

# STATIONARY SOLUTIONS FOR INTEGER-VALUED AUTOREGRESSIVE PROCESSES

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The purpose of this paper is to introduce and develop a family of  $\mathbb{Z}_+$ -valued autoregressive processes of order  $p$  (INAR( $p$ )) by using the generalized multiplication  $\odot_F$  of van Harn and Steutel (1982). We obtain various distributional and regression properties for these models. A number of stationary INAR( $p$ ) processes with specific marginals are presented and are shown to generalize several existing models.

## 1. Introduction

Integer-valued time series have been the object of growing interest in recent years. Models of stationary time series for count data with a given marginal distribution have been developed by several authors. Applications in the areas of model-fitting for count data and simulation of sequences of dependent Poisson and negative binomial (among others) variables have been offered. These models are based on the binomial thinning operator  $\odot$  of Steutel and van Harn [19] which is defined as follows. If  $X$  is a  $\mathbb{Z}_+$ -valued random variable (rv) and  $\alpha \in (0, 1)$ , then

$$\alpha \odot X = \sum_{i=1}^X X_i, \quad (1.1)$$

where  $\{X_i\}$  is a sequence of i.i.d. Bernoulli( $\alpha$ ) rv's independent of  $X$ . The operation  $\odot$  incorporates the discrete nature of the variates and acts as the analogue of the standard multiplication used in the Box-Jenkins models. For example, an integer-valued first-order autoregressive (INAR(1)) process is described by the following equation:

$$X_{n+1} = \alpha \odot X_n + \epsilon_n. \quad (1.2)$$

McKenzie [14, 16] used the binomial thinning operator to construct stationary Poisson and negative binomial autoregressive moving average (namely, AR(1), MA( $q$ ), and ARMA(1,  $q$ )) processes. Al-Osh and Alzaid [1] offered a general theory for integer-valued moving average processes of order  $q$  (INMA( $q$ )). Du and Li [7] (see also Jayakumar [10] and Latour [11]) developed higher-order integer-valued autoregressive processes of order

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$p$  (INAR( $p$ )) processes. Related models that made use of a more general operator were introduced by Aly and Bouzar [2] and Zhu and Joe [22]. We refer to McKenzie [17] for an overview of the work in this area (and for further references).

van Harn et al. [21] introduced the generalized multiplication  $\odot_F$  (see definition below) as an extension of binomial thinning and used it to define concepts of discrete self-decomposability and stability. Subsequently, van Harn and Steutel [20] used  $\odot_F$  to define and solve stability equations involving continuous-time  $\mathbb{Z}_+$ -valued processes with stationary independent increments.

The purpose of this paper is to introduce and develop a family of INAR( $p$ ) processes by using the  $\odot_F$  multiplication in lieu of binomial thinning see Definitions 2.1 and 5.1. We obtain various distributional and regression properties for these models. A number of stationary INAR( $p$ ) processes with specific marginals are presented and are shown to generalize several existing models. The paper is organized as follows. In Section 2, we introduce the  $F$ -INAR(1) process and give several properties. Stationary solutions for  $F$ -INAR(1) processes and characterizations of their marginals are offered in Section 3. In Section 4, we study the question of time-reversibility for a stationary  $F$ -INAR(1) process and we obtain a form for the probability generating function (pgf) of its marginal. Section 5 is devoted to higher-order  $F$ -INAR( $p$ ) models and their properties.

In the rest of this section, we recall some definitions and results that are needed in the sequel. For proofs and further details, we refer to Athreya and Ney (see [5, Chapter 3]), van Harn et al. [21], and van Harn and Steutel [20]. The pgf of a distribution  $(p_n, n \geq 0)$  on  $\mathbb{Z}_+$  is defined by

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad (|z| \leq 1). \quad (1.3)$$

$F := (F_t, t \geq 0)$  will denote a continuous composition semigroup of pgf's such that  $F_t \neq 1$  and  $\delta_F = -\ln F'_1(1) > 0$ . For any  $|z| \leq 1$ ,

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (s, t \geq 0), \quad \lim_{t \downarrow 0} F_t(z) = z, \quad \lim_{t \rightarrow \infty} F_t(z) = 1. \quad (1.4)$$

The infinitesimal generator  $U$  of the semigroup  $F$  is defined by

$$U(z) = \lim_{t \downarrow 0} \frac{F_t(z) - z}{t} \quad (|z| \leq 1), \quad (1.5)$$

and satisfies  $U(z) > 0$  for  $0 \leq z < 1$ . There exist a constant  $a > 0$  and a distribution  $(h_n, n \geq 0)$  on  $\mathbb{Z}_+$  with pgf  $H(z)$  such that  $h_1 = 0$ ,

$$H'(1) = \sum_{n=1}^{\infty} n h_n \leq 1, \quad (1.6)$$

$$U(z) = a\{H(z) - z\}, \quad |z| \leq 1. \quad (1.7)$$

The related  $A$ -function is defined by

$$A(z) = \exp \left\{ - \int_0^z (U(x))^{-1} dx \right\}, \quad z \in [0, 1). \quad (1.8)$$

The functions  $U(z)$  and  $A(z)$  satisfy

$$U(F_t(z)) = U(z)F'_t(z), \quad A(F_t(z)) = e^{-t}A(z) \quad (t \geq 0, 0 \leq z \leq 1). \quad (1.9)$$

Moreover,

$$\delta_F = a(1 - H'(1)) = -U'(1), \quad F'_t(1) = e^{-\delta_F t} \quad (t \geq 0). \quad (1.10)$$

The function  $B(z)$  defined by

$$B(z) = \lim_{t \rightarrow \infty} \frac{F_t(z) - F_t(0)}{1 - F_t(0)} \quad (1.11)$$

is a pgf such that  $B(0) = 0$  and takes the form

$$B(z) = 1 - A(z)^{\delta_F}. \quad (1.12)$$

For a  $\mathbb{Z}_+$ -valued rv  $X$  and  $\eta \in (0, 1)$ , the generalized multiplication  $\eta \odot_F X$  is defined by

$$\eta \odot_F X = \sum_{i=1}^X Y_i, \quad (1.13)$$

where  $(Y_i, i \geq 1)$  is a sequence of i.i.d. rv's independent of  $X$  with common pgf  $F_t$ ,  $t = -\ln \eta$ .

Throughout the paper, stationarity of a stochastic process is considered to be in the strict sense. Finally,  $P_X$  will denote the pgf of the distribution of a  $\mathbb{Z}_+$ -valued rv  $X$ .

## 2. $F$ -INAR(1) processes

*Definition 2.1.* A sequence  $(X_n, n \in \mathbb{Z})$  of  $\mathbb{Z}_+$ -valued rv's is said to be an  $F$ -INAR(1) process if for any  $n \in \mathbb{Z}$ ,

$$X_n = \eta \odot_F X_{n-1} + \epsilon_n, \quad (2.1)$$

where  $0 < \eta < 1$  and  $(\epsilon_n, n \in \mathbb{Z})$  is an i.i.d. sequence of  $\mathbb{Z}_+$ -valued rv's that is assumed to be independent of the  $Y$  variables that define the operator  $\odot_F$  (see below).  $(\epsilon_n, n \in \mathbb{Z})$  is called the innovation sequence.

