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ON CHUNG-TEICHER TYPE STRONG LAW FOR ARRAYS OF VECTOR-VALUED RANDOM VARIABLES

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We study the equivalence between the weak and strong laws of large numbers for arrays of row-wise independent random elements with values in a Banach space $\mathcal{B}$. The conditions under which this equivalence holds are of the Chung or Chung-Teicher types. These conditions are expressed in terms of convergence of specific series and $o(1)$ requirements on specific weighted row-wise sums. Moreover, there are not any conditions assumed on the geometry of the underlying Banach space.

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Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{B}$ be a real separable Banach space with norm $\|\cdot\|$. A strongly measurable transformation from Ω to $\mathcal{B}$ is said to be a $\mathcal{B}$-valued random variable or a random element. If $E\|X\| < \infty$, then the expected value is defined by the Bochner integral.

Let $\{X_n,\ n \geq 1\}$ be a sequence of $\mathcal{B}$-valued random variables. Then $\{X_n,\ n \geq 1\}$ is said to obey the strong law of large numbers (SLLN) if there exist sequences of real numbers $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ such that

$$\sum_{j=1}^{n} a_j (X_j - b_j) \longrightarrow 0 \quad \text{a.s.},\ n \longrightarrow \infty. \tag{1}$$

Sufficient conditions for SLLN use very often the geometry of a Banach space, that is, they assume that $\mathcal{B}$ is a special-type space, for instance $\mathcal{B}$ is of Rademacher type p, $1 < p \leq 2$.

The space $\mathcal{B}$ is of Rademacher type p if there exists a positive constant C such that

$$E\left\|\sum_{n=1}^{\infty} \varepsilon_n x_n\right\|^p \leq C \sum_{n=1}^{\infty} ||x_n||^p \tag{2}$$

for each $(x_1, x_2, \ldots) \in C(B)$, where $\{\varepsilon_n,\ n \geq 1\}$ is a Bernoulli sequence, that is, ε_n, $n \geq 1$, are i.i.d. random variables and $P[\varepsilon_n = 1] = P[\varepsilon_n = -1] = 1/2$, $C(B) = \{(x_1, x_2, \ldots) \in B^\infty : \sum_{n=1}^{\infty} \varepsilon_n x_n$ converges in probability$\}$, $B^\infty = B \times B \times B \times \cdots$.

The sufficient conditions for SLLN for random elements taking value in a space of Rademacher type p were presented by Woyczyński [15], Hoffmann-Jørgensen and Pisier [6], Kuczmaszewska and Szynal [8], and Adler et al. [1].

The type of Marcinkiewicz-Zygmunt SLLN provides that for $1 \le \alpha < 2$ and a sequence $\{X_n,\ n \ge 1\}$ of i.i.d. $\mathcal{B}$-valued random variables,

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} (X_i - EX_i) \longrightarrow 0 \quad \text{a.s.}, \ n \longrightarrow \infty, \tag{3}$$

if and only if $E\|X_1\| < \infty$ and the Banach space $\mathcal{B}$ is of a Rademacher type p for $\alpha < p \le 2$ (cf. [15]).

The classical result of Hoffmann-Jørgensen and Pisier [6] proved that the assumption that a Banach space $\mathcal{B}$ is the space of Rademacher type p, $1 \le p \le 2$, is equivalent to the fact that the condition

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|^p}{n^p} < \infty \tag{4}$$

implies SLLN for a sequence of $\mathcal{B}$-valued independent random variables $\{X_n,\ n \ge 1\}$ with $EX_n = 0$, $n \ge 1$.

In view of many statistical applications, it is important to consider the array-type SLLN.

Let $\{k_n,\ n \ge 1\}$ be a strictly increasing sequence of positive integers. An array of $\mathcal{B}$-valued random variables $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ obeys the general array type of SLLN if

$$\sum_{i=1}^{k_n} a_{ni}(X_{ni} - c_{ni}) \longrightarrow 0 \quad \text{a.s.}, \ n \longrightarrow \infty, \tag{5}$$

where $\{a_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ and $\{c_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ are suitable arrays of constants (weights) and $\mathcal{B}$-valued elements, respectively, and 0 denotes the zero-element in $\mathcal{B}$.

Hu and Taylor [7] considered SLLN for arrays of row-wise independent random variables $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$.

Row-wise independence means that the random elements within each row are independent but no independence is assumed between rows.

In [3] Bozorgnia et al. obtained the Chung-type SLLN for arrays of row-wise independent random elements in a separable Banach space of Rademacher type p, $1 < p \le 2$. They proved the following result.

Theorem 1. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of row-wise independent random elements in a separable Banach space of Rademacher type p, $1 < p \le 2$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a positive, even, and continuous function such that*

$$\frac{\varphi(|x|)}{|x|^r} \nearrow, \quad \frac{\varphi(|x|)}{|x|^{r+p-1}} \searrow \quad \text{as } |x| \nearrow, \tag{6}$$

for some integer $r \ge 2$.

Then the conditions

$$EX_{ni} = 0, \quad 1 \le i \le n,\ n \ge 1,$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\varphi(\|X_{ni}\|)}{\varphi(a_n)} < \infty, \qquad \sum_{n=1}^{\infty}\left[\sum_{i=1}^{n}E\left(\left\|\frac{X_{ni}}{a_n}\right\|^p\right)\right]^{pk} < \infty, \tag{7}$$

for some positive integer k, imply

$$\frac{1}{a_n}\sum_{i=1}^{n}X_{ni} \longrightarrow 0 \quad a.s.,\ n \longrightarrow \infty, \tag{8}$$

where $\{a_n,\ n \ge 1\}$ is a sequence of positive increasing real numbers such that

$$\lim_{n\to\infty} a_n = \infty. \tag{9}$$

This theorem generalizes Hu and Taylor's result (cf. [7]) on the case of $\mathcal{B}$-valued random variables $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ taking value in a Banach space of Rademacher type p. Moreover, the assumptions of the function φ have some relationships with the geometric condition Rademacher type p of the Banach space.

