SWITCHINGS, REALIZATIONS, AND INTERPOLATION THEOREMS FOR GRAPH PARAMETERS

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Interpolation theorems on several graph parameters obtained in the past few years will be reviewed in this paper. Some simplified proofs are provided. Open problems in this direction are reviewed.

1. Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [4]. Let $G = (V,E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We will use the following notation and terminology for a typical graph G. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$. We use $|S|$ to denote the cardinality of a set S and therefore we define $n = |V|$ the *order* of G and $m = |E|$ the *size* of G. To simplify writing, we write $e = uv$ for the edge e that joins the vertex u to the vertex v. The *degree* of a vertex v of a graph G is defined as $d_G(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|$. The maximum degree of a graph G is usually denoted by $\Delta(G)$. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by S in G. For a graph G and $X \subseteq E(G)$, we denote by $G - X$ the graph obtained from G by removing all edges in X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G and $X \subseteq V(G)$, the graph $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G and $X \subseteq E(\overline{G})$, we denote by $G + X$ the graph obtained from G by adding all edges in X. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs G and H, we define $G \cup H$ (their union) by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We can extend this definition to a finite union of pairwise disjoint graphs, since the operation "$\cup$" is associative. For a positive integer p and a graph G, pG is denoted for the union of p copies of G. A graph G is said to be *r-regular* if all of its vertices have degree r. A 3-regular graph is called a *cubic graph*.

Let G be a graph of order n and let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G. The sequence $(d_G(v_1), d_G(v_2), \dots, d_G(v_n))$ is called a *degree sequence* of G. A sequence $\mathbf{d} = (d_1, d_2, \dots, d_n)$ of nonnegative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of $\mathbf{d}$.

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An algorithm for determining whether or not a given sequence of nonnegative integers is graphic was independently obtained by Havel [22] and Hakimi [14]. We state their results in the following theorem.

Theorem 1.1. *Let* $\mathbf{d} = (d_1, d_2, \dots, d_n)$ *be a nonincreasing sequence of nonnegative integers and denote the sequence*

$$(d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n) = \mathbf{d}'. \tag{1.1}$$

Then $\mathbf{d}$ *is graphic if and only if* $\mathbf{d}'$ *is graphic.*

Let G be a graph and let $ab, cd \in E(G)$ be independent, where $ac, bd \notin E(G)$. Put

$$G^{\sigma(a,b;c,d)} = \big(G - \{ab, cd\}\big) + \{ac, bd\}. \tag{1.2}$$

The operation $\sigma(a,b;c,d)$ is called a *switching operation*. It is easy to see that the graph obtained from G by a switching has the same degree sequence as G. The following theorem has been shown by Havel [22] and Hakimi [14].

Theorem 1.2. *Let* $\mathbf{d} = (d_1, d_2, \dots, d_n)$ *be a graphic degree sequence. If* G_1 *and* G_2 *are any two realizations of* $\mathbf{d}$*, then* G_2 *can be obtained from* G_1 *by a finite sequence of switchings.*

As a consequence of Theorem 1.2, Eggleton and Holton [10] defined in 1979 *the graph* $\mathcal{R}(\mathbf{d})$ of realizations of $\mathbf{d}$ whose vertices are the graphs with degree sequence $\mathbf{d}$; two vertices being adjacent in the graph $\mathcal{R}(\mathbf{d})$ if one can be obtained from the other by a switching. They obtained the following theorem.

Theorem 1.3. *The graph* $\mathcal{R}(\mathbf{d})$ *is connected.*

The following theorem was shown by Taylor [38] in 1980.

Theorem 1.4. *For a graphic degree sequence* $\mathbf{d}$*, let* $\mathcal{CR}(\mathbf{d})$ *be the set of all connected realizations of* $\mathbf{d}$*. Then the induced subgraph* $\mathcal{CR}(\mathbf{d})$ *of* $\mathcal{R}(\mathbf{d})$ *is connected.*

Let $\mathbb{G}$ be the class of all simple graphs, a function $f : \mathbb{G} \to \mathbb{Z}$ is called a *graph parameter* if $G \cong H$, then $f(G) = f(H)$. If f is a graph parameter and $\mathbb{J} \subseteq \mathbb{G}$, f is called an *interpolation graph parameter with respect to* $\mathbb{J}$ if there exist integers x and y such that

$$\{f(G) : G \in \mathbb{J}\} = [x, y] = \{k \in \mathbb{Z} : x \le k \le y\}. \tag{1.3}$$

If f is an interpolation graph parameter with respect to $\mathbb{J}$, $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f, \mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f, \mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$. In the case where $\mathbb{J} = \mathcal{R}(\mathbf{d})$, we simply write $\min(f, \mathbf{d})$ and $\max(f, \mathbf{d})$ for $\min(f, \mathcal{R}(\mathbf{d}))$ and $\max(f, \mathcal{R}(\mathbf{d}))$, respectively, and in the case where $\mathbb{J} = \mathcal{CR}(\mathbf{d})$, we write $\operatorname{Min}(f, \mathbf{d})$ and $\operatorname{Max}(f, \mathbf{d})$ for $\min(f, \mathcal{CR}(\mathbf{d}))$ and $\max(f, \mathcal{CR}(\mathbf{d}))$, respectively.

