lambda_1(m),n} - t_{m,\lambda_2(n)} + t_{m,n}\big) \Big\}. \tag{3.13}$$

It remains to observe that by (2.6) we have

$$P_{\lambda_1(m),n} - P_{m,n} \leq P_{\lambda_1(m),\lambda_2(n)} - \triangle(m,n;\lambda_1(m),\lambda_2(n)), \tag{3.14}$$

whence, by (2.8) we obtain

$$\limsup_{m,n\to\infty} \frac{P_{\lambda_1(m),n} - P_{m,n}}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} < \infty; \tag{3.15}$$

and similarly

$$\limsup_{m,n\to\infty} \frac{P_{m,\lambda_2(n)} - P_{m,n}}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} < \infty. \tag{3.16}$$

Combining the last two inequalities with (3.13) and using the convergence of $(t_{m,n})$, we conclude (3.10).

The proof of (3.11) is similar. □

The following corollary is an immediate consequence of Lemma 3.1 by taking $(\lambda_1(m), \lambda_2(n)) = (m+1, n+1)$.

COROLLARY 3.2. *If*

$$\liminf_{m,n\to\infty} \frac{p_{m,n}}{P_{m,n}} > 0, \tag{3.17}$$

then $s_{m,n} \to s(\overline{N}, p)$ *if and only if* $s_{m,n} \to s$.

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PROOF OF THEOREM 2.5. *Necessity.* It follows directly from Lemma 3.1 and the convergence of $(s_{m,n})$. Even more is true, for every pair $(\lambda_1(m), \lambda_2(n)) \in \Lambda_u$, we have

$$\lim_{m,n\to\infty} \frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \sum_{m<k\leq\lambda_1(m)} \sum_{n<\ell\leq\lambda_2(n)} p_{k,\ell}(s_{k,\ell} - s_{m,n}) = 0. \qquad (3.18)$$

An analogous limit relation holds for every pair $(\mu_1(m), \mu_2(n)) \in \Lambda_\ell$.

Sufficiency. Given $\varepsilon > 0$, by (2.14a) there exists a pair $(\lambda_1(m), \lambda_2(n)) \in \Lambda_u$ such that

$$\limsup_{m,n\to\infty} \frac{1}{\triangle(m,n;\lambda_1(m),\lambda_2(n))} \sum_{m<k\leq\lambda_1(m)} \sum_{n<\ell\leq\lambda_2(n)} p_{k,\ell}(s_{k,\ell} - s_{m,n}) \leq \varepsilon. \qquad (3.19)$$

By Lemma 3.1, we conclude that

$$s - \liminf_{m,n\to\infty} s_{m,n} \leq \varepsilon. \qquad (3.20)$$

In a similar way, by (2.14b) and Lemma 3.1, we conclude

$$\limsup_{m,n\to\infty} s_{m,n} - s \leq \varepsilon. \qquad (3.21)$$

Since $\varepsilon > 0$ was arbitrary, (3.20) together with (3.21) yield $s_{m,n} \to s$ as $m, n \to \infty$. $\qquad \square$

PROOF OF THEOREM 2.7. It is omitted since it follows similar arguments as the proof of Theorem 2.5. $\qquad \square$

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Ferenc Móricz: Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

E-mail address: moricz@math.u-szeged.hu

U. Stadtmüller: Abteilung Zahlentheorie und Wahrscheinlichkeitstheorie, Fakultät für Mathematik und Wirtschaftswissenschaften, Universität Ulm, 89069 Ulm, Germany

E-mail address: stamue@mathematik.uni-ulm.de