SEMICOMPACTNESS IN L-TOPOLOGICAL SPACES

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The concepts of semicompactness, countable semicompactness, and the semi-Lindelöf property are introduced in L-topological spaces, where L is a complete de Morgan algebra. They are defined by means of semiopen L-sets and their inequalities. They do not rely on the structure of basis lattice L and no distributivity in L is required. They can also be characterized by semiclosed L-sets and their inequalities. When L is a completely distributive de Morgan algebra, their many characterizations are presented.

1. Introduction

The notion of semicompactness [3] was introduced in L-topological spaces by Kudri. In Kudri's work [6], he followed the lines of his definition of compactness which is equivalent to the notion of strong fuzzy compactness in [7, 8, 13]. However, Kudri's semicompactness relies on the structure of L and L is required to be completely distributive.

In [10, 12], a new definition of fuzzy compactness is presented in L-topological spaces by means of an inequality, which does not depend on the structure of L and no distributivity is required in L. When L is a completely distributive de Morgan algebra, it is equivalent to the notion of fuzzy compactness in [7, 8, 13].

Following the lines of [10, 12], we will introduce a new definition of semicompactness in L-topological spaces by means of semiopen L-sets and their inequality, where L is a complete de Morgan algebra. This definition does not rely on the structure of basis lattice L and no distributivity in L is required. It can also be characterized by semiclosed L-sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented. Moreover, we also will introduce the notions of countable semicompactness and the semi-Lindelöf property and research their properties.

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ is a complete de Morgan algebra, X a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

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An element a in L is called prime element if $a \geq b \wedge c$ implies that $a \geq b$ or $a \geq c$. a in L is called a coprime element if a' is a prime element [5]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero coprime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive de Morgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [7, 13], in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [9]:

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}, \qquad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}. \tag{2.1}$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its quasicomplementation is called a closed L-set.

Definition 2.1 (see [7, 13]). For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all the lower semicontinuous maps from (X, τ) to L, that is, $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X, in this case, $(X, \omega_L(\tau))$ is topologically generated by (X, τ).

Definition 2.2 (see [7, 13]). An L-space $(X, \mathcal{T})$ is weak induced if for all $a \in L$, for all $A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X, \omega_L(\tau))$ is weak induced.

Lemma 2.3 (see [11]). *Let $(X, \mathcal{T})$ be a weakly induced L-space, $a \in L$, $A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.*

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ. $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.4 (see [10, 12]). Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$ is called (countably) compact if for every (countably) family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \tag{2.2}$$

Definition 2.5 (see [10]). Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right). \tag{2.3}$$

Lemma 2.6 (see [10]). *Let L be a complete Heyting algebra, let $f : X \to Y$ be a map, $f_L^{\to} : L^X \to L^Y$ is the extension of f, then for any family $\mathcal{P} \subseteq L^Y$,*

$$\bigvee_{y\in Y}\left(f_L^{\to}(G)(y)\wedge\bigwedge_{B\in\mathcal{P}}B(y)\right)=\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}f_L^{\leftarrow}(B)(x)\right). \tag{2.4}$$

Definition 2.7 (see [1]). An L-set G in an L-space $(X,\mathcal{T})$ is called semiopen if there exists $A\in\mathcal{T}$ such that $A\le G\le \mathrm{cl}(A)$. G is called semiclosed if G' is semiopen.

Definition 2.8. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-spaces. A map $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ is called

(1) semicontinuous [1] if $f_L^{\leftarrow}(G)$ is semiopen in $(X,\mathcal{T}_1)$ for every open L-set G in $(Y,\mathcal{T}_2)$;
(2) irresolute [2] if $f_L^{\leftarrow}(G)$ is semiopen in $(X,!\mathcal{T}_1)$ for every semiopen L-set G in $(Y,\mathcal{T}_2)$.

3. Definition and characterizations of semicompactness

Definition 3.1. Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ is called (countably) semicompact if for every (countable) family $\mathcal{U}$ of semiopen L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\right). \tag{3.1}$$

Definition 3.2. Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ is said to have the semi-Lindelöf property (or be a semi-Lindelöf L-set) if for every family $\mathcal{U}$ of semiopen L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)\le\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\right). \tag{3.2}$$

Example 3.3. Let X be any nonempty set and let A be a $[0,1]$-set on X defined as $A(x)=0.5$, for all $x\in X$. Let $\mathcal{T}=\{\varnothing,X,A\}$. Then the set of all semiopen $[0,1]$-sets in $(X,\mathcal{T})$ is $\mathcal{T}$. In this case, any $[0,1]$-set in $(X,\mathcal{T})$ is semicompact, hence it is countably semicompact and has the semi-Lindelöf property.

