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ON SIMULTANEOUS APPROXIMATION FOR SOME MODIFIED BERNSTEIN-TYPE OPERATORS

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We study the simultaneous approximation for a certain variant of Bernstein-type operators.

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1. Introduction. To approximate Lebesgue integrable functions on the interval $I \equiv [0,1]$, the modified Bernstein operators are defined by

$$\begin{aligned} M_{n,\alpha,\beta}(f,x) &= (n-\alpha+1)\sum_{k=\beta}^{n-\alpha+\beta} p_{n,k}(x)\int_0^1 p_{n-\alpha,k-\beta}(t)f(t)dt \\ &\quad + \sum_{k\in I_n} p_{n,k}(x) f\left(\frac{k}{n}\right), \end{aligned} \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k} \tag{1.2}$$

for $n \ge \alpha$, where α, β are integers satisfying $\alpha \ge \beta \ge 0$ and $I_n \subseteq \{0,1,2,\dots,n\}$ is a certain index set. For $\alpha = \beta = 0$, $I_n = \{0\}$, this definition reduces to the Bernstein-Durrmeyer operators, which were first studied by Derriennic [3]. Also if $\alpha = \beta = 1$, $I_n = \{0\}$, we obtain the recently introduced sequence of Gupta and Maheshwari [4], that is, $M_{n,1,1}(f,x) \equiv P_n(f,x)$ which is defined as

$$\begin{aligned} P_n(f,x) &= \int_0^1 W_n(x,t)f(t)dt \\ &= n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)f(t)dt + (1-x)^n f(0), \quad x \in I \equiv [0,1], \end{aligned} \tag{1.3}$$

where $W_n(x,t) = n\sum_{k=1}^{n} p_{n,k}(x)p_{n-1,k-1}(t) + (1-x)^n\delta(t)$, $\delta(t)$ being a Dirac Delta function.

In [4] Gupta and Maheshwari have estimated the rate of convergence for functions of bounded variation for the operators P_n, $n \in \mathbb{N}$. The approximation properties for different values of α, β were studied by several researchers. Recently Abel [1] obtained the complete asymptotic expansion for the Bernstein-Durrmeyer operators ($\alpha = \beta = 0$, $I_n = \{0\}$) in a concise form in simultaneous approximation. The operators $M_{n,\alpha,\beta}(f,x)$ are linear positive operators but their approximation properties are different with different values of α and β. In the present paper, we study the pointwise convergence and asymptotic formula in simultaneous approximation for the operators $M_{n,1,1}(f,x) \equiv P_n(f,x)$. In the end we give a remark that similar results can be obtained for different values of α and β, for example, we mention the asymptotic formula for another particular case, namely, $M_{n,0,1}(f,x) \equiv B_n(f,x)$. Our main theorems can be stated as follows.

Theorem 1.1. *Let $f \in C[0,1]$ and let $f^{(r)}$ exist at a point $x \in (0,1)$, then*

$$P_n^{(r)}(f,x) = f^{(r)}(t) + o(1) \quad \text{as } n \longrightarrow \infty. \tag{1.4}$$

Theorem 1.2. *Let $f \in C[0,1]$. If $f^{(r+2)}$ exists at a point $x \in (0,1)$, then*

$$\begin{aligned} \lim_{n\to\infty} n\left[P_n^{(r)}(f,x) - f^{(r)}(x)\right] &= x(1-x)f^{(r+2)}(x) \\ &\quad + [r - x(1+2r)]f^{(r+1)}(x) - r^2 f^{(r)}(x). \end{aligned} \tag{1.5}$$

2. Auxiliary results. In this section, we mention some results which are necessary to prove the main theorem.

Lemma 2.1. *For $m \in \mathbb{N}^0$ (the set of nonnegative integers), if the following definition holds:*

$$\begin{aligned} P_n((t-x)^m,x) \equiv \mu_{n,m}(x) &= n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^m dt \\ &\quad + (-x)^m(1-x)^n, \end{aligned} \tag{2.1}$$

then

$$\mu_{n,0}(x) = 1, \qquad \mu_{n,1}(x) = \frac{-x}{(n+1)}, \qquad \mu_{n,2}(x) = \frac{x(1-x)(2n+1) - (1-3x)x}{(n+1)(n+2)} \tag{2.2}$$

and for $m \geq 1$ there holds the recurrence relation

$$\begin{aligned} [n+m+1]\mu_{n,m+1}(x) &= x(1-x)\left[\mu_{n,m}^{(1)}(x) + 2m\mu_{n,m-1}(x)\right] \\ &\quad + [m(1-2x) - x]\mu_{n,m}(x). \end{aligned} \tag{2.3}$$