2. Interpolation theorems

Studying interpolation theorems for graph parameters may be divided into two parts: the first part deals with the question that given a graph parameter f and a subset $\mathbb{J}$ of $\mathbb{G}$, does

f interpolate over $\mathbb{J}$? If f interpolates over $\mathbb{J}$, then $\{f(G) : G \in \mathbb{J}\}$ is uniquely determined by $\min(f,\mathbb{J}) = \min\{f(G) : G \in \mathbb{J}\}$ and $\max(f,\mathbb{J}) = \max\{f(G) : G \in \mathbb{J}\}$; thus the second part of the interpolation theorems for graph parameters is to find the values of $\min(f,\mathbf{d})$ and $\max(f,\mathbf{d})$ for the corresponding interpolation graph parameters and this part is, in fact, the extremal problem in graph theory.

The interest in the interpolation properties of graph parameters was motivated by an open question posed by Chartrand during a conference held at Kalamazoo in 1980. He posed the following question: If a graph G contains spanning trees having m and n end-vertices, with $m < n$, does G contain a spanning tree with k end-vertices for every integer k with $m < k < n$? This question (which was answered affirmatively) led to a host of papers studying the interpolation properties of invariants of spanning trees of a given graph. Details can be found in [15, 16, 17, 18, 19, 20, 21]. Another interpolation theorem for the graph parameter $\chi + \overline{\chi}$ was obtained by Fink [12].

Let $\mathbf{d}$ be a graphic degree sequence. It was shown that the graph $\mathcal{R}(\mathbf{d})$ is connected and its subgraph induced by the set $\mathcal{CR}(\mathbf{d})$ is also connected.

Theorem 2.1. *For a graph G of degree sequence $\mathbf{d}$ and a switching σ if $|f(G) - f(G^\sigma)| \le 1$, then f is an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$.*

Theorem 2.2. *For a graph G of degree sequence $\mathbf{d}$ and a switching σ if $|f(G) - f(G^\sigma)| \le 1$, then f is an interpolation graph parameter with respect to $\mathcal{CR}(\mathbf{d})$.*

Theorem 2.3. *Let $\mathbb{J} \subseteq \mathcal{R}(\mathbf{d})$ and the subgraph of $\mathcal{R}(\mathbf{d})$ induced by $\mathbb{J}$ be connected. For a graph G of degree sequence $\mathbf{d}$ and a switching σ if $|f(G) - f(G^\sigma)| \le 1$, then f is an interpolation graph parameter with respect to $\mathbb{J}$.*

We will now review interpolation results on various graph parameters with respect to $\mathcal{R}(\mathbf{d})$. Since our work started with the graph parameter χ, we first state the definition of χ.

Graph coloring takes its name from the map-coloring application. We assign labels to vertices when the numerical value of labels is unimportant, we call them *colors* to indicate that they may be elements of any set.

A *k-coloring* of a graph $G = (V,E)$ is a partition of its vertex set V as $V_1 \cup V_2 \cup \cdots \cup V_k$ such that no two vertices in V_i $(1 \le i \le k)$ are adjacent. The V_i's are called the *color classes*. A function $f : V \to \{1,2,\ldots,k\}$ such that $f(v) = i$ for each $v \in V_i$ $(1 \le i \le k)$ is called a *color function*. If G has a k-coloring, it is said to be *k-colorable* and the minimum integer k for which G is k-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that G is *k-chromatic*.

Remark 2.4. In a proper coloring, each color class contains no edge, so G is k-colorable if and only if G is a k-partite graph. Thus a graph is 2-colorable if and only if it is bipartite. Thus a graph containing an odd cycle must be at least 3-colorable.

We proved in [25] the following results.

Theorem 2.5. *Let G be a graph and let σ be a switching on G. Then $|\chi(G) - \chi(G^\sigma)| \le 1$.*

Proof. Without loss of generality, we can assume that $\chi(G) < \chi(G^\sigma)$. Since G^σ is the graph obtained from G by a switching, there exist $a,b,c,d \in V(G)$ such that $ab, cd \in E(G)$,

$ac, bd \notin E(G)$, $ac, bd \in E(G^\sigma)$, and $ab, cd \notin E(G^\sigma)$. Let $\pi : V(G) \to \{1,2,\dots,\chi(G)\}$ be a color function of G such that $\pi(a) = i$, $\pi(b) = j$, $\pi(c) = p$, and $\pi(d) = q$. Then we will define $\pi' : V(G^\sigma) \to \{1,2,\dots,\chi(G),\chi(G)+1\}$ by $\pi'(c) = \pi'(d) = \chi(G)+1$ and $\pi'(v) = \pi(v)$ for all $v \in V(G^\sigma) - \{c,d\}$. Thus the graph G^σ is $(\chi(G)+1)$-colorable. This completes the proof. □

A maximal complete subgraph of a graph G is called a *clique* of G. The maximum order of clique of G is called the *clique number* of G and is denoted by $\omega(G)$.

In general, there is no formula for the chromatic number of a graph. In fact, determining the chromatic number of even a relatively small graph is often a challenging problem. However, lower bounds for the chromatic number of a graph G can be given in terms of the clique number of G. That is for any graph G, $\chi(G) \geq \omega(G)$.

We proved in [26]the following results on interpolation theorems on ω.

Theorem 2.6. *Let G be a graph and let σ be a switching on G. Then $|\omega(G) - \omega(G^\sigma)| \leq 1$.*

Proof. Let $\sigma(a,b;c,d) = \sigma$ be a switching on G, such that $ab, cd \in E(G)$, $ac, bd \notin E(G)$. Then G^σ is the graph obtained from G by deleting edges ab, cd and adding edges ac, bd. Since vertices a and c cannot lie in the same complete subgraph of G, $\omega(G^\sigma) \geq \omega(G) - 1$. By symmetry of switching, we may assume that $\omega(G) \geq \omega(G^\sigma)$. Thus $\omega(G^\sigma)$ is either $\omega(G) - 1$ or $\omega(G)$. In both cases, we have $|\omega(G) - \omega(G^\sigma)| \leq 1$. □

Acyclic graph is a graph containing no cycle as its subgraph. An acyclic graph is called a forest. Therefore, each component of an acyclic graph is a tree. Since a tree is connected, every two vertices in a tree are connected by a unique path.