Some results which consider the problem of equivalence between weak law of large numbers (WLLN) and SLLN for a sequence $\{X_n,\ n \ge 1\}$ of independent $\mathcal{B}$-valued random variables can be found in Kuelbs and Zinn [10], de Acosta [4], Etemadi [5], Mikosch and Norvaiša [11, 12], Wang et al. [14], and Kuczmaszewska and Szynal [9].

Now, we recall some definitions and a lemma which will be used in the paper.

Definition 2. A double array $\{a_{ni},\ i \ge 1,\ n \ge 1\}$ of real numbers is said to be a Toeplitz array if $\lim_{n\to\infty} a_{ni} = 0$ for each $i \ge 1$ and $\sum_{i=1}^{\infty}|a_{ni}| \le C$ for all $n \ge 1$, where $C > 0$.

In further consideration, we need an extension of the concept of stochastic domination by a random variable to an array of $\mathcal{B}$-valued random variables.

An array $\{X_{ni},\ i \ge 1,\ n \ge 1\}$ of $\mathcal{B}$-valued random variables is stochastically dominated by the random element X if there exists a constant $D > 0$ such that

$$P[\|X_{ni}\| > x] \le DP[D\|X\| > x] \tag{10}$$

for all $x \ge 0$, $i \ge 1$, and $n \ge 1$.

We also need some inequalities which will be very important in our consideration. The following lemma presents one of them.

Lemma 3 (cf. Yurinskii [16]). *Let $X_1, X_2, \dots, X_n$ be independent $\mathcal{B}$-valued random variables with $E\|X_i\| < \infty$, $i = 1,2,\dots,n$. Let $\mathcal{F}$ be a σ-field generated by $(X_1, X_2, \dots, X_k)$, $k = 1,2,\dots,n$, and let $\mathcal{F}_0 = \{\Omega, \varnothing\}$. Then for $1 \le k \le n$ and $S_n = \sum_{i=1}^{n} X_i$,*

$$\left| E(\|S_n\| \mid \mathcal{F}_k) - E(\|S_n\| \mid \mathcal{F}_{k-1}) \right| \le \|X_k\| + E\|X_k\|. \tag{11}$$

Theorem 4. *Let $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables with $EX_{ni} = 0$ for all $1 \le i \le k_n$, $n \ge 1$, and for some increasing* *sequence* $\{k_n,\ n \geq 1\}$ *of positive integers. Let* $\varphi_{ni} : \mathbb{R} \to \mathbb{R}_+$ *and* $\psi_{ni} : \mathbb{R} \to \mathbb{R}_+$ *be positive, even, and continuous functions, which for constants* $\alpha_{ni} \geq 1$, $0 < \beta_{ni} \leq 2$, $K_{ni} > 0$, *and* $M_{ni} > 0$, $1 \leq i \leq k_n$, $n \geq 1$, *satisfy the following conditions:*

$$|x_1| \leq |x_2| \Longrightarrow \frac{\varphi_{ni}(|x_1|)}{|x_1|^{\alpha_{ni}}} \leq K_{ni}\frac{\varphi_{ni}(|x_2|)}{|x_2|^{\alpha_{ni}}}, \tag{12}$$

$$|x_1| \leq |x_2| \Longrightarrow \frac{|x_1|^{\beta_{ni}}}{\psi_{ni}(|x_1|)} \leq M_{ni}\frac{|x_2|^{\beta_{ni}}}{\psi_{ni}(|x_2|)}. \tag{13}$$

Suppose that for some array $\{a_{ni},\ (1 \leq i \leq k_n,\ n \geq 1)\}$ *of nonzero reals and* $k \geq 1/2$,

$$\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} M_{ni}\frac{\psi_{ni}(\|X_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})}\right)^k < \infty, \tag{14}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[\|X_{ni}\| \geq c_{ni}] < \infty, \tag{15}$$

for some array $\{c_{ni},\ 1 \leq i \leq k_n,\ n \geq 1\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} K_{ni}^2 \cdot \varphi_{ni}(c_{ni})\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty. \tag{16}$$

Then

$$\sum_{i=1}^{k_n} a_{ni}X_{ni} \xrightarrow{P} 0, \quad n \longrightarrow \infty, \tag{17}$$

if and only if

$$\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0 \quad a.s.,\ n \longrightarrow \infty. \tag{18}$$

Proof. Let $X'_{ni} = X_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]$ and $X^*_{ni} = X'_{ni} - EX'_{ni}$.
Now we introduce the following notation:

$$S_n = \sum_{i=1}^{k_n} a_{ni}X_{ni}, \qquad S'_n = \sum_{i=1}^{k_n} a_{ni}X'_{ni}. \tag{19}$$

Note that using this notation, condition (12) on the Borel functions φ_{ni}, and assumptions (15) and (16), we have

$$\begin{aligned}\sum_{n=1}^{\infty} P[S_n \neq S'_n] &= \sum_{n=1}^{\infty} P\left[\left\|\sum_{i=1}^{k_n} a_{ni}X_{ni}I[\|X_{ni}\| > |a_{ni}^{-1}|]\right\| > \varepsilon\right] \\ &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[\|X_{ni}\|I[\|X_{ni}\| > |a_{ni}^{-1}|] \neq 0] \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[\|X_{ni}\| > |a_{ni}^{-1}|]\end{aligned}$$

$$\begin{aligned}
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} EI\big[\|X_{ni}\| > |a_{ni}^{-1}|\big]\cdot I\big[\|X_{ni}\| \ge c_{ni}\big] \\
&\quad + \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} EI\big[\|X_{ni}\| > |a_{ni}^{-1}|\big]\cdot I\big[\|X_{ni}\| < c_{ni}\big] \\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P\big[\|X_{ni}\| \ge c_{ni}\big] \\
&\quad + \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E\left\{\left(\frac{\|X_{ni}\|^{\alpha_{ni}}}{(|a_{ni}^{-1}|)^{\alpha_{ni}}}\right)^2 I\big[\|X_{ni}\| > |a_{ni}^{-1}|\big]\cdot I\big[\|X_{ni}\| < c_{ni}\big]\right\} \\
&\le \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P\big[\|X_{ni}\| \ge c_{ni}\big] + \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} K_{ni}^2\cdot\varphi_{ni}(c_{ni})\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty.
\end{aligned} \tag{20}$$

Thus the two sequences $\{S_n,\ n \ge 1\}$ and $\{S'_n,\ n \ge 1\}$ are equivalent.