In the remainder of this paper, we will at times refer to the single-ended  $F$ -INAR(1) process  $(X_n, n \geq 0)$  that arises when (2.1) is assumed to hold only for  $n \geq 0$ .

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The generalized multiplication  $\eta \odot_F X_{n-1}$  in (2.1) is performed independently for each  $n$ . More precisely, we assume the existence of an array  $(Y_{i,n}, i \geq 0, n \in \mathbb{Z})$  of i.i.d.  $\mathbb{Z}_+$ -valued rv's, independent of  $(\epsilon_n, n \in \mathbb{Z})$ , such that the array's common pgf is  $F_t(z), t = -\ln \eta$ , and (see (1.13))

$$\eta \odot_F X_{n-1} = \sum_{i=1}^{X_{n-1}} Y_{i,n-1}. \quad (2.2)$$

These assumptions clearly make the model (2.1) a Markov chain.

The pgf's  $P_{X_n}(z)$  and  $P_\epsilon(z)$  of the  $F$ -INAR(1) process (2.1) satisfy the equation

$$P_{X_n}(z) = P_{X_{n-1}}(F_t(z))P_\epsilon(z), \quad t = -\ln \eta, \quad n \in \mathbb{Z}. \quad (2.3)$$

By using (2.3) recursively (and the fact that  $F_t(z)$  is a semigroup), it can be shown that an  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  admits the following representation for any  $k \geq 1$ :

$$X_n \stackrel{d}{=} \eta^k \odot_F X_{n-k} + \sum_{i=0}^{k-1} \eta^i \odot_F \epsilon_{n-i}, \quad n \in \mathbb{Z}. \quad (2.4)$$

Further distributional and correlation properties of  $F$ -INAR(1) processes are gathered in the following proposition.

**PROPOSITION 2.2.** *Assume  $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$ . Let  $(X_n, n \in \mathbb{Z})$  be an  $F$ -INAR(1) process (for some  $0 < \eta < 1$ ) such that  $E(X_n) < \infty$  and  $E(X_n^2) < \infty$  for any  $n \in \mathbb{Z}$ ,  $\mu_\epsilon = E(\epsilon_0) < \infty$  and  $\sigma_\epsilon^2 = \text{Var}(\epsilon_0) < \infty$ .*

(i) *The regression of  $X_n$  on  $X_{n-1}$  is linear:*

$$E(X_n | X_{n-1}) = \eta^{\delta_F} X_{n-1} + \mu_\epsilon, \quad n \in \mathbb{Z}. \quad (2.5)$$

(ii) *The conditional variance of  $X_n$  given  $X_{n-1}$  is linear:*

$$\text{Var}(X_n | X_{n-1}) = A X_{n-1} + \sigma_\epsilon^2, \quad n \in \mathbb{Z}, \quad (2.6)$$

where  $A = (1 - U''(1)/U'(1))\eta^{\delta_F}(1 - \eta^{\delta_F})$ .

(iii) *For any  $n \in \mathbb{Z}$  and  $k \geq 0$ , the covariance at lag  $k$ ,  $\Gamma_n(k) = \text{cov}(X_{n-k}, X_n)$  of  $\{X_n\}$ , is*

$$\Gamma_n(k) = \eta^{k\delta_F} \text{Var}(X_{n-k}). \quad (2.7)$$

(iv) *For any  $n \in \mathbb{Z}$  and  $k \geq 0$ ,*

$$E(X_n) = \eta^{k\delta_F} E(X_{n-k}) + \mu_\epsilon \sum_{i=0}^{k-1} \eta^{i\delta_F}, \quad (2.8)$$

$$\text{Var}(X_n) = \eta^{2k\delta_F} \text{Var}(X_{n-k}) + A \sum_{i=1}^k \eta^{2(i-1)\delta_F} E(X_{n-i}) + \sigma_\epsilon^2 \sum_{i=1}^k \eta^{2(i-1)\delta_F},$$

where  $A$  is as in (2.6) above.

*Proof.* First, we note (by (1.7)) that  $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$  implies that  $U''(1)$  exists. By (2.2) and (1.10),  $E(\eta \circledast_F X_{n-1} | X_{n-1}) = \eta^{\delta_F} X_{n-1}$  which leads to (2.5). By differentiating twice (with respect to  $z$ ) the expression  $U(F_t(z)) = F'_t(z)U(z)$  ( $t = -\ln \eta$ ) and letting  $z \rightarrow 1$ , we obtain  $F''_t(1) = \eta^{\delta_F} (\eta^{\delta_F} - 1)U''(1)/U'(1)$ . Again, by (2.2) and (1.10),  $E((\eta \circledast_F X_{n-1})^2 | X_{n-1}) = \text{Var}(Y_{1,n-1})X + \eta^{\delta_F} X^2$ . Noting that

$$\text{Var}(Y_{1,n-1}) = F''_t(1) + F'_t(1) - F'_t(1)^2 = \eta^{\delta_F} (1 - \eta^{\delta_F}) \left( 1 - \frac{U''(1)}{U'(1)} \right), \quad (2.9)$$

(2.6) follows by direct calculations. Equation (2.7) is obtained by applying a conditioning argument to (2.4). Finally, (2.8) are easily derived from (2.5) and (2.6).  $\square$

The following result due to Latour [11] insures the existence of a stationary  $F$ -INAR(1) process.

**PROPOSITION 2.3.** *Given  $0 < \eta < 1$  and a sequence  $(\epsilon_n, n \in \mathbb{Z})$  of i.i.d.  $\mathbb{Z}_+$ -valued rv's with finite mean  $\mu_\epsilon$  and finite variance  $\sigma_\epsilon^2$ , there exists a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  satisfying (2.1) and such that  $\text{cov}(X_m, \epsilon_n) = 0, m < n$ .*

Next, we explore the relationship between discrete self-decomposability and stationary  $F$ -INAR(1) processes. A distribution on  $\mathbb{Z}_+$  with pgf  $P(z)$  is said to be  $F$ -self-decomposable (van Harn et al. [21]) if for any  $t > 0$ , there exists a pgf  $P_t(z)$  such that

$$P(z) = P(F_t(z))P_t(z), \quad |z| \leq 1. \quad (2.10)$$

Any  $F$ -self-decomposable distribution can arise as the marginal distribution of a stationary  $F$ -INAR(1) process. More precisely, we have the following result.

**PROPOSITION 2.4.** *Let  $P(z)$  be the pgf of an  $F$ -self-decomposable distribution. For any  $\eta \in (0, 1)$ , there exists a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  whose marginal distribution has pgf  $P(z)$ .*

*Proof.* Using (2.10), one can construct for every  $\eta \in (0, 1)$ , a single-ended  $F$ -INAR(1) process  $(X_n, n \geq 0)$  of the form (2.1) whose innovation sequence  $(\epsilon_n, n \geq 0)$  has common pgf  $P_t(z)$  (where  $t = -\ln \eta$ ) and such that  $X_0$  has pgf  $P(z)$ . It follows from (2.3) and (2.10) that the  $X_n$ 's are identically distributed, which implies that  $(X_n, n \geq 0)$  is stationary since it is a Markov chain. The double-ended version is obtained by sliding  $(X_n, n \geq 0)$  to the left.  $\square$

Next, we state a representation theorem for stationary  $F$ -INAR(1) processes. The proof follows easily from (2.4) and is omitted.