Let G be a graph and $F \subseteq V(G)$, F is called an *induced forest* of G, if the induced subgraph $G[F]$ of G contains no cycle. For a graph G, we define $I(G)$ as

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}. \tag{2.1}$$

We proved in [28] the following results on interpolation theorems on the graph parameter I.

Theorem 2.7. *If S is any subset of vertices of G such that $G[S]$ is a forest, and σ is any switching on G, then $G^\sigma[S]$ contains at most one cycle.*

Corollary 2.8. *If σ is any switching on G, then $|I(G) - I(G^\sigma)| \leq 1$.*

The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is difficult even for some simply defined graphs. For a graph G, this minimum number is known as the *decycling number* of G, and is denoted by $\phi(G)$. It is easy to see that for a graph G of order n, $\phi(G) + I(G) = n$. Thus the interpolation result for ϕ in $\mathcal{R}(\mathbf{d})$ is easily obtained.

A subset U of the vertex set V of a graph $G = (V,E)$ is said to be an *independent set* of G if the induced subgraph $G[U]$ of G is an empty graph. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices of a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha_0(G)$.

It is clear that $\alpha_0(G)$ is a graph parameter and $\alpha_0(G) = \omega(\overline{G})$, for any graph G. Observe that for a graph G and a switching σ on G, $\overline{G}^{\sigma} = \overline{G^{\sigma}}$. Thus the interpolation result for α_0 in $\mathcal{R}(\mathbf{d})$ follows directly from the graph parameter ω.

A vertex of a graph $G = (V, E)$ is said to cover the edges incident with it. A *vertex cover* of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the *vertex covering number* of G and is denoted by $\beta_0(G)$.

Within a set of people, some pairs are compatible as roommates; under what condition can we pair them all up? Many applications of graphs involve such pairings. Problem of job assignment with qualified applicants is also an example of matching.

A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have a common vertex. A matching M is *maximum* in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of a maximum matching of G is denoted by $\alpha_1(G)$ and is called the *matching number* of G.

There is an analogous covering concept for edges.

An edge of a graph $G = (V, E)$ is said to cover the two vertices incident with it. An *edge cover* of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of G is called the *edge covering number* of G and is denoted by $\beta_1(G)$.

Theorem 2.9 [31]. *If G is a graph with $\alpha_1(G) = \alpha_1$ and σ is a switching on G, then $\alpha_1(G^{\sigma}) \geq \alpha_1 - 1$.*

Proof. Let M be an independent set of edges of E with $|M| = \alpha_1(G)$. Let $\sigma(a,b;c,d) = \sigma$ be a switching on G. If $\{ab, cd\} \cap M = \emptyset$, then $|M| = |M^{\sigma}|$. If $\{ab, cd\} \subseteq M$, then $|M| = |M^{\sigma}|$. Finally, if M contains exactly one edge from the set $\{ab, cd\}$, then $|M^{\sigma}| = |M| - 1$. Therefore, $\alpha_1(G^{\sigma}) \geq \alpha_1 - 1$. □

Corollary 2.10 [31]. *If σ is a switching on G, then $|\alpha_1(G) - \alpha_1(G^{\sigma})| \leq 1$.*

Let n be a positive integer. The *star* of order $n+1$ is the complete bipartite graph $K_{n,1}$. The edges covered by one vertex in a vertex cover are the edges incident to it; they form a star. The vertex cover problem can be described as covering the edge set with the fewest number of stars. This is equivalent to our next graph parameter.

A *dominating set* of a graph $G = (V, E)$ is a subset D of V such that each vertex of $V - D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the cardinality of a minimal dominating set with the least number of elements.

Theorem 2.11 [31]. *If G is a graph with $\gamma(G) = \gamma$ and σ is a switching on G, then $\gamma(G^{\sigma}) \leq \gamma + 1$.*

Proof. Let D be a dominating set of vertices of V with $|D| = \gamma(G)$. Let $\sigma(a,b;c,d) = \sigma$ be a switching on G. If $\{a,b,c,d\} \cap D = \emptyset$, then $|D| = |D^{\sigma}|$. If $\{a,b,c,d\} \subseteq D$, then $|D| = |D^{\sigma}|$. If $a \in D$ and $b,c,d \in V - D$, then $D \cup \{b\}$ is a dominating set of G^{σ}. If $a,b \in D$ and $c,d \in V - D$, then D is a dominating set of G^{σ}. If $a,c \in D$, then $D \cup \{b\}$ is a dominating set of G^{σ}. Finally, if $a,b,c \in D$ and $d \in V - D$, then $D \cup \{d\}$ is a dominating set of G^{σ}. Thus $\gamma(G^{\sigma}) \leq \gamma(G) + 1$. □

Corollary 2.12 [31]. *If σ is a switching on G, then $|\gamma(G) - \gamma(G^{\sigma})| \leq 1$.*

Proof. Since a switching is symmetric, we may assume that $\gamma(G) \le \gamma(G^\sigma)$. By Theorem 2.11, $\gamma(G^\sigma)$ is either $\gamma(G)+1$ or $\gamma(G)$. In both cases, we have $|\gamma(G)-\gamma(G^\sigma)| \le 1$. □

Gallai [13] and Norman-Rabin [23] proved the following results concerning relationship between α_0 and β_0, and α_1, and β_1, respectively.