Now we must prove that

$$E\|S'_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{21}$$

First we will show that

$$\|S'_n\| - E\|S'_n\| \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{22}$$

Using the Markov inequality, the Marcinkiewicz-Zygmunt inequality in its Banach space version (cf. de Acosta [4] or Berger [2]), and assumptions (12) and (14), for any $\varepsilon > 0$, we get

$$\begin{aligned}
P\big[\big|\|S'_n\| - E\|S'_n\|\big| > \varepsilon\big] &\le \varepsilon^{-2k} E\big|\|S'_n\| - E\|S'_n\|\big|^{2k} \\
&\le \varepsilon^{-2k} A_k E\left(\sum_{i=1}^{k_n}\|a_{ni}X'_{ni}\|^2\right)^k = \varepsilon^{-2k} A_k E\left(\sum_{i=1}^{k_n}\frac{\|X'_{ni}\|^2}{(a_{ni}^{-1})^2}\right)^k \\
&= \varepsilon^{-2k} A_k E\left(\sum_{i=1}^{k_n}\frac{\|X'_{ni}\|^{\beta_{ni}}}{(|a_{ni}^{-1}|)^{\beta_{ni}}}\cdot\frac{\|X'_{ni}\|^{2-\beta_{ni}}}{(|a_{ni}^{-1}|)^{2-\beta_{ni}}}\right)^k \\
&\le \varepsilon^{-2k} A_k E\left(\sum_{i=1}^{k_n} M_{ni}\frac{\psi_{ni}(\|X'_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})}\right)^k \\
&\quad \times \varepsilon^{-2k} A_k E\left(\sum_{i=1}^{k_n} M_{ni}\frac{\psi_{ni}(\|X_{ni}\|)}{\psi_{ni}(a_{ni}^{-1})}\right)^k = o(1).
\end{aligned} \tag{23}$$

Thus we conclude that (22) holds and, together with (17) and the equivalence between $\{S_n,\ n \ge 1\}$ and $\{S'_n,\ n \ge 1\}$, gives (21).

Now, we will show that $\|S'_n\| \to 0$ a.s., as $n \to \infty$. By (21) it is enough to prove that

$$||S'_n|| - E||S'_n|| \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{24}$$

As before, using the Markov inequality, the Marcinkiewicz-Zygmunt inequality, condition (13), and assumption (14), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left[\,\left|\,||S'_n|| - E||S'_n||\,\right| > \varepsilon\right] &\le \varepsilon^{-2k} \sum_{n=1}^{\infty} E\left|\,||S'_n|| - E||S'_n||\,\right|^{2k} \\
&\le \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} \frac{||X'_{ni}||^2}{(a_{ni}^{-1})^2}\right)^k \\
&= \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} \frac{||X'_{ni}||^{\beta_{ni}}}{(|a_{ni}^{-1}|)^{\beta_{ni}}} \cdot \frac{||X'_{ni}||^{2-\beta_{ni}}}{(|a_{ni}^{-1}|)^{2-\beta_{ni}}}\right)^k \\
&\le \varepsilon^{-2k} A_k \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} M_{ni} \frac{\psi_{ni}(||X_{ni}||)}{\psi_{ni}(a_{ni}^{-1})}\right)^k < \infty.
\end{aligned} \tag{25}$$

Hence, by the Borel-Cantelli lemma, we obtain (24), which, by the equivalence between $\{S_n,\ n \ge 1\}$ and $\{S'_n,\ n \ge 1\}$, completes the proof. □

Note that if we put in Theorem 4 $\varphi_{ni} = \varphi$ and $\psi_{ni} = \psi$, where $\varphi : \mathbb{R} \to \mathbb{R}_+$, $\psi : \mathbb{R} \to \mathbb{R}_+$ are positive, even, and continuous functions such that

$$\frac{\varphi(|x|)}{|x|^{\alpha}} \nearrow, \quad \frac{\psi(|x|)}{|x|^{\beta}} \searrow \quad \text{as } |x| \nearrow, \tag{26}$$

for some $\alpha \ge 1$ and $0 < \beta \le 2$, we get the following result.

Corollary 5. *Let $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables with $EX_{ni} = 0$ for all $1 \le i \le k_n$, $n \ge 1$, and for any increasing sequence $\{k_n,\ n \ge 1\}$ of positive integers. Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ and $\psi : \mathbb{R} \to \mathbb{R}_+$ be positive, even, and continuous functions satisfying (26) for some $\alpha \ge 1$ and $0 < \beta \le 2$.*

Suppose that for some array $\{a_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ of nonzero reals and $k \ge 1/2$,

$$\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} \frac{\psi(||X_{ni}||)}{\psi(a_{ni}^{-1})}\right)^k < \infty, \qquad \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P\left[||X_{ni}|| \ge c_{ni}\right] < \infty, \tag{27}$$

for some array $\{c_{ni},\ (1 \le i \le 1,\ n \ge 1)\}$ of positive numbers such that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \varphi(c_{ni}) \frac{E\varphi(||X_{ni}||)}{\varphi^2(a_{ni}^{-1})} < \infty. \tag{28}$$

Then (17) is equivalent to (18).

Putting $\psi(x) = |x|^p$, $1 < p \le \beta \le 2$, and $c_{ni} = |a_{ni}^{-1}|$, we obtain the following result for a separable Banach space of Rademacher type p.