**PROPOSITION 2.5.** *Any stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  admits the following (infinite-order) moving average representation for some  $0 < \eta < 1$ :*

$$X_n \stackrel{d}{=} \sum_{i=0}^{\infty} \eta^i \circledast_F \epsilon_{n-i}, \quad n \in \mathbb{Z}, \quad (2.11)$$

where the convergence of the series is in the weak sense.

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The mean, variance, and autocorrelation function (ACRF) of a stationary  $F$ -INAR(1) process follow straightforwardly from Proposition 2.2.

**COROLLARY 2.6.** *Assume  $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$ . Let  $(X_n, n \in \mathbb{Z})$  be a stationary  $F$ -INAR(1) process (for some  $0 < \eta < 1$ ) such that  $E(X_0) < \infty$ ,  $E(X_0^2) < \infty$ ,  $\mu_\epsilon = E(\epsilon_0) < \infty$ , and  $\sigma_\epsilon^2 = \text{Var}(\epsilon_0) < \infty$ . Then*

(i) *for any  $n \in \mathbb{Z}$ ,*

$$\begin{aligned} E(X_n) &= \mu_\epsilon (1 - \eta^{\delta_F})^{-1}, \\ \text{Var}(X_n) &= \frac{(1 - U''(1)/U'(1))\eta^{\delta_F}\mu_\epsilon + \sigma_\epsilon^2}{1 - \eta^{2\delta_F}}; \end{aligned} \quad (2.12)$$

(ii) *for any  $k \geq 0$  and  $n \in \mathbb{Z}$ , the correlation coefficient of  $(X_{n-k}, X_n)$  is*

$$\rho(k) = \eta^{k\delta_F}. \quad (2.13)$$

We note that the ACRF of a stationary  $F$ -INAR(1) process, as given by (2.13), has the same form as that of the standard AR(1) process. It decays exponentially at lag  $k$ . However, unlike the standard AR(1) case,  $\rho(k)$  remains always positive.

*Remark 2.7.* Assume that  $(X_n, n \geq 0)$  is a one-sided  $F$ -INAR(1) process. If the pgf's  $P_\epsilon(z)$  and  $F_t(z)$  ( $t = -\ln \eta$ ) satisfy

$$\int_0^1 \frac{1 - P_\epsilon(x)}{F_t(x) - x} dx < \infty, \quad t = -\ln \eta, \quad (2.14)$$

then by Foster and Williamson [9],  $(X_n, n \geq 0)$  admits a limiting distribution. Endowing  $X_0$  with the limiting distribution leads to stationarity (since  $(X_n, n \geq 0)$  is a Markov chain).

### 3. Stationary solutions of $F$ -INAR(1) processes

In this section, we present several stationary solutions of  $F$ -INAR(1) processes.

A  $\mathbb{Z}_+$ -valued rv  $X$  is said to have an  $F$ -stable distribution with exponent  $\gamma > 0$  if there exists a sequence of i.i.d. rv's  $(X_i, i \geq 0)$ ,  $X_i \stackrel{d}{=} X$  for all  $i$ , such that for any  $n > 0$ ,  $X \stackrel{d}{=} n^{-1/\gamma} \odot_F \sum_{i=1}^n X_i$  (see van Harn et al. [21]).  $F$ -stable distributions are  $F$ -self-decomposable and exist only when  $0 < \gamma \leq \delta_F$ . Moreover, the pgf  $P(z)$  of an  $F$ -stable distribution with exponent  $\gamma \in (0, \delta_F]$  admits the canonical representation

$$P(z) = \exp[-\lambda A(z)^\gamma] \quad (3.1)$$

for some  $\lambda > 0$ , where  $A(z)$  is given in (1.8).

It follows by Proposition 2.4 that for every  $0 < \eta < 1$ , there exists a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  with an  $F$ -stable marginal distribution with exponent  $\gamma$  ( $0 < \gamma \leq \delta_F$ ). The marginal distribution of the innovation sequence  $(\epsilon_n, n \in \mathbb{Z})$ , obtained by solving for  $P_\epsilon$  in (2.3) and by using (1.9), is also  $F$ -stable with exponent  $\gamma$  and has pgf

$$P_\epsilon(z) = \exp[-\lambda(1 - \eta^\gamma)A(z)^\gamma]. \quad (3.2)$$

Moreover, it can be shown (see van Harn et al. [21]) that stationary  $F$ -INAR(1) processes whose marginal is  $F$ -stable with finite mean arise only in the case  $\gamma = \delta_F$  and  $B'(1) < \infty$  (where  $B(z)$  is given by (1.12)). The process has finite variance if  $B''(1) < \infty$ .

We have shown above (by letting  $\eta = e^{-t}$ ) that the pgf  $P(z)$  of the marginal distribution of a stationary  $F$ -stable  $F$ -INAR(1) process satisfies the following property. For any  $t > 0$ , there exist  $c(t) \in (0, 1)$  such that

$$P[F_t(z)] = P(z)^{c(t)}, \quad 0 \leq z \leq 1. \quad (3.3)$$

It turns out that this property characterizes such processes.

**PROPOSITION 3.1.** *Let  $P(z)$  be a pgf such that  $P(z) \neq 0$  for all  $0 \leq z \leq 1$ . Then  $P(z)$  is  $F$ -stable, with some exponent  $\gamma \in (0, \delta_F]$ , if and only if for any  $t > 0$ , there exists  $c(t) \in (0, 1)$  such that (3.3) holds. The function  $c(t)$  is necessarily of the form  $c(t) = e^{-\gamma t}$ .*

*Proof.* We need only to prove the “if” part. Let  $\psi(z) = \ln P(z)$ . It follows by (3.3) that for any  $t > 0$ , there exists  $c(t) \in (0, 1)$  such that

$$\psi(F_t(z)) = c(t)\psi(z). \quad (3.4)$$

Letting  $\psi_1(z) = \psi(z)/\psi(0)$  and noting that  $c(t) = \psi(F_t(0))/\psi(0)$ , (3.4) becomes

$$\psi_1(F_t(z)) = \psi_1(F_t(0))\psi_1(z). \quad (3.5)$$

By differentiating (3.5) with respect to  $t$ , we obtain

$$\frac{\partial}{\partial t} F_t(z) \psi_1'(F_t(z)) = \frac{\partial}{\partial t} F_t(0) \psi_1'(F_t(0)) \psi_1(z). \quad (3.6)$$

Using  $(\partial/\partial t)F_t(z) = U(F_t(z))$  and letting  $t \downarrow 0$ , it follows by (1.4) that

$$\frac{\psi_1'(z)}{\psi_1(z)} = \frac{U(0)}{U(z)} \psi_1'(0), \quad (3.7)$$

whose solution is  $\psi_1(z) = A(z)^\gamma$ , where  $\gamma = -\psi_1'(0)U(0) > 0$ . Hence,  $P(z)$  has the form (3.1). Since  $P$  is a pgf,  $\gamma$  must satisfy  $\gamma \leq \delta_F$  (cf. van Harn and Steutel [20, the proof of Lemma 4.2]). The form of  $c(t)$  follows from its uniqueness and the “only if” part.  $\square$

Next, we present a stationary  $F$ -INAR(1) process with an  $F$ -geometric stable marginal distribution.