Proof. The values of $\mu_{n,0}(x)$ and $\mu_{n,1}(x)$ can easily follow from the definition. We prove the recurrence relation as follows:

$$\begin{aligned} x(1-x)\mu_{n,m}^{(1)}(x) &= n\sum_{k=1}^{n} x(1-x)p_{n,k}^{(1)}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^m dt \\ &\quad - mn\sum_{k=1}^{n} x(1-x)p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^{m-1} dt \\ &\quad - \left\{n(-x)^m(1-x)^{n-1} + m(-x)^{m-1}(1-x)^n\right\}x(1-x). \end{aligned} \tag{2.4}$$

Now using the identity $x(1-x)p_{n,k}^{(1)}(x) = (k-nx)p_{n,k}(x)$, we obtain

$$\begin{aligned}
x(1-x)&\left[\mu_{n,m}^{(1)}(x)+m\mu_{n,m-1}(x)\right]\\
&= n\sum_{k=1}^{n}(k-nx)p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^m\,dt+n(-x)^{m+1}(1-x)^n\\
&= n\sum_{k=1}^{n}p_{n,k}(x)\int_0^1\left[k-1-(n-1)t+(n-1)(t-x)\right.\\
&\qquad\qquad\left.+(1-x)\right]p_{n-1,k-1}(t)(t-x)^m\,dt\\
&\quad+n(-x)^{m+1}(1-x)^n\\
&= n\sum_{k=1}^{n}p_{n,k}(x)\int_0^1 t(1-t)p_{n-1,k-1}^{(1)}(t)(t-x)^m\,dt\\
&\quad+(n-1)\mu_{n,m+1}(x)+(1-x)\mu_{n,m}(x)-(-x)^m(1-x)^n\\
&= n\sum_{k=1}^{n}p_{n,k}(x)\int_0^1\left[(1-2x)(t-x)+(t-x)^2+x(1-x)\right]\\
&\qquad\qquad\times p_{n-1,k-1}^{(1)}(t)(t-x)^m\,dt\\
&\quad+(n-1)\mu_{n,m+1}(x)+(1-x)\mu_{n,m}(x)-(-x)^m(1-x)^n\\
&= -(m+1)(1-2x)\left[\mu_{n,m}(x)-(-x)^m(1-x)^n\right]\\
&\quad+(m+2)\left[\mu_{n,m+1}(x)-(-x)^{m+1}(1-x)^n\right]\\
&\quad-x(1-x)m\left[\mu_{n,m-1}(x)-(-x)^{m-1}(1-x)^n\right]+(n-1)\mu_{n,m+1}(x)\\
&\quad+(1-x)\mu_{n,m}(x)-(-x)^m(1-x)^n\\
&= \left[(1-x)-(m+1)(1-2x)\right]\mu_{n,m}(x)+(n+m+1)\mu_{n,m+1}(x)\\
&\quad-mx(1-x)\mu_{n,m-1}(x).
\end{aligned}\tag{2.5}$$

This completes the proof of the recurrence relation. □

The value of $\mu_{n,2}(x)$ can be easily obtained from the above recurrence relation.

Remark 1. For each fixed $x\in[0,1]$, it follows from the above lemma that

$$P_n(\psi_x^s,x)=O\left(n^{-[(s+1)/2]}\right),\quad n\longrightarrow\infty,\tag{2.6}$$

where $\psi_x = t-x$.

Lemma 2.2. *For $m \in \mathbb{N} \cup \{0\}$, if the mth-order moment is defined as*

$$U_{n,m}(x) = \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{v}{n} - x\right)^m, \tag{2.7}$$

then $U_{n,0}(x) = 1$, $U_{n,1}(x) = 0$, and

$$nU_{n,m+1}(x) = x(1-x)\left[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)\right]. \tag{2.8}$$

Consequently,

$$U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.9}$$

Lemma 2.3 [5]. *There exist the polynomials $Q_{i,j,r}(x)$ independent of n and v such that*

$$\{x(1-x)\}^r D^r\left[p_{n,k}(x)\right] = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i (k-nx)^j Q_{i,j,r}(x) p_{n,k}(x), \quad D \equiv \frac{d}{dx}. \tag{2.10}$$

3. Proofs of theorems

Proof of Theorem 1.1. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r, \tag{3.1}$$

where $\varepsilon(t,x) \to 0$ as $t \to \infty$.