Corollary 6. *Let $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables in a separable Banach space of Rademacher type p, $1 < p \le \beta \le 2$, with $EX_{ni} = 0$ for all $1 \le i \le k_n$, $n \ge 1$, and for some increasing sequence $\{k_n,\ n \ge 1\}$ of positive integers. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a positive, even, and continuous function such that*

$$\frac{\varphi(|x|)}{|x|^{\alpha}} \nearrow \quad \text{as } |x| \nearrow, \tag{29}$$

for some $\alpha \ge 1$.

Then, for some array $\{a_{ni},\ (1 \le i \le k_n,\ n \ge 1)\}$ of nonzero reals and some integer $k \ge 1$, the conditions

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{k_n}\frac{E(||X_{ni}||^p)}{|a_{ni}|^{-p}}\right]^k < \infty, \tag{30}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{E\varphi(||X_{ni}||)}{\varphi(a_{ni}^{-1})} < \infty \tag{31}$$

imply (18).

Proof. Putting $M_{ni} = K_{ni} = 1$ and using (20), we see that it is enough to show that Theorem 4 holds for $\{X_{ni},\ 1 \le i \le k_n,\ n \ge 1\}$ and $k \ge 2$.

Indeed, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n}\frac{||X'_{ni}||^p}{|a_{ni}^{-1}|^p}\right)^k \\
&\quad \le \sum_{n=1}^{\infty}\sum{}^{*}\binom{k}{s_1,\dots,s_{k_n}} E\left(\frac{||X'_{n1}||^p}{|a_{n1}^{-1}|^p}\right)^{s_1} E\left(\frac{||X'_{n2}||^p}{|a_{n2}^{-1}|^p}\right)^{s_2}\cdots E\left(\frac{||X'_{nk_n}||^p}{|a_{nk_n}^{-1}|^p}\right)^{s_{k_n}},
\end{aligned} \tag{32}$$

where the sum $\sum^{*}\binom{k}{s_1,\dots,s_{k_n}}$ is over all choices of $\{s_1, s_2, \dots, s_{k_n}\}$, $s_i \in \{0,1,2,\dots,k\}$, such that $\sum_{i=1}^{k_n} s_i = k$. Choose n sufficiently large so that $k_n > k$. Let $m = m(s_1, s_2, \dots, s_{k_n})$ be a number of $s_i \ne 0$. We see that m takes all the values from the set $\{1,2,\dots,k\}$. Changing the order in our sum, we can express the right-hand side of (32) in the following form:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{\substack{1\le i_j\le k_n,\\ j=1,2,\dots,m\\ i_j\ne i_k\ \forall k\ne j}}{}^{*}\binom{k}{s_{i_1},\dots,s_{i_m}} E\left(\frac{||X'_{ni_1}||^p}{|a_{ni_1}^{-1}|^p}\right)^{s_{i_1}} E\left(\frac{||X'_{ni_2}||^p}{|a_{ni_2}^{-1}|^p}\right)^{s_{i_2}}\cdots E\left(\frac{||X'_{ni_m}||^p}{|a_{ni_m}^{-1}|^p}\right)^{s_{i_m}} \\
&\le \sum_{n=1}^{\infty}\left\{\sum_{1\le i_1<i_2<\cdots<i_k\le k_n} E\left(\frac{||X'_{ni_1}||^p}{|a_{ni_1}^{-1}|^p}\right)E\left(\frac{||X'_{ni_2}||^p}{|a_{ni_2}^{-1}|^p}\right)\cdots E\left(\frac{||X'_{ni_k}||^p}{|a_{ni_k}^{-1}|^p}\right)\right.
\end{aligned}$$

$$+\left(\sum_{m=1}^{k-1}\sum_{\substack{1\le i_j\le k_n,\\ j=1,2,\dots,m\\ i_j\ne i_k\ \forall k\ne j}}{}^{*}\binom{k}{s_1,\dots,s_{i_m}}\left(\prod_{h=1\, s_{i_{j_h}}\ge 2}^{L} E\left(\frac{\left\|X'_{ni_{j_h}}\right\|^p}{\left|a_{ni_{j_h}}^{-1}\right|^p}\right)^{s_{i_{j_h}}}\right)\right.$$
$$\left.\cdot\left(\prod_{j=1}^{N} E\frac{\left\|X'_{ni_{h_j}}\right\|^p}{\left|a_{ni_{h_j}}^{-1}\right|^p}\right)\right\}, \tag{33}$$

where $L=$ number of $s_i\ge 2$, $N=$ number of $s_i=1$, and $\{s_{i_1},\dots,s_{i_m}\}=\{s_{i_{j_h}},h=1,\dots,L\}\cup\{s_{i_{h_j}},s_{i_{h_j}}=1,h=1,\dots,N\}$, $\{s_{i_{j_h}},h=1,\dots,L\}\cap\{s_{i_{h_j}},s_{i_{h_j}}=1,h=1,\dots,N\}=\varnothing$.

But

$$\left(\frac{\left\|X'_{ni_j}\right\|^p}{\left|a_{ni_j}^{-1}\right|^p}\right)^{s_{i_j}}\le\frac{\left\|X'_{ni_j}\right\|^p}{\left|a_{ni_j}^{-1}\right|^p},\qquad E\frac{\left\|X'_{ni_j}\right\|^p}{\left|a_{ni_j}^{-1}\right|^p}\le\sum_{i=1}^{k_n}\frac{E\left\|X'_{ni_j}\right\|^p}{\left|a_{ni_j}^{-1}\right|^p}\quad\text{for } 1\le i_j\le k_n, \tag{34}$$

so the right-hand side of (33) can be estimated as follows:

$$\begin{aligned}C\sum_{n=1}^{\infty}&\left\{\left(\sum_{i=1}^{k_n}E\frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p}\right)^k+\left(\sum_{i=1}^{k_n}E\frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p}\right)^L\cdot\left(\sum_{i=1}^{k_n}E\frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p}\right)^M\right\}\\ &\le C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k_n}E\frac{\|X'_{ni}\|^p}{|a_{ni}^{-1}|^p}\right)^k<\infty.\end{aligned} \tag{35}$$