A  $\mathbb{Z}_+$ -valued rv  $X$  is said to have an  $F$ -geometric stable distribution if for any  $p \in (0, 1)$ , there exists  $\alpha(p) \in (0, 1)$  such that  $X \stackrel{d}{=} \alpha(p) \odot_F \sum_{i=1}^{N_p} X_i$ , where  $(X_i, i \geq 1)$  is a sequence of i.i.d. rv's,  $X_i \stackrel{d}{=} X$ ,  $N_p$  has the geometric distribution with parameter  $p$ , and  $(X_i, i \geq 1)$  and  $N_p$  are independent (see Bouzar [6]).  $F$ -geometric stable distributions are  $F$ -self-decomposable and their pgf's admit the canonical representation

$$P(z) = (1 + \lambda A(z)^\gamma)^{-1} \quad (3.8)$$

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for  $0 < \gamma \leq \delta_F$  and  $\lambda > 0$ . We will refer to distributions with pgf (3.8) as  $F$ -geometric stable distributions with exponent  $\gamma$ .

By Proposition 2.4, there exists, for every  $\eta \in (0, 1)$ , a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  with an  $F$ -geometric stable marginal distribution with pgf (3.8). Its innovation sequence  $(\epsilon_n, n \in \mathbb{Z})$  has marginal pgf (obtained by solving (2.3) for  $P_\epsilon(z)$  and by using (1.9))

$$P_\epsilon(z) = \eta^\gamma + (1 - \eta^\gamma)(1 + \lambda A(z)^\gamma)^{-1}, \quad 0 < \gamma \leq \delta_F, \lambda > 0. \quad (3.9)$$

This implies that a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  with an  $F$ -geometric stable distribution can be written as

$$X_n = \eta \odot_F X_{n-1} + I_n E_n, \quad n \in \mathbb{Z}, \quad (3.10)$$

where  $(I_n, n \in \mathbb{Z})$  and  $(E_n, n \in \mathbb{Z})$  are independent sequences of i.i.d. rv's such that  $I_n$  is Bernoulli( $1 - \eta^\gamma$ ) and  $E_n$  has the same distribution as  $X_n$ . Moreover, a stationary  $F$ -INAR(1) process with an  $F$ -geometric stable marginal has finite mean only if  $\gamma = \delta_F$  and  $B'(1) < \infty$ . It has a finite variance if  $B''(1) < \infty$ .

We have in fact shown by the above argument (and by letting  $\eta = e^{-t}$ ) that the pgf  $P(z)$  of the marginal distribution of a stationary  $F$ -geometric stable  $F$ -INAR(1) process satisfies the following property. For any  $t > 0$ , there exists  $c(t) \in (0, 1)$  such that

$$P(z) = P(F_t(z))(c(t) + (1 - c(t))P(z)). \quad (3.11)$$

We show next that the converse is true.

**PROPOSITION 3.2.** *Let  $P(z)$  be the pgf of a nondegenerate distribution on  $\mathbb{Z}_+$ . Then  $P(z)$  is  $F$ -geometric stable with some exponent  $\gamma \in (0, \delta_F]$  if and only if for any  $t > 0$ , there exists  $c(t) \in (0, 1)$  such that (3.11) holds. The function  $c(t)$  is necessarily of the form  $c(t) = e^{-\gamma t}$ .*

*Proof.* We only need to show the “if” part. Rewriting  $P(z) = (1 + \psi(z))^{-1}$ , it follows by (3.11) that for  $t > 0$ , there exists  $c(t) \in (0, 1)$  such that  $\psi(F_t(z)) = c(t)\psi(z)$ . Using the exact same argument as the one in the proof of Proposition 3.1 (following (3.5)), we have  $\psi(z) = \lambda A(z)^\gamma$  for some  $0 < \gamma \leq \delta_F$  and  $\lambda > 0$ . The form of  $c(t)$  follows from its uniqueness and the “only if” part.  $\square$

We define next a compound discrete Linnik distribution and construct the corresponding stationary  $F$ -INAR(1) process.

A  $\mathbb{Z}_+$ -valued rv  $X$  is said to have an  $F$ -compound discrete Linnik distribution if its pgf has the form

$$P(z) = (1 + \lambda A(z)^\gamma)^{-r} \quad (3.12)$$

for some  $0 < \gamma \leq \delta_F$ ,  $\lambda > 0$ , and  $r > 0$ . Note that  $P(z)$  indeed results from the compounding of i.i.d. rv's (with the common pgf  $B(z)$  of (1.12)) by a discrete Linnik distribution (with pgf  $G(z) = (1 + \lambda(1 - z)^{\gamma/\delta_F})^{-r}$ ). The special case  $r = 1$  in (3.12) gives the  $F$ -geometric stable distribution. van Harn and Steutel [20] showed that  $F$ -compound discrete Linnik



distributions are  $F$ -self-decomposable and arise as solutions to stability equations for  $\mathbb{Z}_+$ -valued processes with stationary independent increments.

Again by self-decomposability, for every  $\eta \in (0, 1)$ , there exists a stationary  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  with an  $F$ -compound Linnik marginal distribution (with pgf (3.12)). Its innovation sequence  $\{\epsilon_n\}$  has pgf

$$P_\epsilon(z) = \left( \frac{1 + \lambda \eta^\gamma A(z)^\gamma}{1 + \lambda A(z)^\gamma} \right)^r. \tag{3.13}$$

It can be shown by a straightforward pgf argument that  $\{\epsilon_n\}$  has the representation

$$\epsilon \stackrel{d}{=} \sum_{i=1}^N (\eta^{U_i}) \odot_F W_i, \tag{3.14}$$

where  $(W_i, i \geq 0)$  is a sequence of i.i.d.  $F$ -geometric stable rv's (with pgf (3.8)),  $\{U_i\}$  are i.i.d. uniform  $(0, 1)$  rv's, and  $N$  is Poisson with mean  $-r\gamma \ln \eta$ , with all these variables independent. This allows for a shot-noise interpretation of the process that is similar to the one given by Lawrance [12] for the gamma AR(1) process (see also McKenzie [15] for the case of the negative binomial INAR(1) process). A shot-noise process is defined by

$$X(t) = \sum_{m=N(-\infty)}^{N(t)} \eta^{t-\tau_m} \odot W_m, \tag{3.15}$$

where  $(W_m, m \geq 0)$  are  $\mathbb{Z}_+$ -valued i.i.d. rv's (amplitudes of the shots) and  $(N(t), t \geq 0)$  is a Poisson process with occurrence times at  $\tau_m$ . If the  $W_m$ 's have their common pgf given by (3.8) and  $N(t)$  has rate  $-r\gamma \ln \eta$ , then  $X(t)$  of (3.15), sampled at  $n = 0, \pm 1, \pm 2, \dots$  gives another representation of the stationary  $F$ -INAR(1) process (2.1) with marginal pgf (3.12). The proof of this fact is an adaptation of Lawrance's [12] argument and the details are omitted.