Obviously, we have the following theorem.

Theorem 3.4. *Semicompactness implies countably semicompactness and the semi-Lindelöf property. Moreover, an L-set having the semi-Lindelöf property is semicompact if and only if it is countably semicompact.*

Since an open L-set must be semiopen, we have the following theorem.

Theorem 3.5. *Semicompactness implies compactness, countably semicompactness implies countably compactness, and the semi-Lindelöf property implies the Lindelöf property.*

From Definitions 3.1 and 3.2, we can obtain the following two theorems by using quasicomplementation.

Theorem 3.6. *Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ is (countably) semicompact if and only if for every (countable) family $\mathcal{B}$ of semiclosed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{B}}B(x)\right)\geq\bigwedge_{\mathcal{F}\in 2^{(\mathcal{B})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right). \tag{3.3}$$

Theorem 3.7. *Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ has the semi-Lindelöf property if and only if for every family $\mathcal{B}$ of semiclosed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{B}}B(x)\right)\geq\bigwedge_{\mathcal{F}\in 2^{[\mathcal{B}]}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right). \tag{3.4}$$

In order to present characterizations of semicompactness, countable semicompactness and the semi-Lindelöf property, we generalize the notions of a–shading and a–R-neighborhood family in [10, 12] as follows.

Definition 3.8. Let $(X,\mathcal{T})$ be an L-space, $a\in L\backslash\{1\}$, and $G\in L^X$. A family $\mathcal{A}\subseteq L^X$ is said to be

(1) an a–shading of G if for any $x\in X$, $(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x))\nleq a$;
(2) a strong a–shading of G if $\bigwedge_{x\in X}(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x))\nleq a$;
(3) an a–R-neighborhood family of G if for any $x\in X$, $(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x))\ngeq a$;
(4) a strong a–R-neighborhood family of G if $\bigvee_{x\in X}(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x))\ngeq a$.

It is obvious that a strong a–shading of G is an a–shading of G, a strong a–R-neighborhood family of G is an a–R-neighborhood family of G, and $\mathcal{P}$ is a strong a–R-neighborhood family of G if and only if $\mathcal{P}'$ is a strong a–shading of G.

Definition 3.9. Let $a\in L\backslash\{0\}$ and $G\in L^X$. A subfamily $\mathcal{A}$ of L^X is said to have weak a–nonempty intersection in G if $\bigvee_{x\in X}(G(x)\wedge\bigwedge_{A\in\mathcal{A}}A(x))\geq a$. $\mathcal{A}$ is said to have the finite (countable) weak a–intersection property in G if every finite (countable) subfamily $\mathcal{F}$ of $\mathcal{A}$ has weak a–nonempty intersection in G.

From Definitions 3.1, 3.2, Theorems 3.5 and 3.6, we immediately obtain the following two results.

Theorem 3.10. *Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. Then the following conditions are equivalent.*

(1) *G is (countably) semicompact.*
(2) *For any $a\in L\backslash\{1\}$, each (countable) semiopen strong a–shading $\mathcal{U}$ of G has a finite subfamily which is a strong a–shading of G.*
(3) *For any $a\in L\backslash\{0\}$, each (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a–R-neighborhood family of G.*
(4) *For any $a\in L\backslash\{0\}$, each (countable) family of semiclosed L-sets which has the finite weak a–intersection property in G has weak a–nonempty intersection in G.*

Theorem 3.11. *Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G has the semi-Lindelöf property.*

(2) *For any $a \in L\backslash\{1\}$, each semiopen strong a–shading $\mathcal{U}$ of G has a countable subfamily which is a strong a–shading of G.*

(3) *For any $a \in L\backslash\{0\}$, each semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a countable subfamily which is a strong a–R-neighborhood family of G.*

(4) *For any $a \in L\backslash\{0\}$, each family of semiclosed L-sets which has the countable weak a–intersection property in G has weak a–nonempty intersection in G.*

4. Properties of (countable) semicompactness

Theorem 4.1. *Let L be a complete Heyting algebra. If both G and H are (countably) semicompact, then $G \vee H$ is (countably) semicompact.*

Proof. For any (countable) family $\mathcal{P}$ of semiclosed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
&\quad=\left\{\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\\
&\quad\geq\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\vee\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\\
&\quad=\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right).
\end{aligned}\tag{4.1}$$

This shows that $G \vee H$ is (countably) semicompact. □

Analogously, we have the following result.