Therefore, assumption (14) of Theorem 4 holds. Moreover, by (31), we get (15) and (16) of Theorem 4. We also note that if $\mathcal{B}$ is a Banach space of Rademacher type p, $1<p\le 2$, we have the following estimation:

$$P[\|S_n\|\ge\varepsilon]\le\varepsilon^{-p}E\|S_n\|^p\le\varepsilon\sum_{i=1}^{k_n}E\|a_{ni}X_{ni}\|^p\le\varepsilon^{-p}\sum_{i=1}^{k_n}\frac{E(\|X_{ni}\|^p)}{|a_{ni}|^{-p}}=o(1). \tag{36}$$

This fact, together with (20), completes the proof. □

Now we present the result which gives the sufficient conditions for the equivalence of (17) and (18) in the Chung-Teicher terms (cf. Teicher [13]).

Theorem 7. *Let $\{X_{ni},\ 1\le i\le k_n,\ n\ge 1\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables with $EX_{ni}=0$ for all $1\le i\le k_n$, $n\ge 1$, and for some increasing sequence $\{k_n,\ n\ge 1\}$ of positive integers. Let $\varphi_{ni}:\mathbb{R}\to\mathbb{R}_+$ be positive, even, and* *continuous functions which, for constants* $\alpha_{ni} \geq 1$, $0 < \beta_{ni} \leq 2$, $K_{ni} > 0$, *and* $M_{ni} > 0$, $n \geq 1$, $i \geq 1$, *satisfy (12) and*

$$|x_1| \leq |x_2| \Longrightarrow \frac{|x_1|^{\beta_{ni}}}{\varphi_{ni}(|x_1|)} \leq M_{ni}\frac{|x_2|^{\beta_{ni}}}{\varphi_{ni}(|x_2|)}. \tag{37}$$

Suppose that for some array $\{a_{ni},\ (1 \leq i \leq k_n,\ n \geq 1)\}$ *of nonzero reals,*

$$\sum_{n=1}^{\infty}\sum_{i=2}^{k_n} M_{ni}\frac{E\varphi_{ni}(\|X_{ni}\|)}{P\varphi_{ni}(a_{ni}^{-1})}\sum_{j=1}^{i-1} M_{nj}\frac{E\varphi_{nj}(\|X_{nj}\|)}{\varphi_{nj}(a_{nj}^{-1})} < \infty, \tag{38}$$

$$\sum_{i=1}^{k_n} M_{ni}\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1), \tag{39}$$

$$\sum_{i=1}^{k_n} K_{ni}\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1), \tag{40}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[\|X_{ni}\| \geq c_{ni}] < \infty, \tag{41}$$

for some array $\{c_{ni},\ i \geq 1,\ n \geq 1\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} M_{ni}^2 \cdot \varphi_{ni}(c_{ni})\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty, \tag{42}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} K_{ni}^2 \cdot \varphi_{ni}(c_{ni})\frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty. \tag{43}$$

Then (17) is equivalent to (18).

Proof. Let $X'_{ni} = X_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]$, $X^*_{ni} = X'_{ni} - EX'_{ni}$, $S'_n = \sum_{i=1}^{k_n} a_{ni}X'_{ni}$, and $S^*_n = \sum_{i=1}^{k_n} a_{ni}X^*_{ni}$. By (20), we state that $\{S_n,\ n \geq 1\}$ and $\{S'_n,\ n \geq 1\}$ are equivalent.

Moreover, we have by (40)

$$\begin{aligned}
\left\|\sum_{i=1}^{k_n} a_{ni}EX_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]\right\| &\leq \sum_{i=1}^{k_n} |a_{ni}| \cdot \|EX_{ni}I[\|X_{ni}\| \leq |a_{ni}^{-1}|]\| \\
&= \sum_{i=1}^{k_n} |a_{ni}| \cdot \|EX_{ni}I[\|X_{ni}\| > |a_{ni}^{-1}|]\| \\
&\leq \sum_{i=1}^{k_n} \frac{E\|X_{ni}\|I[\|X_{ni}\| > |a_{ni}^{-1}|]}{|a_{ni}^{-1}|} \\
&\leq \sum_{i=1}^{k_n} \frac{E\|X_{ni}\|^{\alpha_{ni}}I[\|X_{ni}\| > |a_{ni}^{-1}|]}{|a_{ni}^{-1}|^{\alpha_{ni}}} \\
&\leq \sum_{i=1}^{k_n} K_{ni} \cdot \frac{E\varphi_{ni}(\|X_{ni}\|)}{\varphi_{ni}(a_{ni}^{-1})} = o(1).
\end{aligned} \tag{44}$$

Now we define

$$Y_{ni} = E(\|S_n^*\| \mid \mathcal{F}_{ni}) - E(\|S_n^*\| \mid \mathcal{F}_{ni-1}), \tag{45}$$

where $\mathcal{F}_{ni} = \sigma(X_{n1}^*, X_{n2}^*, \dots, X_{ni}^*)$ and $\mathcal{F}_{n0} = \{\emptyset, \Omega\}$. Then we have

$$\|S_n^*\| - E\|S_n^*\| = \sum_{i=1}^{k_n} Y_{ni} \tag{46}$$

and we note that $\{Y_{ni},\ 1 \le i \le k_n\}$ is a sequence of martingale differences for a fixed n.