Finally, as above, a stationary  $F$ -INAR(1) process with an  $F$ -compound discrete Linnik marginal has finite mean only if  $\gamma = \delta_F$  and  $B'(1) < \infty$ . It has a finite variance if  $B''(1) < \infty$ .

van Harn et al. [21] give some rich examples of continuous composition semigroups of pgf's from which one can generate  $F$ -INAR(1) processes. We mention the parametrized family of semigroups  $(F^{(\theta)}, \theta \in [0, 1])$  described by

$$F_t^{(\theta)}(z) = 1 - \frac{\bar{\theta} e^{-\bar{\theta}t}(1-z)}{\bar{\theta} + \theta(1 - e^{-\bar{\theta}t})(1-z)}, \quad t \geq 0, |z| \leq 1, \bar{\theta} = 1 - \theta. \tag{3.16}$$

In this case, we have  $\delta_{F^{(\theta)}} = \bar{\theta}$ ,  $U_{F^{(\theta)}}(z) = (1-z)(1-\theta z)$  and  $A_{F^{(\theta)}}(z) = ((1-z)/(1-\theta z))^{1/\bar{\theta}}$ . We note that for  $\theta = 0$ ,  $F^{(\theta)}$  corresponds to the standard semigroup  $F_t^{(0)}(z) = 1 - e^{-t} + e^{-t}z$  and the multiplication  $\odot_{F^{(0)}}$  becomes the binomial thinning operator of Steutel and van Harn [19].

The Poisson AR(1) process of McKenzie [16] is the stationary  $F^{(0)}$ -INAR(1) process with an  $F^{(0)}$ -stable marginal. More generally, the Poisson geometric INAR(1) process of

Aly and Bouzar [2] arises as the stationary  $F^{(\theta)}$ -INAR(1) process with an  $F^{(\theta)}$ -stable marginal. The discrete Mittag-Leffler  $F$ -INAR(1) process of Pillai and Jayakumar [10] (and, in particular, the geometric INAR(1) of McKenzie [14]) is the stationary  $F^{(0)}$ -INAR(1) process with an  $F^{(0)}$ -geometric stable marginal. The discrete Linnik INAR(1) process of Aly and Bouzar [3] (and, in particular, the negative binomial INAR(1) of McKenzie [14]) is the stationary  $F^{(0)}$ -INAR(1) process with an  $F^{(0)}$ -compound Linnik marginal.

Finally, we note that Zhu and Joe [22] used a reparametrized version of the semi-group  $F^{(\theta)}$  to construct a continuous-time  $\mathbb{Z}_+$ -valued Markov process  $(X(t), t \geq 0)$  via the equation

$$X(t) = e^{-\mu(t-s)} \odot_{F^{(\theta)}} X(s) + \epsilon(s, t), \quad s < t, \quad (3.17)$$

where  $\mu > 0$  and  $\epsilon(s, t)$  is  $\mathbb{Z}_+$ -valued and independent of  $X(s)$ .

#### 4. Time-reversibility of stationary $F$ -INAR(1) processes

A stochastic process  $(X_n, n \in \mathbb{Z})$  is said to be time-reversible if for any  $n \in \mathbb{Z}$  and  $k \geq 0$ ,  $(X_n, X_{n+1}, \dots, X_{n+k})$  and  $(X_{n+k}, X_{n+k-1}, \dots, X_n)$  have the same joint distribution.

Let  $(X_n, n \in \mathbb{Z})$  be an  $F$ -INAR(1) process. By the Markov property,  $(X_n, n \in \mathbb{Z})$  is time-reversible if and only if for any  $n \in \mathbb{Z}$ ,  $(X_{n-1}, X_n)$  and  $(X_n, X_{n-1})$  have the same joint distribution. In terms of the joint pgf  $g_n(z_1, z_2)$  of  $(X_{n-1}, X_n)$  which is defined by

$$g_n(z_1, z_2) = E\left(z_1^{X_{n-1}} z_2^{X_n}\right) \quad (|z_1| \leq 1, |z_2| \leq 1), \quad (4.1)$$

$(X_n, n \in \mathbb{Z})$  is time-reversible if and only if  $g_n(z_1, z_2) = g_n(z_2, z_1)$  for all  $n \in \mathbb{Z}$ ,  $|z_1| \leq 1$ , and  $|z_2| \leq 1$ .

By (2.1) and a conditioning argument, it is easily shown that

$$g_n(z_1, z_2) = P_\epsilon(z_2) P_{X_{n-1}}(z_1 F_t(z_2)) \quad (t = -\ln \eta, 0 < \eta < 1). \quad (4.2)$$

By Proposition 2.2(i), a time-reversible  $F$ -INAR(1) process  $(X_n, n \in \mathbb{Z})$  (such that  $E(X_n) < \infty$  and  $E(\epsilon_n) < \infty$ ) possesses the property of linear backward regression. That is, there exist  $c > 0$  and  $d \geq 0$  such that for any  $n \in \mathbb{Z}$ ,

$$E(X_{n-1} | X_n) = d + cX_n. \quad (4.3)$$

We show next that under an  $x \ln x$  condition (condition (4.4) below), a stationary  $F$ -INAR(1) process with finite mean and finite variance has the property of backward linear regression only if its pgf admits a certain form.

PROPOSITION 4.1. *Assume that the distribution  $(h_n, n \geq 0)$  satisfies*

$$\sum_{n=2}^{\infty} h_n n \ln n < \infty. \quad (4.4)$$

*Let  $(X_n, n \in \mathbb{Z})$  be a stationary  $F$ -INAR(1) process with finite mean and finite variance with the property of linear backward regression (4.3). Then the pgf  $P(z)$  of the marginal*

distribution of  $(X_n, n \in \mathbb{Z})$  has the form

$$P(z) = \exp\left(-\lambda \int_z^1 \frac{B(x)}{x} dx\right), \quad (4.5)$$

where  $\lambda > 0$  and  $B(z)$  is the pgf of (1.12).

*Proof.* Let  $n \geq 1$  and let  $g(z_1, z_2)$  be the joint pgf of  $(X_{n-1}, X_n)$ . We have by (2.3) and (4.2)

$$g(z_1, z_2) = \frac{P(z_1 F_t(z_2)) P(z_2)}{P(F_t(z_2))} \quad (t = -\ln \eta, 0 < \eta < 1). \quad (4.6)$$

Differentiating  $g$  with respect to  $z_1$ , then setting  $z_1 = 1$  and  $z_2 = z$ , it follows that for any  $n \in \mathbb{Z}$ ,

$$E(X_{n-1} z^{X_n}) = \frac{F_t(z) P(z) P'(F_t(z))}{P(F_t(z))}. \quad (4.7)$$

Now by (4.3) we have for some  $c > 0$  and  $d \geq 0$ ,

$$E(X_{n-1} z^{X_n}) = E(z^{X_n} E(X_{n-1} | X_n)) = cz E(X_n z^{X_n-1}) + d E(z^{X_n}) \quad (4.8)$$

for any  $n \geq 1$ . Letting  $Q(z) = zP'(z)/P(z)$  and combining (4.7) and (4.8) (note that  $E(X_n z^{X_n-1}) = P'(z)$ ), we obtain  $cQ(z) + d = Q(F_t(z))$ , and therefore, by differentiation,  $cQ'(z) = F_t'(z)Q'(F_t(z))$ . Noting that  $Q'(1) = \text{Var}(X_n) \neq 0$ , it follows that  $c = F_t'(1) = e^{-\delta_F t}$  (with the second equation following from (1.10)) which implies

$$Q'(z) = e^{\delta_F t} F_t'(z) Q'(F_t(z)). \quad (4.9)$$

An induction argument yields for any  $n \geq 1$ ,

$$Q'(z) = e^{n\delta_F t} \prod_{j=0}^{n-1} F_t'(F_{jt}(z)) Q'(F_{nt}(z)). \quad (4.10)$$