Theorem 2.13 [13]. *For any graph G of order n, $\alpha_0+\beta_0 = n$.*

Theorem 2.14 [23]. *For any graph G of order n and $\delta \ge 1$, $\alpha_1+\beta_1 = n$.*

The interpolation property of the graph parameters β_0 and β_1 follows from the last two theorems. Thus combining the results in this section, we can conclude the following results.

Theorem 2.15. *Let $\mathbf{d} = (d_1,d_2,\dots,d_n)$, $d_1 \ge d_2 \ge \cdots \ge d_n \ge 1$ be a graphic degree sequence. Then χ, ω, I, ϕ, α_0, α_1, β_0, β_1, and γ are interpolation graph parameters with respect to $\mathcal{R}(\mathbf{d})$.*

Theorem 2.16. *Let $f \in \{\chi,\omega,I,\phi,\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(f)$ and $b := b(f)$ such that $\mathbf{d}$ has a realization G with $f(G) = c$ if and only if c is an integer satisfying $a \le c \le b$.*

3. Extremal graph parameters

In this section, we discuss the second part of interpolation theorems for graph parameters. We first recall what we mean by an extremal problem in graph theory. An *extremal problem* asks for minimum and maximum values of a function over a class of objects.

Remark 3.1. Proving that A is the minimum of $f(G)$ for graphs in a class $\mathbb{J}$ requires showing two things:

(1) $f(G) \ge A$ for all $G \in \mathbb{J}$;
(2) $f(G) = A$ for some $G \in \mathbb{J}$.

The proof of the bound must apply to every $G \in \mathbb{J}$. For equality it suffices to obtain an example in $\mathbb{J}$ with the desired value of f.

Changing "$\ge$" to "$\le$" yields the criteria for a maximum.

It will be seen that we have produced several results in the extremal graph theory for graph parameters χ, ω, I, ϕ, and α_1 in the class of all r-regular graphs of order n and some other related classes.

We start with the graph parameter χ. In the graph-theoretic colloquium at Smolenice in 1963, Dirac conjectured that the chromatic number of a proper regular subgraph of a complete n-gon is at most $3n/5$. Erdős and Gallai answered this conjecture immediately and presented their result during the conference. Their article was entitled "Solution to a problem of Dirac," and it was appeared in the proceedings of the symposium, Smolenice, in 1964. In fact, the result was more than of what Dirac asked. They have, in addition, found all regular graphs reaching the upper bound as stated in the following theorem.

Theorem 3.2. *An r-regular graph G of order $n > r+1$ has chromatic number $k \le 3n/5$, with equality holds if and only if the complementary graph $\overline{G}$ of G is a union of 5 cycles.*

In their proof, they used the concept of the *covering number* of a graph G. A *covering* of a graph G is a partition P of $V(G)$ such that for each $V_i \in P$, the induced subgraph $G[V_i]$ in G is a complete graph. The covering number of G, denoted by $c(G)$, is defined by

$$c(G) := \min\{|P| \mid P \text{ is a covering of G}\}. \tag{3.1}$$

It is easy to see that for any graph G, $c(G) = \chi(\bar{G})$. We extended their result by improving the upper bound in [24].

Let f be an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$. It is clear that $\{f(G) : G \in \mathcal{R}(\mathbf{d})\}$ is uniquely determined by $\min(f,\mathbf{d})$ and $\max(f,\mathbf{d})$. We will discuss in this section those extremal values for various kinds of graph parameters. For simplicity reason, we will consider the class of regular graphs and some other related classes of graphs.

For the graph parameter χ, we obtained in [25] the following results.

Theorem 3.3. *If $r \geq 2$ and $n \geq 2r$, then*

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 3 & \text{if } n \text{ is odd}. \end{cases} \tag{3.2}$$

Theorem 3.4. *If $r \geq 2$, then*

$$\begin{aligned} \min(\chi, r^{r+1}) &= \max(\chi, r^{r+1}) = r+1, \\ \min(\chi, r^{r+2}) &= \max(\chi, r^{r+2}) = \frac{r+2}{2}. \end{aligned} \tag{3.3}$$

The proofs of Theorems 3.3 and 3.4 were based on graph constructions.

Theorem 3.5. *For any $r \geq 4$ and odd integer s such that $3 \leq s \leq r$, let q and t be integers satisfying $r+s = sq+t, 0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases} \tag{3.4}$$

Theorem 3.6. *For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r+s = sq+t$, $0 \leq t < s$. Then*

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases} \tag{3.5}$$

The proofs of Theorems 3.5 and 3.6 were based on the concept of the covering number of graphs and graph constructions depending on the parity of s.

Theorem 3.7. *Let* $r \geq 2$. *Then*

(1)

$$\max(\chi, r^{2r}) = r, \tag{3.6}$$

(2)

$$\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases} \tag{3.7}$$

(3)

$$\max(\chi, r^{n}) = r + 1 \quad \text{for } n \geq 2r + 2. \tag{3.8}$$

Brooks' theorem [5] and graph constructions were applied to prove this theorem.