Now we are going to prove that

$$E\|S_n^*\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{47}$$

First we will show that

$$\|S_n^*\| - E\|S_n^*\| \xrightarrow{P} 0, \quad n \longrightarrow \infty. \tag{48}$$

Using Chebyshev's inequality, Lemma 3, and assumption (39), we get, for any $\varepsilon > 0$,

$$\begin{aligned} &P\big[\,\big|\|S_n^*\| - E\|S_n^*\|\big| > \varepsilon\big] \\ &\quad\le \varepsilon^{-2} E\big(\|S_n^*\| - E\|S_n^*\|\big)^2 = \varepsilon^{-2} E\left(\sum_{i=1}^{k_n} Y_{ni}\right)^2 = \varepsilon^{-2} \sum_{i=1}^{k_n} E\big(Y_{ni}^2\big) \\ &\quad\le \varepsilon^{-2} \sum_{i=1}^{k_n} E\big(\|a_{ni}X_{ni}^*\| + E\|a_{ni}X_{ni}^*\|\big)^2 \le 8\varepsilon^{-2} \sum_{i=1}^{k_n} a_{ni}^2 E\|X_{ni}'\|^2 \\ &\quad= 8\varepsilon^{-2} \sum_{i=1}^{k_n} E \frac{\|X_{ni}'\|^{\beta_{ni}}}{|a_{ni}^{-1}|^{\beta_{ni}}} \cdot \frac{\|X_{ni}'\|^{2-\beta_{ni}}}{|a_{ni}^{-1}|^{2-\beta_{ni}}} \le 8\varepsilon^{-2} \sum_{i=1}^{k_n} M_{ni} \frac{E\varphi_{ni}(\|X_{ni}'\|)}{\varphi_{ni}(a_{ni})} = o(1). \end{aligned} \tag{49}$$

Thus, we conclude that (48) holds and, together with (17), (20), and (44), gives (47).

Now we want to show that $\|S_n^*\| \to 0$ a.s., as $n \to \infty$.

By (47) it is enough to prove that

$$\|S_n^*\| - E\|S_n^*\| \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{50}$$

Taking into account the identity

$$\big(\|S_n^*\| - E\|S_n^*\|\big)^2 = \sum_{i=1}^{k_n} Y_{ni}^2 + 2\sum_{i=2}^{k_n} Y_{ni} \sum_{j=1}^{i-1} Y_{nj} \tag{51}$$

and using the notation

$$Z_{ni} = Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] - E\big(Y_{ni}^2 I[\|X_{ni}\| < c_{ni}] \mid \mathcal{F}_{ni-1}\big), \quad 1 \le i \le k_n, \tag{52}$$

we have, by Chebyshev's inequality, Lemma 3, and assumption (42),

$$\begin{aligned}\sum_{n=1}^{\infty} P\left[\left|\sum_{i=1}^{k_n} Z_{ni}\right| > \varepsilon\right] &\le \varepsilon^{-2}\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{k_n} Z_{ni}\right)^2 = \varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E(Z_{ni}^2)\\
&\le C\cdot\varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E(Y_{ni}^4 I[||X_{ni}|| < c_{ni}])\\
&\le C\cdot\varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E||a_{ni}X'_{ni}||^4 I[||X_{ni}|| < c_{ni}]\\
&\le C\cdot\varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E\left(\frac{||X'_{ni}||^{2\beta_{ni}}}{|a_{ni}^{-1}|^{2\beta_{ni}}}\cdot\frac{||X'_{ni}||^{4-2\beta_{ni}}}{|a_{ni}^{-1}|^{4-2\beta_{ni}}} I[||X_{ni}|| < c_{ni}]\right)\\
&\le C\cdot\varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} M_{ni}\cdot\varphi_{ni}(c_{ni})\cdot\frac{E\varphi_{ni}(||X_{ni}||)}{\varphi_{ni}^2(a_{ni}^{-1})} < \infty.\end{aligned} \tag{53}$$

Hence, by the Borel-Cantelli Lemma, we obtain

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[||X_{ni}|| < c_{ni}] - \sum_{i=1}^{k_n} E(Y_{ni}^2 I[||X_{ni}|| < c_{ni}] \mid \mathcal{F}_{ni-1}) \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{54}$$

Using Lemma 3, we can note by assumption (39) that

$$\begin{aligned}&\sum_{i=1}^{k_n} E(Y_{ni}^2 I[||X_{ni}|| < c_{ni}] \mid \mathcal{F}_{ni-1})\\
&\quad\le 8\sum_{i=1}^{k_n} E\left(||a_{ni}X'_{ni}||^2\right) \le \sum_{i=1}^{k_n} E\left(\frac{||X'_{ni}||^{\beta_{ni}}}{|a_{ni}^{-1}|^{\beta_{ni}}}\cdot\frac{||X'_{ni}||^{2-\beta_{ni}}}{|a_{ni}^{-1}|^{2-\beta_{ni}}}\right)\\
&\quad\le \sum_{i=1}^{k_n} M_{ni}\cdot\frac{E\varphi_{ni}(||X_{ni}||)}{\varphi_{ni}(a_{ni}^{-1})} = o(1),\end{aligned} \tag{55}$$

which, together with (54), allows us to state that

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[||X_{ni}|| < c_{ni}] \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{56}$$

To prove that

$$\sum_{i=1}^{k_n} Y_{ni}^2 \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty, \tag{57}$$

we only need to show that

$$\sum_{i=1}^{k_n} Y_{ni}^2 I[||X_{ni}|| \ge c_{ni}] \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{58}$$

Indeed, by (41) and Lemma 3, we have

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{i=1}^{k_n} Y_{ni}^2 I[||X_{ni}|| \geq c_{ni}]\right| \geq \varepsilon\right]$$
$$\leq \varepsilon^{-1}\sum_{n=1}^{\infty} E\left|\sum_{i=1}^{k_n} Y_{ni}^2 I[||X_{ni}|| \geq c_{ni}]\right| \leq C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} E||a_{ni}X'_{ni}||^2 I[||X_{ni}|| \geq c_{ni}] \tag{59}$$
$$\leq C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} EI[||X_{ni}|| \geq c_{ni}] \leq \varepsilon^{-1}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[||X_{ni}|| \geq c_{ni}] < \infty,$$

and, by the Borel-Cantelli Lemma, we get (57). To close this proof, we must show that

$$\sum_{i=2}^{k_n} Y_{ni}\sum_{j=1}^{i-1} Y_{nj} \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{60}$$