By the semigroup property and (1.9), we have  $F_t'(F_{jt}(z)) = U(F_{(j+1)t}(z))/U(F_{jt}(z))$ ,  $j = 0, \dots, n-1$ . Therefore,

$$Q'(z) = e^{n\delta_F t} \frac{U(F_{nt}(z))}{U(z)} Q'(F_{nt}(z)). \quad (4.11)$$

From the semigroup properties (1.4), (1.10), and (1.11), we have

$$\lim_{n \rightarrow \infty} F_{nt}(z) = 1, \quad \lim_{n \rightarrow \infty} \frac{U(F_{nt}(z))}{F_{nt}(z) - 1} = U'(1) = -\delta_F, \quad \lim_{n \rightarrow \infty} \frac{F_{nt}(z) - 1}{F_{nt}(0) - 1} = 1 - B(z). \quad (4.12)$$

Moreover, (4.4) implies (see van Harn et al. [21])

$$\lim_{n \rightarrow \infty} e^{n\delta_F t} (F_{nt}(0) - 1) = -1. \quad (4.13)$$

Therefore, by letting  $n \rightarrow \infty$  in (4.11), we obtain

$$Q'(z) = \delta_F Q'(1) \frac{1 - B(z)}{U(z)}. \quad (4.14)$$

Since (by (1.8) and (1.12))  $1/U(z) = -A'(z)/A(z)$  and  $1 - B(z) = A(z)^{\delta_F}$ , we have (note that  $Q(0) = 0$ )

$$Q(z) = \int_0^z Q'(x) dx = Q'(1)(1 - A(z)^{\delta_F}) = Q'(1)B(z), \quad (4.15)$$

which implies that  $P'(z)/P(z) = Q'(1)B(z)/z$  or

$$\ln P(z) = -Q'(1) \int_z^1 \frac{B(x)}{x} dx. \quad (4.16)$$

□

We note that Proposition 4.1 remains valid if the property of backward linear regression is replaced by the (stronger) assumption of time-reversibility.

For the family of semigroups  $(F^{(\theta)}, \theta \in [0, 1])$  of (3.16), the condition (4.4) is satisfied (since  $h_n = 0$  for  $n \geq 3$ ). In this case, the pgf  $P(z)$  of (4.5) is shown to be

$$P(z) = \begin{cases} e^{-\lambda(1-z)} & \text{if } \theta = 0 \ (\lambda > 0), \\ \left(\frac{\bar{\theta}}{1-\theta z}\right)^r & \text{if } 0 < \theta < 1 \ (r > 0). \end{cases} \quad (4.17)$$

The Poisson distribution (resp., the negative binomial distribution with probability of success  $\theta$ ,  $0 < \theta < 1$ ) is the only distribution that arises as the marginal of a stationary  $F^{(0)}$ -INAR(1) (resp.,  $F^{(\theta)}$ -INAR(1)) process with finite mean and finite variance and with the property of backward linear regression. These results were established by Alzaid and Al-Osh [4] (for  $\theta = 0$ ) and by Aly and Bouzar [2] (for  $0 < \theta < 1$ ).

## 5. An $F$ -INAR( $p$ ) process

Lawrance and Lewis [13] introduced the mixed autoregressive process of order  $p$  (AR( $p$ ))

$$X_n = \sum_{i=1}^p I(\xi_n = i) \eta_i X_{n-i} + \epsilon_n, \quad (5.1)$$

where  $I(A)$  is the indicator function of the event  $A$ ,  $\{\xi_n\}$  and  $\{\epsilon_n\}$  are two independent sequences of i.i.d. rv's,  $0 < \eta_i < 1$ ,  $P(\xi_n = i) = c_i$ ,  $i = 1, 2, \dots, p$ , and  $\sum_{i=1}^p c_i = 1$ . The authors obtained the distribution of the innovation rv  $\epsilon_n$  for the stationary AR( $p$ ) process with an exponential marginal. Pillai and Jayakumar [18] went a bit further by deriving the distribution of  $\epsilon_n$  for the stationary AR( $p$ ) process with the Mittag-Leffler marginal on  $\mathbb{R}_+$ . Using the binomial thinning operator of Steutel and van Harn [19], Jayakumar [10] defined the discrete analogue of (5.1) and constructed the discrete Mittag-Leffler INAR( $p$ ) process.

In this section, we present a generalized INAR( $p$ ) process by using the  $\odot_F$  operator. In particular, we will derive the marginal distribution of the stationary INAR( $p$ ) process with an  $F$ -geometric stable marginal.

*Definition 5.1.* A sequence  $\{X_n\}$  of  $\mathbb{Z}_+$ -valued rv's is said to be an  $F$ -INAR( $p$ ) process if for any  $n \in \mathbb{Z}$ ,

$$X_n = \sum_{i=1}^p I(\xi_n = i) \eta_i \odot_F X_{n-i} + \epsilon_n, \tag{5.2}$$

where  $(\xi_n, n \in \mathbb{Z})$  and  $(\epsilon_n, n \in \mathbb{Z})$  are two independent sequences of i.i.d.  $\mathbb{Z}_+$ -valued rv's,  $0 < \eta_i < 1$ ,  $P(\xi_n = i) = c_i$ ,  $i = 1, 2, \dots, p$ , and  $\sum_{i=1}^p c_i = 1$ .

The generalized multiplication  $\eta_i \odot_F X_{n-i}$  in (5.2) is performed independently for each  $i$ . More precisely, we assume the existence of  $p$  independent arrays  $(Y_{i,n}^{(j)}, i \geq 0, n \in \mathbb{Z})$ ,  $j = 1, 2, \dots, p$ , of i.i.d.  $\mathbb{Z}_+$ -valued rv's, independent of  $(\xi_n, n \in \mathbb{Z})$  and  $(\epsilon_n, n \in \mathbb{Z})$ , such that for each  $j = 1, 2, \dots, p$ , the array's common pgf is  $F_{t_j}(z)$ ,  $t_j = -\ln \eta_j$ , and

$$\eta_j \odot_F X_{n-j} = \sum_{i=1}^{X_{n-j}} Y_{i,n-j}^{(j)}. \tag{5.3}$$

In terms of pgf's, it follows from (5.2) that

$$P_{X_n}(z) = \left( \sum_{i=1}^p c_i P_{X_{n-i}}(F_{t_i}(z)) \right) P_\epsilon(z), \quad t_i = -\ln \eta_i. \tag{5.4}$$

The autocorrelation structure of a stationary  $F$ -INAR( $p$ ) process is given in the following proposition and its corollary.