Theorem 3.8. *For any* r *and* s *such that* $3 \leq s \leq r - 1$,

(1)

$$\max(\chi, r^{r+s}) \geq \frac{r+s}{2} \quad \text{if } r + s \text{ is even}, \tag{3.9}$$

(2)

$$\max(\chi, r^{r+s}) \geq \frac{r+s-1}{2} \quad \text{if } r + s \text{ is odd}. \tag{3.10}$$

Purely graph constructions were used in the proof of this theorem. The exact values of $\max(\chi, r^n)$ are not easy to obtain when $r + 3 \leq n \leq 2r - 1$. Theorem 3.2 was proved by Erdős and Gallai [11] to provide an upper bound in the class of connected proper regular graphs of order n.

It should be noted that the bound given by Theorem 3.2 does not depend on r and this bound may be very far from the actual values of the chromatic numbers. Also the exact value of $\max(\chi, r^{r+3})$ was obtained by Theorem 3.2. We gave in [24] the following definition.

Let j be a positive integer. An *$F(j)$-graph* is a $(j-1)$-regular graph G of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$.

It is easy to see that $F(3)$-graph is a unique graph C_5 and $f(3) = 5$. We will see later that $F(j)$-graphs, $j \geq 5$, are not unique.

We found $F(j)$-graphs for all odd integers j as we state in the following theorem.

Theorem 3.9 [24]. *For odd integer* j *with* $j \geq 3$, $f(j) = (5/2)(j-1)$ *if* $j \equiv 3 (\text{mod}\, 4)$ *and* $f(j) = 1 + (5/2)(j-1)$ *if* $j \equiv 1 (\text{mod}\, 4)$.

Theorem 3.10 [24]. *Any* r*-regular graph of order* n *with* $n - r = j$ *odd and* $j \geq 3$ *has chromatic number at most* $((f(j)+1)/2f(j)) \cdot n$, *and this bound is achieved precisely for those graphs with complement equal to a disjoint union of* $F(j)$*-graphs.*

Note that our results in Theorems 3.9 and 3.10 can be considered as a generalization of Theorem 3.2 as $F(3)$-graph is C_5 and $f(3) = 5$.

Problem 3.11. Find an $F(j)$-graph for even integer $j \geq 4$.

Problem 3.12. Find the value of $\max(\chi, r^{r+j})$ for even integer j and $4 \leq j \leq r-2$.

For the graph parameter ω, we completely answered in [26] the second part of interpolation theorem with respect to the class of r-regular graphs of order n as stated in the following results.

THEOREM 3.13. *Let G be a graph of order n and maximum degree Δ. Then $\omega(G) \geq n/(n-\Delta)$.*

Since the complete graph K_n is a clique, $\min(\omega, r^{r+1}) = \max(\omega, r^{r+1}) = r+1$. Given positive integers n and k with $k \leq n$, there exists a connected graph G of order n with $\omega(G) = k$. However, if G is regular, we have the following remarkable result.

THEOREM 3.14. *Let $\mathbf{d} = r^n$ be a graphic degree sequence with $r+2 \leq n \leq 2r+1$. Then $\max(\omega, r^n) = \lfloor n/2 \rfloor$.*

THEOREM 3.15. *For any $r \geq 6$ and odd integer s such that $5 \leq s < r$, let q and t be integers satisfying $r+s = sq+t$, $0 \leq t < s$. Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases} \tag{3.11}$$

THEOREM 3.16. *For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r+s = sq+t$, $0 \leq t < s$. Then*

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases} \tag{3.12}$$

Observe that results of Theorems 3.15 and 3.16, respectively, coincide with Theorems 3.5 and 3.6 above.

For graph parameter I, we found in [27] a lower bound of $\min(I, \mathbf{d})$ by using *the probabilistic method*. In particular, we proved that if G is a graph with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots d_n \geq 1$, then

$$I(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i+1}. \tag{3.13}$$

As an application, we found all minimum values of the order of maximum induced forest in a (Δ, n)-graph.

First of all we would like to introduce basic tools from discrete probability theory which will be used for calculation of some bounds of graph parameters.

Let (Ω, p) be a finite *probability space*, where Ω is a finite set and p is a probability function that maps from Ω into the interval $[0,1]$ with

$$\sum_{\omega \in \Omega} p(\omega) = 1. \tag{3.14}$$

A *random variable* X on Ω is a mapping $X : \Omega \to \mathbb{R}$. We define a probability space on the image set $X(\Omega)$ as

$$p(X = x) := \sum_{X(\omega)=x} p(\omega). \tag{3.15}$$

The *expectation* $E(X)$ of X is

$$E(X) = \sum_{\omega\in\Omega} p(\omega)X(\omega). \tag{3.16}$$

Now suppose X and Y are two random variables on Ω, then the sum $X + Y$ is again a random variable, sharing a property $E(X + Y) = E(X) + E(Y)$.

Clearly, this can be extended to any finite linear combination of random variables (the linearity of expectation). Note that it is not necessary that the random variables are "independent."

Thus we have the following theorems.