Theorem 4.2. *Let L be a complete Heyting algebra. If both G and H have the semi-Lindelöf property, then $G \vee H$ has the semi-Lindelöf property.*

Theorem 4.3. *If G is (countably) semicompact and H is semiclosed, then $G \wedge H$ is (countably) semicompact.*

Proof. For any (countable) family $\mathcal{P}$ of semiclosed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\wedge H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
&\quad=\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}\bigcup\{H\}}B(x)\right)\\
&\quad\geq\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P}\cup\{H\})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{ \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \Big) \Big\} \wedge \Big\{ \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \Big(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \Big) \Big\} \\
&= \Big\{ \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \Big(G(x) \wedge H(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \Big) \Big\} \\
&= \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \Big((G \wedge H)(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \Big).
\end{aligned}
\tag{4.2}
$$

This shows that $G \wedge H$ is (countably) semicompact. □

Analogously, we have the following result.

Theorem 4.4. *If G has the semi-Lindelöf property and H is semiclosed, then $G \wedge H$ has the semi-Lindelöf property.*

Theorem 4.5. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an irresolute map. If G is a semicompact (resp., countably semicompact, semi-Lindelöf) L-set in $(X, \mathcal{T}_1)$, then so is $f_L^{\to}(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. We only prove that the theorem is true for semicompactness. Suppose that $\mathcal{P}$ is a family of semiclosed L-sets in $(Y, \mathcal{T}_2)$, by Lemma 2.6 and semicompactness of G, we have that

$$
\begin{aligned}
&\bigvee_{y\in Y} \Big(f_L^{\to}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y) \Big) \\
&\qquad = \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x) \Big) \\
&\qquad \geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \Big(G(x) \wedge \bigwedge_{B\in\mathcal{F}} f_L^{\leftarrow}(B)(x) \Big) \\
&\qquad = \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{y\in Y} \Big(f_L^{\to}(G)(y) \wedge \bigwedge_{B\in\mathcal{F}} B(y) \Big).
\end{aligned}
\tag{4.3}
$$

Therefore $f_L^{\to}(G)$ is semicompact. □

Analogously, we have the following result.

Theorem 4.6. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a semi-continuous map. If G is a semicompact (resp., countably semicompact, semi-Lindelöf) L-set in $(X, \mathcal{T}_1)$, then $f_L^{\to}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y, \mathcal{T}_2)$.*

Definition 4.7. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called strongly irresolute if $f_L^{\leftarrow}(G)$ is open in $(X, \mathcal{T}_1)$ for every semiopen L-set G in $(Y, \mathcal{T}_2)$.

It is obvious that a strongly irresolute map is irresolute.

Analogously, we have the following result.

THEOREM 4.8. *Let L be a complete Heyting algebra and let $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ be a strongly irresolute map. If G is a compact (resp., countably compact, Lindelöf) L-set in $(X,\mathcal{T}_1)$, then $f_L^{\to}(G)$ is a semicompact (countably semicompact, semi-Lindelöf) L-set in $(Y,\mathcal{T}_2)$.*

5. Further characterizations of semicompactness and goodness

In this section, we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of β_a–open cover and Q_a–open cover [10] as follows.

Definition 5.1. Let $(X,\mathcal{T})$ be an L-space, $a\in L\backslash\{0\}$, and $G\in L^X$. A family $\mathcal{U}\subseteq L^X$ is called a β_a–cover of G if for any $x\in X$, it follows that $a\in\beta(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x))$. $\mathcal{U}$ is called a strong β_a–cover of G if $a\in\beta(\bigwedge_{x\in X}(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)))$.

It is obvious that a strong β_a–cover of G must be a β_a–cover of G.

Definition 5.2. Let $(X,\mathcal{T})$ be an L-space, $a\in L\backslash\{0\}$, and $G\in L^X$. A family $\mathcal{U}\subseteq L^X$ is called a Q_a–cover of G if for any $x\in X$, it follows that $G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\geq a$.