**PROPOSITION 5.2.** Assume  $\sum_{n=2}^\infty n(n-1)h_n < \infty$ . Let  $(X_n, n \in \mathbb{Z})$  be a stationary  $F$ -INAR( $p$ ) process such that  $E(X_0) < \infty$ ,  $E(X_0^2) < \infty$ ,  $\mu_\epsilon = E(\epsilon_0) < \infty$ , and  $\sigma_\epsilon^2 = \text{Var}(\epsilon_0) < \infty$ . Let  $c_i$  and  $\eta_i$  ( $i = 1, \dots, p$ ) be as in (5.2). Then

(i) for any  $n \in \mathbb{Z}$ ,

$$E(X_n) = \left( 1 - \sum_{i=1}^p c_i \eta_i^{\delta_F} \right)^{-1} \mu_\epsilon; \tag{5.5}$$

(ii) the autocovariance function  $\Gamma(k) = \text{Cov}(X_{n-k}, X_n)$  of  $(X_n, n \in \mathbb{Z})$  is given by

$$\Gamma(k) = \begin{cases} \sum_{i=1}^p c_i \eta_i^{\delta_F} \Gamma(i) + B & \text{if } k = 0, \\ \sum_{i=1}^p c_i \eta_i^{\delta_F} \Gamma(k-i) & \text{if } k \geq 1, \end{cases} \tag{5.6}$$

with  $\Gamma(k) = \Gamma(-k)$ ,  $k \geq 1$ , and

$$B = \sigma_\epsilon^2 + \frac{1 - U''(1)}{U'(1)} E(X_0) \sum_{i=1}^p c_i \eta_i^{\delta_F} (1 - \eta_i^{\delta_F}) + E(X_0^2) \sum_{i=1}^p (c_i - c_i^2) \eta_i^{2\delta_F}. \quad (5.7)$$

*Proof.* Stationarity implies  $\Gamma(k) = \Gamma(-k)$ . Using (5.2), (1.10), and a conditioning argument yields (5.5) and

$$E(X_{n-k}(\eta_i \odot_F X_{n-i})) = \eta_i^{\delta_F} E(X_{n-k} X_{n-i}) \quad (i = 1, 2, \dots, p). \quad (5.8)$$

It follows that  $E(X_n X_{n-k}) = \sum_{i=1}^p c_i \eta_i^{\delta_F} E(X_{n-k} X_{n-i}) + E(\epsilon_n) E(X_{n-k})$ . Combining this last equation with (5.5) gives the formula for  $\Gamma(k)$ ,  $k \geq 1$ , in (5.6). Direct calculations lead to (note again that  $U''(1)$  exists because  $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$ )

$$\Gamma(0) = \sum_{i=1}^p c_i^2 \eta_i^{2\delta_F} \Gamma(0) + 2 \sum_{1 \leq i < j \leq p} c_i c_j \eta_i^{\delta_F} \eta_j^{\delta_F} \Gamma(j-i) + B, \quad (5.9)$$

where  $B$  is as in (5.7). This in turn leads to  $\Gamma(0)$  of (5.6).  $\square$

The sequence  $(\Gamma(k), k \in \mathbb{Z})$  is completely determined by (5.6) once  $\Gamma(k)$ ,  $k = 0, \dots, p$ , are found. The latter are solution of the linear system (5.6) restricted to  $k = 0, 1, \dots, p$ . Since  $\sum_{i=1}^p c_i \eta_i^{\delta_F} < 1$ , it follows by Latour [11, Lemma 2.1] that this system has a unique solution given by

$$\mathbf{\Gamma}_p = B(\mathbf{I} - \mathbf{C})^{-1} \mathbf{c}, \quad (5.10)$$

where  $\mathbf{\Gamma}_p = (\Gamma(0), \Gamma(1), \dots, \Gamma(p))$ ,  $\mathbf{I}$  is the  $(p+1) \times (p+1)$  identity matrix,  $\mathbf{C}$  is also  $(p+1) \times (p+1)$  whose  $(i, j)$  entry is equal to the coefficient of  $\Gamma(i)$  on the right-hand side of the  $j$ th equation of (5.6),  $i, j = 0, 1, \dots, p$ ,  $\mathbf{c} = (1, 0, \dots, 0)$  (of length  $p+1$ ), and  $B$  is given by (5.7).

**COROLLARY 5.3.** *Assume  $\sum_{n=2}^{\infty} n(n-1)h_n < \infty$ . Let  $(X_n, n \in \mathbb{Z})$  be a stationary  $F$ -INAR( $p$ ) process such that  $E(X_0) < \infty$ ,  $E(X_0^2) < \infty$ ,  $\mu_\epsilon = E(\epsilon_0) < \infty$  and  $\sigma_\epsilon^2 = \text{Var}(\epsilon_0) < \infty$ . Then the autocorrelation coefficient  $\rho_k$  at lag  $k$ ,  $k \geq 1$ , is given by*

$$\rho_k = \sum_{i=1}^p c_i \eta_i^{\delta_F} \rho_{k-i}, \quad (5.11)$$

where  $\rho_0 = 1$ ,  $\rho_k = \rho_{-k}$ , and  $c_i$  and  $\eta_i$  ( $i = 1, \dots, p$ ) are as in (5.2).

Next, we determine the marginal distribution of the innovation sequence of a stationary  $F$ -INAR( $p$ ) process with an  $F$ -geometric stable marginal distribution (whose pgf is given by (3.8)). We need a lemma first.

LEMMA 5.4. For  $i = 1, 2$ , let  $f_i(t) = \sum_{j=1}^{n_i} c_{ij}(a_{ij}/(a_{ij} + t))$ ,  $t \geq 0$ ,  $a_{ij} > 0$ ,  $c_{ij} \geq 0$ , and  $\sum_{j=1}^{n_i} c_{ij} = 1$ . If  $\max_{1 \leq i \leq n_1} a_{1i} < \min_{1 \leq i \leq n_2} a_{2i}$ , there exist  $c'_k \geq 0$ ,  $d_k > 0$ , with  $\sum_{k=0}^{n_1+n_2-1} c'_k = 1$  and  $d_k$  strictly increasing (with the  $d_k$ 's being the  $a_{1k}$ 's in ascending order for  $k = 1, \dots, n_1$ ) such that

$$\frac{f_1(t)}{f_2(t)} = c'_0 + \sum_{k=1}^{n_1+n_2-1} c'_k \frac{d_k}{d_k + t}. \tag{5.12}$$

*Proof.* Note that  $f_i(t)$  is the Laplace-Stieltjes transform of a mixture of exponential distributions. By grouping and reordering if necessary, we may assume without loss of generality that  $0 < a_{ij} < \dots < a_{in_i}$ ,  $i = 1, 2$ . Using a result in Feller [8, pages 438–439], we have

$$f_i(t) = C_i \cdot \frac{b_{i1} + t}{a_{i1} + t} \cdot \dots \cdot \frac{b_{i,n_i-1} + t}{a_{i,n_i-1} + t} \cdot \frac{1}{a_{in_i} + t}, \tag{5.13}$$

where  $0 < a_{i1} < b_{i1} < a_{i2} < b_{i2} < \dots < b_{i,n_i-1} < a_{in_i}$  and  $C_i > 0$ . Letting  $d_k = a_{1k}$ ,  $k = 1, \dots, n_1$ , and  $d_{n_1+k} = b_{2k}$ ,  $k = 1, \dots, n_2 - 1$ , a straightforward partial fraction expansion leads to (5.12). The fact that  $a_{1n_1} < a_{21}$  and  $f_1(0)/f_2(0) = 1$  implies that  $c'_k \geq 0$ ,  $\sum_{k=0}^{n_1+n_2-1} c'_k = 1$ , and  $d_k$  is increasing by definition.  $\square$

PROPOSITION 5.5. If  $(X_n, n \in \mathbb{Z})$  is a stationary  $F$ -INAR( $p$ ) process with an  $F$ -geometric stable marginal (with pgf given by (3.8)), then

$$\epsilon_n \stackrel{d}{=} \sum_{j=0}^p I(V_n = j) \beta_j \circ_F E_n, \tag{5.14}$$

where  $(V_n, n \in \mathbb{Z})$  and  $(E_n, n \in \mathbb{Z})$  are independent sequences of i.i.d. rv's,  $P(V_n = j) = c'_j$ ,  $j = 0, \dots, p$ ,  $\sum_{j=0}^p c'_j = 1$ ,  $E_n$  has the same distribution as  $X_n$ ,  $\beta_0 = 0$ ,  $\beta_1 = 1$ , and  $0 < \beta_i < 1$  for  $i = 2, \dots, p$ .