Theorem 3.17 [27]. *Let G be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \dots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 1. \tag{3.17}$$

Then

$$I(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}. \tag{3.18}$$

Proof. Let G be an arbitrary graph on the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Denote the degree of v_i by d_i. We choose with equal probability $1/n!$ a random permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of V and construct the following set F_π. We put π_i into F_π if and only if π_i is adjacent to at most one π_j $(j < i)$. It is clear that F_π is an induced forest of G. Let $X(\pi) = |F_\pi|$ be the corresponding random variable. We have $X = \sum_{i=1}^{n} X_i$, where X_i is the indicator random variable of the vertex v_i, in other words $X_i = 1$ if $v_i \in F_\pi$ and $X_i = 0$ if $v_i \notin F_\pi$. In order to calculate $E(X_i)$, we first calculate the number of permutations π which contain the vertex v_i. Since v_i has d_i neighbors, we partition $n!$ permutations on V into $\binom{n}{d_i+1}$ classes according to $d_i + 1$ positions that v_i and its neighbors appeared. It is clear that each class contains $(d_i + 1)!(n - d_i - 1)!$ permutations and $2d_i!(n - d_i - 1)!$ of which have the property that $v_i \in F_\pi$. Thus

$$E(X_i) = \sum_{\pi} p(\pi)X_i(\pi) = \frac{2}{d_i + 1}. \tag{3.19}$$

Therefore,

$$E(X) = \sum_{i=1}^{n} E(X_i) = 2\sum_{i=1}^{n} \frac{1}{d_i + 1}. \tag{3.20}$$

□

Corollary 3.18. *If the average degree of G is at most d, then $I(G) \geq 2n/(d+1)$.*

We now have the following corollary as a result of Caro [7].

Corollary 3.19. *Let G be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \dots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 0. \tag{3.21}$$

Then

$$\alpha_0(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}, \tag{3.22}$$

where $\alpha_0(G)$ is the independence number of G.

Proof. The proof follows easily from the fact that a forest is bipartite. □

Corollary 3.20. *Let G be a graph with degree sequence*

$$\mathbf{d} = (d_1, d_2, \dots, d_n), \quad d_1 \geq d_2 \geq \cdots d_n \geq 0. \tag{3.23}$$

Then

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i}, \tag{3.24}$$

where $\omega(G)$ is the clique number of G.

Proof. The proof follows from the fact that $\omega(G) = \alpha_0(\overline{G})$. □

Let G be a graph. The problem of determining the decycling number $\phi(G)$ of G is equivalent to finding the greatest order of an induced forest $I(G)$ of the graph G and the sum of the two numbers equals the order of the graph. Observe that for a minimum decycling set S of a graph G, if $v \in S$, then there exists a connected component C of $G - S$ such that v is adjacent to at least two vertices of C. Thus $\Delta(G[S]) \leq \Delta(G) - 2$. With this observation, we find that if G is an r-regular graph and S is a minimum decycling set of G, the graph $G[S]$ may not be an $(r-2)$-regular graph. This causes a difficulty in finding $\max(\phi, r^n)$ if we consider only the class of regular graphs. It is reasonable to enlarge the class of regular graphs into the following class of graphs. Let Δ be a nonnegative integer and let n be a positive integer such that $n \geq \Delta + 1$. Let $\mathbb{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ. *The (Δ, n)-graph* is a graph having $\mathbb{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

Lemma 3.21 [27]. *The (Δ, n)-graph is connected.*

Proof. For any graph $G \in \mathbb{G}(\Delta, n)$, if $F = K_{1,\Delta} \cup (n - \Delta - 1)K_1$ and $G \neq F$, G can be obtained from F by a finite sequence of adding edges. The proof is complete. □

Theorem 3.22 [27]. *Let* $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ *be a graphic degree sequence and* $d_1 + 1 \leq n \leq 2d_1 + 1$. *Then*

(1) $\min(I, \mathbf{d}) = 2$ *if and only if* $d_1 = d_2 = d_3 = \cdots = d_n$ *and* $n = d_1 + 1$,

(2) *if* $\mathbf{d}$ *does not have a complete graph as its realization, then* $\min(I, \mathbf{d}) = 3$ *if and only if* $\overline{\mathbf{d}}$ *has a union of stars as its realization.*

Theorem 3.23 [27]. *Let* $n = (\Delta + 1)q + t$, $0 \leq t \leq \Delta$. *Then*

(1) $\min(I, \mathbb{G}(\Delta, n)) = 2q$ *if* $t = 0$,

(2) $\min(I, \mathbb{G}(\Delta, n)) = 2q + 1$ *if* $t = 1$,

(3) $\min(I, \mathbb{G}(\Delta, n)) = 2q + 2$ *if* $2 \leq t \leq \Delta$.

We obtained in [28] the values of $\max(I, r^n)$, for all r and n as stated in the following theorems.

Theorem 3.24.

$$\max(I, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \left\lfloor \dfrac{nr - 2}{2(r - 1)} \right\rfloor & \text{if } n \geq 2r. \end{cases} \tag{3.25}$$

The values of $\min(I, r^n)$, for all r and n, were obtained in [33] in terms of the graph parameter ϕ as stated in the following theorems. Note that $\min(I, r^n) + \max(\phi, r^n) = n$.

Theorem 3.25. *For* $r \geq 3$, *and* $n = r + j$, $1 \leq j \leq r + 1$,

(1) $\min(I, r^n) = 2$ *if and only if* $n = r + 1$,

(2) $\min(I, r^n) = 3$ *if and only if* $n = r + 2$,

(3) $\min(I, r^n) = 4$ *for all even integers* n, $r + 3 \leq n$,

(4) $\min(I, r^n) = 4$ *for all odd integers* n, $r + 3 \leq n$ *and* $n \geq f(j)$,

(5) $\min(I, r^n) = 5$ *for all odd integers* n, $r + 3 \leq n$ *and* $n < f(j)$,

where

$$\begin{aligned} f(j) &= \frac{5}{2}(j - 1) \quad \text{if } j \equiv 3 (\operatorname{mod} 4), \\ f(j) &= 1 + \frac{5}{2}(j - 1) \quad \text{if } j \equiv 1 (\operatorname{mod} 4). \end{aligned} \tag{3.26}$$

Theorem 3.26. *For* $n \geq 2r + 2$ *and* $r \geq 3$, *write* $n = (r + 1)q + t$, $q \geq 2$ *and* $0 \leq t \leq r$. *Then*

(1) $\min(I, r^n) = 2q$ *if* $t = 0$,

(2) $\min(I, r^n) = 2q + 1$ *if* $t = 1$,

(3) $\min(I, r^n) = 2q + 2$ *if* $2 \leq t \leq r - 1$,

(4) $\min(I, r^n) = 2q + 3$ *if* $t = r$.