It is obvious that a β_a–cover of G must be a Q_a–cover of G.

Analogous to the proof of [10, Theorem 2.9], we can obtain the following theorem.

THEOREM 5.3. *Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. Then the following conditions are equivalent.*

(1) *G is (countably) semicompact.*

(2) *For any $a\in L\backslash\{0\}$, each (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a–R-neighborhood family of G.*

(3) *For any $a\in L\backslash\{0\}$, each (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a finite subfamily which is an a–R-neighborhood family of G.*

(4) *For any $a\in L\backslash\{0\}$ and any (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b\in\beta(a)$ such that $\mathcal{F}$ is a strong b–R-neighborhood family of G.*

(5) *For any $a\in L\backslash\{0\}$ and any (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b\in\beta(a)$ such that $\mathcal{F}$ is a b–R-neighborhood family of G.*

(6) *For any $a\in M(L)$, each (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a finite subfamily which is a strong a–R-neighborhood family of G.*

(7) *For any $a\in M(L)$, each (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G has a finite subfamily which is an a–R-neighborhood family of G.*

(8) *For any $a\in M(L)$ and any (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b\in\beta^*(a)$ such that $\mathcal{F}$ is a strong b–R-neighborhood family of G.*

(9) *For any $a\in M(L)$ and any (countable) semiclosed strong a–R-neighborhood family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b\in\beta^*(a)$ such that $\mathcal{F}$ is a b–R-neighborhood family of G.*

(10) *For any $a\in L\backslash\{1\}$, each (countable) semiopen strong a–shading $\mathcal{U}$ of G has a finite subfamily which is a strong a–shading of G.*

(11) *For any $a \in L\backslash\{1\}$, each (countable) semiopen strong a–shading $\mathcal{U}$ of G has a finite subfamily which is an a–shading of G.*

(12) *For any $a \in L\backslash\{1\}$ and any (countable) semiopen strong a–shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a strong b–shading of G.*

(13) *For any $a \in L\backslash\{1\}$ and any (countable) semiopen strong a–shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a b–shading of G.*

(14) *For any $a \in P(L)$, each (countable) semiopen strong a–shading $\mathcal{U}$ of G has a finite subfamily which is a strong a–shading of G.*

(15) *For any $a \in P(L)$, each (countable) semiopen strong a–shading $\mathcal{U}$ of G has a finite subfamily which is an a–shading of G.*

(16) *For any $a \in P(L)$ and any (countable) semiopen strong a–shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a strong b–shading of G.*

(17) *For any $a \in P(L)$ and any (countable) semiopen strong a–shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a b–shading of G.*

(18) *For any $a \in L\backslash\{0\}$, each (countable) semiopen strong β_a–cover $\mathcal{U}$ of G has a finite subfamily which is a strong β_a–cover of G.*

(19) *For any $a \in L\backslash\{0\}$, each (countable) semiopen strong β_a–cover $\mathcal{U}$ of G has a finite subfamily which is a β_a–cover of G.*

(20) *For any $a \in L\backslash\{0\}$ and any (countable) semiopen strong β_a–cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a strong β_b–cover of G.*

(21) *For any $a \in L\backslash\{0\}$ and any (countable) semiopen strong β_a–cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a β_b–cover of G.*

(22) *For any $a \in M(L)$, each (countable) semiopen strong β_a–cover $\mathcal{U}$ of G has a finite subfamily which is a strong β_a–cover of G.*

(23) *For any $a \in M(L)$, each (countable) semiopen strong β_a–cover $\mathcal{U}$ of G has a finite subfamily which is a β_a–cover of G.*

(24) *For any $a \in M(L)$ and any (countable) semiopen strong β_a–cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a strong β_b–cover of G.*

(25) *For any $a \in M(L)$ and any (countable) semiopen strong β_a–cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a β_b–cover of G.*

(26) *For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each (countable) semiopen Q_a–cover of G has a finite subfamily which is a Q_b–cover of G.*

(27) *For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each (countable) semiopen Q_a–cover of G has a finite subfamily which is a β_b–cover of G.*

(28) *For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each (countable) semiopen Q_a–cover of G has a finite subfamily which is a strong β_b–cover of G.*

(29) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each (countable) semiopen Q_a–cover of G has a finite subfamily which is a Q_b–cover of G.*

(30) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each semiopen Q_a–cover of G has a finite subfamily which is a β_b–cover of G.*

(31) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each (countable) semiopen Q_a–cover of G has a finite subfamily which is a strong β_b–cover of G.*

Analogously, we also can present characterizations of the semi-Lindelöf property.