*Proof.* Using (1.9), (3.8), and (5.4), we have

$$P_\epsilon(z) = \frac{(1 + \lambda A(z)^\gamma)^{-1}}{\sum_{i=1}^p c_i a_i (a_i + \lambda A(z)^\gamma)^{-1}}, \tag{5.15}$$

with  $\lambda > 0$ ,  $0 < \gamma \leq \delta_F$ , and  $a_i = \bar{\gamma}_i^{-\gamma} > 1$ ,  $i = 1, \dots, p$ . By applying Lemma 5.4 to (5.15) with  $n_1 = 1$ ,  $n_2 = p$ ,  $a_{11} = 1$ ,  $a_{2j} = a_j$ , and  $t = \lambda A(z)^\gamma$ , we obtain

$$P_\epsilon(z) = c'_0 + c'_1 (1 + \lambda A(z)^\gamma)^{-1} + \sum_{j=2}^p c'_j (1 + \lambda \beta_j^\gamma A(z)^\gamma)^{-1} \tag{5.16}$$

with  $\beta_j^\gamma = d_j^{-1}$ ,  $d_j > 1$  (by Lemma 5.4), for  $j = 2, \dots, p$ . The representation (5.14) then follows easily from (5.16).  $\square$

The proof of Proposition 5.5 shows that in the case of a stationary  $F$ -INAR( $p$ ) process with an  $F$ -geometric stable marginal, the solution (5.16) of (5.4) exists for the full range of  $p$ ,  $\{t_i\}$ , and  $\{c_i\}$ . It turns out that the converse is true. That is, if  $P(z)$  is the pgf of a nondegenerate distribution such that (5.16) is the solution of (5.4) (where  $P_{X_k}(z) = P(z)$  for all  $k$ 's) for the full range of  $p$ ,  $\{t_i\}$ , and  $\{c_i\}$ , then  $P(z)$  is necessarily  $F$ -geometric stable. This is a direct consequence of Proposition 3.2, since, applied to the case  $p = 1$ , the assumption reduces to (3.11).

Finite mixtures of  $F$ -geometric stable distributions can arise as marginals of stationary  $F$ -INAR( $p$ ) processes. Such distributions have a pgf of the form

$$P(z) = \sum_{j=1}^m q_j (1 + \lambda_j A(z)^\gamma)^{-1}, \quad (5.17)$$

where  $0 < \gamma \leq \delta_F$ ,  $\lambda_j > 0$ ,  $q_j \geq 0$ , and  $\sum_{j=1}^m q_j = 1$ . Solving (5.4) for  $P_\epsilon$ , with  $P_X$  as in (5.17), gives

$$P_\epsilon(z) = \frac{\sum_{j=1}^m q_j b_j (b_j + A(z)^\gamma)^{-1}}{\sum_{i=1}^p \sum_{j=1}^m c_i q_j b_{ij} (b_{ij} + A(z)^\gamma)^{-1}}, \quad (5.18)$$

where  $b_j = 1/\lambda_j$  and  $b_{ij} = b_j \eta_i^{-\gamma}$ . If

$$\frac{\max_{1 \leq i \leq m} \lambda_i}{\min_{1 \leq i \leq m} \lambda_i} < \min_{1 \leq i \leq p} \eta_i^{-\gamma}, \quad (5.19)$$

then a straightforward application of Lemma 5.4 ( $n_1 = m$ ,  $n_2 = p$ , the  $a_{1k}$ 's are the  $b_k$ 's, the  $a_{2k}$ 's are the  $b_{ij}$ 's, and  $t = A(z)^\gamma$ ) shows that  $P_\epsilon(z)$  is the pgf of a mixture of  $F$ -geometric stable distributions and the degenerate distribution at 0.

One can use the family of semigroups ( $F^{(\theta)}$ ,  $\theta \in [0, 1)$ ) of (3.16) to construct stationary  $F$ -INAR( $p$ ) processes with  $F$ -geometric stable (or finite mixtures of  $F$ -geometric stable) marginals. We note that Jayakumar's [10] discrete Mittag-Leffler INAR( $p$ ) process is the  $F^{(0)}$ -INAR( $p$ ) process with an  $F^{(0)}$ -geometric stable marginal. As seen above, finite mixtures of discrete Mittag-Leffler distributions can arise as the marginal of a stationary  $F^{(0)}$ -INAR( $p$ ) process (under condition (5.19)). This is a counterexample to Jayakumar's [10, Theorem 2.2] that states that only discrete Mittag-Leffler distributions can arise as marginals of stationary INAR( $p$ ) processes.

Finally, we mention that a more general  $F$ -INAR( $p$ ) process can be constructed using Latour's [11] definition of an INAR( $p$ ) process. In this case, (5.2) becomes

$$X_n = \sum_{i=1}^p \eta_i \odot_F X_{n-i} + \epsilon_n, \quad (5.20)$$

where  $0 < \eta_i < 1$ ,  $i = 1, \dots, p$ , and  $(\epsilon_n, n \in \mathbb{Z})$  is an i.i.d. sequence of  $\mathbb{Z}_+$ -valued rv's with finite mean  $\mu_\epsilon$  and finite variance  $\sigma_\epsilon^2$ . By Latour [11], a finite mean stationary INAR( $p$ )



process of type (5.20) exists if and only if

$$\sum_{i=1}^p \eta_i^{\delta_F} < 1. \quad (5.21)$$

Proposition 5.2 and Corollary 5.3 remain valid with the following modified formulas for the mean:

$$E(X_n) = \left(1 - \sum_{i=1}^p \eta_i^{\delta_F}\right)^{-1} \mu_\epsilon, \quad (5.22)$$

the autocovariance function (with  $B_1 = \sigma_\epsilon^2 + (1 - U''(1)/U'(1))E(X_0) \sum_{i=1}^p \eta_i^{\delta_F} (1 - \eta_i^{\delta_F})$ )

$$\Gamma(k) = \begin{cases} \sum_{i=1}^p \eta_i^{\delta_F} \Gamma(i) + B_1 & \text{if } k = 0, \\ \sum_{i=1}^p \eta_i^{\delta_F} \Gamma(k - i) & \text{if } k \geq 1, \end{cases} \quad (5.23)$$

and the autocorrelation coefficient

$$\rho_k = \sum_{i=1}^p \eta_i^{\delta_F} \rho_{k-i}. \quad (5.24)$$

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