In [32], we investigated the values of $\min(\alpha_1, r^n)$ and $\max(\alpha_1, r^n)$ for all r and n. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, from now on we will consider $r \geq 2$ and $n \geq r + 1$.

We first investigate the value of $\max(\alpha_1, r^n)$ by the following theorem.

Theorem 3.27. *For* $r \geq 2$, $n \geq r + 1$, *and* $nr \equiv 0 (\operatorname{mod} 2)$, *there exists an* r*-regular Hamiltonian graph of order* n. *In particular,* $\max(\alpha_1, r^n) = \lfloor n/2 \rfloor$.

Proof. Since there is a well-known result by Dirac (cited from [4]) that an r-regular graph of order n with $r \geq n/2$ is Hamiltonian, we need to consider only when $r < n/2$. It is easy to construct a Hamiltonian graph with 10 or fewer vertices. It is also easy to construct such a graph with $r = 2$ or 3. Let $X = \{x_0, x_1, \ldots, x_{t-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{t-1}\}$, where $t \geq 5$. For an integer r with $1 \leq r \leq t$, take the edge set

$$E = \{x_i y_{i+j} : i = 0,1,2,\ldots,t-1,\ j = 0,1,2,\ldots,r-1\}, \tag{3.27}$$

where all subscripts are taken mod t. It is clear that the graph $B(r;X,Y) = (X \cup Y, E)$ is an r-regular bipartite graph and it is Hamiltonian if $r \geq 2$. Suppose that $n = 2t+1$ and r is an even integer with $4 \leq r \leq t$, the graph $G = (X \cup Y \cup \{u\}, E')$, where $E' = [E(B(r-2;X,Y)) - \{x_i y_i : i = 0,1,2,\ldots,r/2\}] \cup \{x_i x_{i+1}, y_i y_{i+1} : i = 0,1,2,\ldots,t-1\} \cup \{ux_i, uy_i : i = 0,1,2,\ldots,r/2\}$, is an r-regular Hamiltonian graph. □

We will use the generalized result of Tutte and the result of Wallis [41] to obtain all values of $\min(\alpha_1, r^n)$.

A 1-*factor* of a graph G is a 1-regular spanning subgraph of G. A 1-*factorization* of G is a set of pairwise edge-disjoint 1-factors which together contain each edge of G.

It is well known that K_{2n} and $K_{n,n}$ have 1-factorizations for all positive integers n. The question of which graphs contain 1-factors is one that has attracted considerable attention. For a comprehensive review, we refer to the survey of Akiyama and Kano [1] and of Wallis in [9].

A necessary and sufficient condition for a graph to have a perfect matching was obtained by Tutte [40]. A component of a graph is *odd* or *even* according as it has an odd or even number of vertices. We denote by $O(G)$ the number of odd components of G. The following theorem is due to Tutte [40]

Theorem 3.28. *A graph G has a perfect matching if and only if*

$$O(G-S) \leq |S| \quad \forall S \subseteq V. \tag{3.28}$$

Berge [3] generalized Tutte's result and it makes easier for application.

Theorem 3.29 [3]. *The number of edges in a maximum matching of a graph G is $(1/2)(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)}\{O(G-S) - |S|\}$.*

Let $F(r,d)$ be the minimum order of an r-regular graph G with $\alpha_1(G) = (1/2)(|V(G)| - d)$. It is clear that $|V(G)| \equiv d \pmod 2$. Wallis [41] found $F(r,2)$ for all $r \geq 3$. In other words, he proved the following theorem.

Theorem 3.30. *Let G be an r-regular graph with no 1-factor and no odd component. Then*

$$|V(G)| \geq \begin{cases} 3r+7 & \text{if } r \text{ is odd},\ r \geq 3, \\ 3r+4 & \text{if } r \text{ is even},\ r \geq 6, \\ 22 & \text{if } r = 4. \end{cases} \tag{3.29}$$

Furthermore, no such graphs exist for $r = 1$ or 2.

Suppose that G is an r-regular graph with $\alpha_1(G) = (1/2)(|V(G)| - d)$. There exists a k-subset K of $V(G)$ such that $O(G-K) = k+d$. If $k = 0$, then r is even, G contains d odd components, and each component of G has order at least $r+1$. Suppose $k \geq 1$ and $G-K$ has an odd component with p vertices, where p is less than or equal to r. The number of edges within the component is at most $(1/2)p(p-1)$. This means that the sum of degree of these p vertices in $G-K$ is at most $p(p-1)$. But G is an r-regular graph, so the sum of these p vertices in G is pr. The number of edges joining to the component K must be at least $pr - p(p-1)$. For a fixed r and for integer p satisfying $1 \leq p \leq r$, the function $f(p) = pr - p(p-1)$, $1 \leq p \leq r$, has minimum value $f(1) = f(r) = r$. So any odd component with r or less vertices is joined to K by r or more edges.