Lemma 5.4. *Let $(X,\omega(\tau))$ be generated topologically by (X,τ). If A is a semiopen L-set in (X,τ), then χ_A is a semiopen set in $(X,\omega(\tau))$. If B is a semiopen L-set in $(X,\omega(\tau))$, then $B_{(a)}$ is a semiopen set in (X,τ) for every $a \in L$.*

Proof. If A is a semiopen set in (X,τ), then there exists $D \in \tau$ such that $D \subseteq A \subseteq \mathrm{cl}(D)$. Thus we have that

$$\chi_D \le \chi_A \le \chi_{\mathrm{cl}(D)} = \mathrm{cl}\left(\chi_D\right). \tag{5.1}$$

This shows that χ_A is semiopen.

If B is a semiopen L-set in $(X,\omega(\tau))$, then there exists $E \in \omega(\tau)$ such that $E \le B \le \mathrm{cl}(E)$. Thus we have that $E_{(a)} \subseteq B_{(a)} \subseteq \mathrm{cl}(E)_{(a)}$. From [9], we can obtain that $\mathrm{cl}(E)_{(a)} \subseteq \mathrm{cl}(E_{(a)})$. Hence by Lemma 2.3, we know that $B_{(a)}$ is a semiopen set in (X,τ). □

The following two theorems show that semicompactness, countable semicompactness and the semi-Lindelöf property are good extensions.

Theorem 5.5. *Let (X,τ) be a topological space and let $(X,\omega(\tau))$ be generated topologically by (X,τ). Then $(X,\omega(\tau))$ is (countably) semicompact if and only if (X,τ) is (countably) semicompact.*

Proof. Necessity. Let $\mathcal{A}$ be a (countable) semiopen cover of (X,τ). Then $\{\chi_A \mid A \in \mathcal{A}\}$ is a family of semiopen L-sets in $(X,\omega(\tau))$ with $\bigwedge_{x\in X}(\bigvee_{A\in\mathcal{U}}\chi_A(x)) = 1$. From (countable) semicompactness of $(X,\omega(\tau))$, we know that

$$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(\bigvee_{A\in\mathcal{V}} \chi_A(x)\right) = \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(\bigvee_{A\in\mathcal{V}} \chi_A(x)\right) = 1. \tag{5.2}$$

This implies that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\bigwedge_{x\in X}(\bigvee_{A\in\mathcal{V}}\chi_A(x)) = 1$. Hence, $\mathcal{V}$ is a cover of (X,τ). Therefore (X,τ) is (countably) semicompact.

Sufficiency. Let $\mathcal{U}$ be a (countable) family of semiopen L-sets in $(X,\omega(\tau))$ and let $\bigwedge_{x\in X}(\bigvee_{B\in\mathcal{U}}B(x)) = a$. If $a = 0$, then obviously we have that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) \le \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} B(x)\right). \tag{5.3}$$

Now we suppose that $a \ne 0$. In this case, for any $b \in \beta(a)\backslash\{0\}$, we have that

$$b \in \beta\left(\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right)\right) \subseteq \bigcap_{x\in X}\beta\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = \bigcap_{x\in X}\bigcup_{B\in\mathcal{U}}\beta(B(x)). \tag{5.4}$$

From Lemma 5.4, this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is a semiopen cover of (X,τ). From (countable) semicompactness of (X,τ), we know that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\{B_{(b)} \mid B \in \mathcal{V}\}$ is a cover of (X,τ). Hence $b \le \bigwedge_{x\in X}(\bigvee_{B\in\mathcal{V}}B(x))$. Further, we have that

$$b \le \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right) \le \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right). \tag{5.5}$$

This implies that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}}B(x)\right)=a=\bigvee\{b\mid b\in\beta(a)\}\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}}B(x)\right).\tag{5.6}$$

Therefore, $(X,\omega(\tau))$ is (countably) semicompact. □

Analogously, we have the following result.

Theorem 5.6. *Let (X,τ) be a topological space and let $(X,\omega(\tau))$ be generated topologically by (X,τ). Then $(X,\omega(\tau))$ has the semi-Lindelöf property if and only if (X,τ) has the semi-Lindelöf property.*

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