Hence

$$\begin{aligned} P_n^{(r)}(f,x) &= \int_0^1 W_n^{(r)}(t,x) f(t)\,dt \\ &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^1 W_n^{(r)}(t,x)(t-x)^i dt + \int_0^1 W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt \\ &= R_1 + R_2. \end{aligned} \tag{3.2}$$

First to estimate R_1, using binomial expansion of $(t-x)^m$ and Lemma 2.1, we have

$$\begin{aligned} R_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \binom{i}{v} (-x)^{i-v} \frac{\partial^r}{\partial x^r} \int_0^1 W_n(t,x) t^v dt \\ &= \frac{f^{(r)}(x)}{r!} \frac{\partial^r}{\partial x^r} \int_0^1 W_n(t,x) t^r dt = f^{(r)}(x) + o(1), \quad n \longrightarrow \infty. \end{aligned} \tag{3.3}$$

Next using Lemma 2.3 we obtain

$$\begin{aligned}|R_2| &\le n \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{\{x(1-x)\}^r} \sum_{k=1}^{n} |k-nx|^j p_{n,k}(x)\\ &\quad\times \int_0^1 p_{n-1,k-1}(t)\,|\varepsilon(t,x)|\,(t-x)^r dt + \frac{n!}{(n-r)!}(1-x)^{n-r}|\varepsilon(0,x)|x^r\\ &= R_3 + R_4.\end{aligned} \tag{3.4}$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$ for a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. Thus for some $M_1>0$, we can write

$$\begin{aligned}R_3 &\le nM_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{n} p_{n,k}(x)|k-nx|^j\\ &\quad\times\Big\{\varepsilon\int_{|t-x|<\delta} p_{n-1,k-1}(t)|t-x|^r\\ &\qquad+\int_{|t-x|\ge\delta} p_{n-1,k-1}(t)M_2|t-x|^r dt\Big\} = R_5+R_6,\end{aligned} \tag{3.5}$$

where

$$M_1 = \sup_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{\{x(1-x)\}^r}. \tag{3.6}$$

and M_2 is independent of t. Applying the Schwarz inequality for integration and summation respectively, we obtain

$$\begin{aligned}R_5 &\le \varepsilon\cdot M_1 n \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{n} p_{n,k}(x)|k-nx|^j\Big(\int_0^1 p_{n-1,k-1}(t)dt\Big)^{1/2}\\ &\qquad\times\Big(\int_0^1 p_{n-1,k-1}(t)(t-x)^{2r}dt\Big)^{1/2}\\ &\le \varepsilon\cdot M_1 n \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{n} p_{n,k}(x)\Big(\sum_{k=1}^{n} p_{n,k}(x)(k-nx)^{2j}\Big)^{1/2}\\ &\qquad\times\Big(\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^{2r}dt\Big)^{1/2}.\end{aligned} \tag{3.7}$$

Using Lemmas 2.2 and 2.1, we get

$$R_5 \le \varepsilon\cdot M_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2})O(n^{-r/2}) = O(1). \tag{3.8}$$

Again using the Schwarz inequality and Lemmas 2.2 and 2.1, we get

$$
\begin{aligned}
R_6 &\le nM_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{n} p_{n,k}(x)|k-nx|^j \int_{|t-x|\ge\delta} p_{n-1,k-1}(t)|t-x|^r dt \\
&\le nM_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \sum_{k=1}^{k} p_{n,k}(x)|k-nx|^j \left(\int_{|t-x|\ge\delta} p_{n-1,k-1}(t)dt\right)^{1/2} \\
&\qquad\qquad \times \left(\int_{|t-x|\ge\delta} p_{n-1,k-1}(t)(t-x)^{2r}dt\right)^{1/2} \\
&\le nM_2 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \left(\sum_{k=1}^{n} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2} \\
&\qquad\qquad \times \left(\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)(t-x)^{2r}dt\right)^{1/2} \\
&= \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2})O(n^{-r/2}) = O(n^{(j-r)/2}) = O(1).
\end{aligned}
\tag{3.9}
$$

Thus, due to arbitrariness of $\varepsilon > 0$, it follows that $R_3 = o(1)$. Also $R_4 \to 0$ as $n \to \infty$ and hence $R_2 = o(1)$. Collecting the estimates of R_1 and R_2, we get the required result. □

Proof of Theorem 1.2. Using Taylor's expansion of f, we have

$$
f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2}, \tag{3.10}
$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Applying Lemma 2.1, we have

$$
\begin{aligned}
n\left[P_n^{(r)}(f(t),x) - f^{(r)}(x)\right] &= n\left[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}\int_0^1 W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)\right] \\
&\quad + \left[n\int_0^1 W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}dt\right] \\
&= E_1 + E_2,
\end{aligned}
$$

$$
\begin{aligned}
E_1 &= n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^1 W_n^{(r)}(t,x)t^j dt - nf^{(r)}(x) \\
&= \frac{f^{(r)}(x)}{r!} n\left[P_n^{(r)}(t^r,x) - (r!)\right] + \frac{f^{(r+1)}(x)}{(r+1)!} \\
&\quad \times n\left[(r+1)(-x)P_n^{(r)}(t^r,x) + P_n^{(r)}(t^{r+1},x)\right] \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!} n\left[\frac{(r+2)(r+1)}{2}x^2 P_n^{(r)}(t^r,x)\right. \\
&\qquad\qquad \left. + (r+2)(-x)P_n^{(r)}(t^{r+1},x) + P_n^{(r)}(t^{r+2},x)\right].
\end{aligned}
\tag{3.11}
$$