Using the fact that $\{Y_{ni}\sum_{j=1}^{i-1} Y_{nj},\ 2 \leq i \leq k_n\}$ is a sequence of martingale differences for each n, we have, by Chebyshev's inequality, Lemma 3, and assumption (38),

$$\sum_{n=1}^{\infty} P\left[\left|\sum_{i=2}^{k_n} Y_{ni}\sum_{j=1}^{i-1} Y_{nj}\right| > \varepsilon\right]$$
$$\leq \varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=2}^{k_n} E\left[\left(||a_{ni}X^*_{ni}|| + E||a_{ni}X^*_{ni}||\right)^2 \cdot \left(\sum_{j=1}^{i-1} Y_{nj}\right)^2\right]$$
$$\leq \varepsilon^{-2}\sum_{n=1}^{\infty}\sum_{i=2}^{k_n} E\left(||a_{ni}X^*_{ni}|| + E||a_{ni}X^*_{ni}||\right)^2 \cdot \sum_{j=1}^{i-1} E(Y_{nj}^2) \tag{61}$$
$$\leq C\sum_{n=1}^{\infty}\sum_{i=2}^{k_n} E||a_{ni}X'_{ni}||^2 \sum_{j=1}^{i-1} E||a_{nj}X'_{nj}||^2 C\sum_{n=1}^{\infty}\sum_{i=2}^{k_n} M_{ni}\frac{E\varphi_{ni}(||X_{ni}||)}{\varphi_{ni}(a_{ni}^{-1})}$$
$$\times\sum_{j=1}^{i-1} M_{nj}\frac{E\varphi_{nj}(||X_{nj}||)}{\varphi_{nj}(a_{nj}^{-1})} < \infty,$$

which, together with the Borel-Cantelli Lemma, implies (60).

Thus, we have proved that

$$\sum_{i=1}^{k_n} a_{ni}(X'_{ni} - EX'_{ni}) \longrightarrow 0 \quad \text{a.s., } n \longrightarrow \infty. \tag{62}$$

But $\{\sum_{i=1}^{k_n} a_{ni}X'_{ni},\ n \geq 1\}$ and $\{\sum_{i=1}^{k_n} a_{ni}X_{ni},\ n \geq 1\}$ are equivalent and (44) holds, so we get (18). □

Now we consider an array $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ of independent random elements which are stochastically dominated by a random element X in the sense of (10).

Corollary 8. *Let $\{X_{ni},\ 1 \leq i \leq k_n,\ n \geq 1\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables with $EX_{ni} = 0$ for all $1 \leq i \leq k_n$, $n \geq 1$, and some increasing* *sequence* $\{k_n,\ n \geq 1\}$ *of positive integers stochastically dominated by a random element* X *in the sense of (10). Let* $E\|X\|^p < \infty$, $1 < p \leq 2$. *Assume that* $\{a_{ni},\ 1 \leq i \leq k_n,\ n \geq 1\}$ *is an array of nonzero reals.*

Suppose that for some increasing sequence $\{k_n,\ n \geq 1\}$ *of positive integers,*

$$\sum_{n=1}^{\infty}\sum_{i=2}^{k_n}|a_{ni}|^p\sum_{j=1}^{i-1}|a_{nj}|^p < \infty, \tag{63}$$

$$\sum_{i=1}^{k_n}|a_{ni}|^p = o(1), \tag{64}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}P[\|X\| \geq c_{ni}] < \infty, \tag{65}$$

for some array $\{c_{ni},\ (1 \leq i \leq 1,\ n \geq 1)\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}c_{ni}^p \cdot |a_{ni}|^{2p} < \infty. \tag{66}$$

Then (17) is equivalent to (18).

Proof. Put $\varphi_{ni}(x) = |x|^p$, $1 < p \leq 2$, for all $1 \leq i \leq k_n$, $n \geq 1$. Then there exist α and β such that $1 \leq \alpha < p < \beta \leq 2$, for which (12) and (37) hold with $\alpha_{ni} = \alpha$, $\beta_{ni} = \beta$, and $M_{ni} = K_{ni} = 1$, $i \geq 1$, $n \geq 1$. Moreover, (10) and $E\|X\|^p < \infty$ imply $E\|X_{ni}\|^p < \infty$ for all $1 \leq i \leq k_n$ and $n \geq 1$. Therefore, we see that assumptions (38)–(43) of Theorem 7 are fulfilled ((38) is fulfilled by (63), (39) and (40) by (64), (41) by (65), and (42)-(43) by (66)). □

Corollary 9. *Let* $\{X_{ni},\ (1 \leq i \leq n,\ n \geq 1)\}$ *be an array of row-wise independent* $\mathcal{B}$*-valued random variables with* $EX_{ni} = 0$ *for* $1 \leq i \leq n$, $n \geq 1$, *stochastically dominated by a random element* X *in the sense of (10). Assume that* $\{a_{ni},\ i \geq 1,\ n \geq 1\}$ *is a Toeplitz array such that* $a_{ni} \neq 0$, $i \geq 1$, $n \geq 1$, *and for some* γ *such that* $\gamma p > 2$, $1 < p \leq 2$,

$$\sup_{i\geq 1}|a_{ni}| = O(n^{-\gamma}). \tag{67}$$

If $E\|X\|^p < \infty$, *then (17) is equivalent to (18).*

Proof. To prove this result, it is enough to show that under assumption (67), conditions (14), (15), and (16) of Theorem 4 are fulfilled.

Indeed, we see that for $\psi_{ni}(x) = \varphi_{ni}(x) = |x|^p$ and $M_{ni} = K_{ni} = 1$, $i \geq 1$, $n \geq 1$, $k = 1$, and $k_n = n$, $n \geq 1$, we have

$$\sum_{n=1}^{\infty}E\left(\sum_{i=1}^{n}|a_{ni}|^p\|X_{ni}\|^p\right) \leq C\sum_{n=1}^{\infty}\left(\sup_{i\geq 1}|a_{ni}|\right)^p\sum_{i=1}^{n}E\|X\|^p \leq C\sum_{n=1}^{\infty}\frac{1}{n^{\gamma p-1}} < \infty. \tag{68}$$

So, condition (14) is fulfilled.

For $c_{ni} = D/a_{ni}$, $i \ge 1$, $n \ge 1$, we get (15) and (16):

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left[\|X_{ni}\| \ge \frac{D}{a_{ni}}\right] \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left[D\|X\| \ge \frac{D}{a_{ni}}\right] \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} |a_{ni}|^p E\|X\|^p \le C\sum_{n=1}^{\infty}\frac{1}{n^{\gamma p-1}} < \infty, \tag{69}$$

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\left|\frac{D}{a_{ni}}\right|^p |a_{ni}|^{2p} E\|X_{ni}\|^p \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}|a_{ni}|^p E\|X\|^p \le C\sum_{n=1}^{\infty}\frac{1}{n^{\gamma p-1}} < \infty.$$

□

Similarly, we can prove the next corollary.

Corollary 10. *Let $\{X_{ni},\ (1 \le i \le n,\ n \ge 1)\}$ be an array of row-wise independent $\mathcal{B}$-valued random variables with $EX_{ni} = 0$ for $1 \le i \le k_n$, $n \ge 1$, and some increasing sequence $\{k_n,\ n \ge 1\}$ of positive integers stochastically dominated by a random element X in the sense of (10). Assume that $\{a_{ni},\ i \ge 1,\ n \ge 1\}$ is a Toeplitz array such that $a_{ni} \ne 0$, $i \ge 1$, $n \ge 1$, and for some γ such that $\gamma q > 2$, $0 < q \le 2$,*

$$\sum_{i=1}^{k_n} |a_{ni}|^q = O(n^{-\gamma}). \tag{70}$$

If $E\|X\|^q < \infty$, then (17) is equivalent to (18).

Using the fact that $\mathcal{B}$ is a Banach space of Rademacher type p, we see that (67) implies (17) under $E\|X\|^p < \infty$. Indeed, we see that

$$P\left[\left\|\sum_{i=1}^{n} a_{ni}X_{ni}\right\| \ge \varepsilon\right] \le \varepsilon^{-p}\sum_{i=1}^{n} E\|a_{ni}X_{ni}\|^p \le C\sum_{i=1}^{n}|a_{ni}|^p E\|X\|^p \le \frac{C}{n^{\gamma p-1}} = o(1). \tag{71}$$

So, we can formulate the next result.

Corollary 11. *Let $\{X_{ni},\ 1 \le i \le n,\ n \ge 1\}$ be an array of row-wise independent random elements taking values in a Banach space of Rademacher type p, $1 < p \le 2$, with $EX_{ni} = 0$ for $i \ge 1$, $n \ge 1$, stochastically dominated by a random element X in the sense of (10) and $E\|X\|^p < \infty$. Assume that $\{a_{ni},\ i \ge 1,\ n \ge 1\}$ is a Toeplitz array. If for some γ satisfying $\gamma p > 2$ and $a_{ni} \ne 0$, $i \ge 1$, $n \ge 1$, (67) holds, then (18) is fulfilled.*

Example 12. Let $\{X_n,\ n \ge 1\}$ be a sequence of independent random variables which are stochastically dominated by a random variable X such that $EX = 0$ and $E|X|^\gamma < \infty$ for some γ, $1 \le \alpha < \gamma < \beta \le 2$. Set $X_{ni} = X_i^i/n^{(1+\varepsilon)i/\gamma}$ and $a_{ni} = 1/n^{i/\gamma}$, for $1 \le i \le n$, $n \ge 1$.

We will verify that conditions (14), (15), and (16) of Theorem 4 hold with $k_n = n$, $n \ge 1$, $c_{ni} = n^{i/\gamma}$, and $\varphi_{ni}(x) = \psi_{ni}(x) = |x|^{\gamma/i}$, $1 \le i \le n$, $n \ge 1$, $k = 1$.

We note that the functions φ_{ni} and ψ_{ni}, $1 \le i \le n, n \ge 1$, satisfy conditions (12) and (13) with $\alpha_{ni} = \alpha/i$, $\beta_{ni} = \beta/i$, and $M_{ni} = K_{ni} = 1$.

To verify (14), note that

$$\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{n} M_{ni}\frac{\psi_{ni}(|X_{ni}|)}{\psi_{ni}(a_{ni}^{-1})}\right) \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_{ni}|^{\gamma}}{n} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_i|^{\gamma}}{n^{1+\varepsilon}\cdot n} \le C\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < \infty, \tag{72}$$

and to verify (15), note that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left[|X_{ni}| > n^{i/\gamma}\right]\\ &\quad\le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left[\frac{|X_i|^{i}}{n^{(1+\varepsilon)i/\gamma}} > n^{i/\gamma}\right] \le \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left[|X_i|^{i} > n^{(2+\varepsilon)i/\gamma}\right] \le C\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < \infty.\end{aligned} \tag{73}$$

Moreover, we see that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{i=1}^{n} K_{ni}^{2}\cdot\varphi_{ni}(c_{ni})\frac{E\varphi_{ni}(|X_{ni}|)}{\varphi_{ni}^{2}(a_{ni}^{-1})}\\ &\quad\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{nE|X_{ni}|^{\gamma/i}}{n^{2}} \le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E|X_i|^{\gamma}}{n^{2+\varepsilon}} \le \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < \infty.\end{aligned} \tag{74}$$

To complete the proof, we note that the real number space is of Rademacher type p with $p = 2$, so by the estimations (36), (72), and (73), we have (17). Thus, by Theorem 4, we have

$$\sum_{i=1}^{\infty} a_{ni}X_{ni} = \sum_{i=1}^{n} n^{-(2+\varepsilon)i/\gamma} X_i^{i} \longrightarrow 0, \quad \text{a.s., } n \longrightarrow \infty. \tag{75}$$

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