It is easily verified from Lemma 2.1 that for each $x \in (0,1)$,

$$P_n(t^v,x) = \frac{(n!)^2}{(n-v)!(n+v)!}x^v + v(v-1)\frac{(n!)^2}{(n-v+1)!(n+v)!}x^{v-1} + O(n^{-2}). \quad (3.12)$$

Therefore

$$\begin{aligned}
E_1 = {} & nf^{(r)}(x)\left[\frac{(n!)^2}{(n-r)!(n+r)!} - 1\right] \\
& + n\frac{f^{(r+1)}(x)}{(r+1)!}\Bigg[(r+1)(-x)(r!)\left\{\frac{(n!)^2}{(n-r)!(n+r)!}\right\} \\
& \qquad + \left\{\frac{(n!)^2}{(n-r-1)!(n+r+1)!}(r+1)!x \right.\\
& \qquad\quad \left. + r(r+1)\frac{(n!)^2}{(n-r)!(n+r+1)!}(r!)\right\}\Bigg] \\
& + n\frac{f^{(r+2)}(x)}{(r+2)!}\Bigg[\frac{(r+2)(r+1)x^2}{2}(r!)\frac{(n!)^2}{(n-r)!(n+r)!} \\
& \qquad + (r+2)(-x)\left\{\frac{(n!)^2}{(n-r-1)!(n+r+1)!}(r+1)!x \right.\\
& \qquad\quad \left. + r(r+1)\frac{(n!)^2}{(n-r)!(n+r+1)!}(r!)\right\} \\
& \qquad + \left\{\frac{(n!)^2}{(n-r-2)!(n+r+2)!}\right\}\frac{(r+2)!}{2}x^2 \\
& \qquad + (r+1)(r+2)\frac{(n!)^2}{(n-r-1)!(n+r+2)!} \\
& \qquad \times (r+1)!x + O(n^{-2})\Bigg].
\end{aligned} \quad (3.13)$$

In order to complete the proof of the theorem, it is sufficient to show that $\{x(1+x)\}^r E_2 \to 0$ as $n \to \infty$, which can easily be proved along the lines of the proof of Theorem 1.1 and by using Lemmas 2.1, 2.2, and 2.3. □

Remark 2. Just like the operators in (1.3), very recently Abel and Gupta [2] considered the following operators:

$$B_n(f,x) = (n+1)\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n,k-1}(t)f(t)\,dt + (1-x)^n f(0), \quad x \in I \equiv [0,1], \quad (3.14)$$

where $p_{n,k}(x)$ is as defined by (1.3). These operators are $M_{n,0,1}(f,x) \equiv B_n(f,x)$.

For these operators, we can easily verify the following: for $m \in \mathbb{N}^0$ (the set of nonnegative integers), if we define

$$B_n((t-x)^m,x) \equiv \phi_{n,m}(x) = (n+1)\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n,k-1}(t)(t-x)^m dt \\ +(-x)^m(1-x)^n, \tag{3.15}$$

then for $m \geq 1$ there holds the recurrence relation

$$[n+m+2]\phi_{n,m+1}(x) = x(1-x)[\phi^{(1)}_{n,m}(x)+2m\phi_{n,m-1}(x)] \\ +[m(1-2x)-2x]\phi_{n,m}(x). \tag{3.16}$$

Also, it is easily verified that

$$B_n(t^v,x) = \frac{n!(n+1)!}{(n-v)!(n+v+1)!}x^v + v(v-1)\frac{n!(n+1)!}{(n-v+1)!(n+v+1)!}x^{v-1} \\ +O(n^{-2}). \tag{3.17}$$

Thus we have the following asymptotic formula for the operators B_n.

Theorem 3.1. *Let $f \in C[0,1]$. If $f^{(r+2)}$ exists at a point $x \in (0,1)$, then*

$$\lim_{n\to\infty} n[B_n^{(r)}(f,x)-f^{(r)}(x)] \\ = x(1-x)f^{(r+2)}(x)+[r-2x(1+r)]f^{(r+1)}(x)-r(r+1)f^{(r)}(x). \tag{3.18}$$

The proof of Theorem 3.1 is parallel to that of Theorem 1.2; we just have to use the above estimates for the operators.

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