Suppose that there are O_+ odd components of $G-K$ with more than r vertices and O_- odd components with less than or r vertices. Thus

$$O_+ + O_- = k+d, \qquad O_+ + rO_- \leq kr. \tag{3.30}$$

From these two relations, we have $O_+ \geq \lceil rd/(r-1) \rceil = d + \lceil d/(r-1) \rceil$ and $k \geq \lceil d/(r-1) \rceil$. Thus we have the following results.

Theorem 3.31 [32]. *Let r be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$.*

Corollary 3.32 [32]. *Let r be an even integer, $r \geq 2$. If $n = (r+1)d + e$, $0 \leq e \leq r$, then* $\min(\alpha_1, r^n) = dr/2 + \lfloor (1+e)/2 \rfloor$.

Suppose that r is odd and $r \geq 3$. Let G be an r-regular graph of order n such that $\alpha_1(G) = (1/2)(n-d)$. Then d must be even. Put $d = 2q$. There exists a nonempty subset K of $V(G)$ of cardinality k such that $O(G-K) = k+2q$. By (3.30), we have

$$n \geq k + (r+2)O_+ \geq \left\lceil \frac{2q}{r-1} \right\rceil + (r+2)\left(2q + \left\lceil \frac{2q}{r-1} \right\rceil\right) = \left\lceil \frac{2q}{r-1} \right\rceil (r+3) + 2q(r+2). \tag{3.31}$$

Wallis [41] defined $G(x,y)$ to be a graph with $x+y$ vertices, x being of degree $x+y-3$ and y of degree $x+y-2$. $G(x,y)$ exists if and only if y is even and $y \geq 2$. It is noted that for any graph $G(x,y)$, it has y vertices of degree r and x vertices of degree $r-1$. Let $x_i, y_i, i = 1,2,\dots,m$, be integers such that $G(x_i, y_i)$ exists for all $i = 1,2,\dots,m$. We construct the graph

$$G(x_1,y_1) * G(x_2,y_2) * \cdots * G(x_m,y_m) \tag{3.32}$$

from disjoint copies of the graphs by inserting a new vertex, say u, and joining u to all vertices of $G(x_i, y_i)$ which have the smallest degree, for $i = 1,2,\dots,m$.

Using this notation, we see that for an odd integer $r \geq 3$ and $q = 1,2,\dots,(r-1)/2$, for any odd positive integers $a_i, i = 1,2,\dots,1+2q$, whose sum is r,

$$G_q = G(a_1, r+2-a_1) * G(a_2, r+2-a_2) * \cdots * G(a_{1+2q}, r+2-a_{1+2q}) \tag{3.33}$$

is an r-regular graph on $(r+2)(1+2q)+1$ vertices with $\alpha_1(G_q) = (1/2)(|V(G_q)| - 2q)$. We have the following results.

Theorem 3.33 [32]. *For an odd integer $r \geq 3$, then*

(1) $F(r,2q) = (r+2)(1+2q)+1$, *for* $q = 1,2,\dots,(r-1)/2$,

(2) *if* $q = ((r-1)/2)s+t$, $0 \leq t < (r-1)/2$, *then* $F(r,2q) = sF(r,r-1)+F(r,2t)$, *where* $F(r,0) = 0$.

Corollary 3.34 [32]. *Let r be an odd integer, $r \geq 3$. If $F(r,2q) \leq n < F(r,2(q+1))$, then* $\min(\alpha_1, r^n) = (1/2)(n-2q)$.

4. Induced subgraphs of $\mathcal{R}(\mathbf{d})$

It was observed by Eggleton and Holton [10] that the graph $\mathcal{R}(\mathbf{d})$ is connected. It is natural to investigate connected subgraphs of $\mathcal{R}(\mathbf{d})$.

Let P be a property which a graph may possess. Denote by $\mathcal{R}(\mathbf{d};P)$ the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by those vertices which correspond to graphs with property P. Property P for which $\mathcal{R}(\mathbf{d};P)$ is connected is called *complete*. If a property P is complete, we may find all the graphs of a given degree sequence with the property by switching constrained to graphs with the property.

Colbourn [8] showed that the property of being a tree is complete and Syslo [37] extended this to the property of being unicyclic. Taylor [38] generalized these results by showing that the property of being connected is complete. The property of being 2-connected was shown to be complete also by Taylor [39].

By using results in Section 2 and the results of Taylor, we immediately obtain the following interpolation theorems.

Theorem 4.1. *Let $f \in \{\chi, I, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(f)$ and $b := b(f)$ such that $\mathbf{d}$ has a connected realization G with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

Theorem 4.2. *Let $f \in \{\chi, I, \phi, \omega, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma\}$. Then for any graphic degree sequence $\mathbf{d}$, there exist integers $a := a(f)$ and $b := b(f)$ such that $\mathbf{d}$ has a* 2-connected realization G with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.

Let f be an interpolation graph parameter with respect to $\mathcal{R}(\mathbf{d})$. Then $\{f(G) : G \in \mathcal{R}(\mathbf{d})\}$ is uniquely determined by $\min(f,\mathbf{d})$ and $\max(f,\mathbf{d})$. In particular, if $f = \chi$, we can define the *chromatic range*, $\chi(\mathbf{d})$, as the interval of integers as

$$\chi(\mathbf{d}) := [a,b] = \{c \in \mathbb{Z} : a \leq c \leq b\}, \tag{4.1}$$

where $a = \min(\chi,\mathbf{d})$ and $b = \max(\chi,\mathbf{d})$. The following results concerning connected subgraphs of $\mathcal{R}(\mathbf{d})$ were obtained in [35].

We consider the problem of determining the structure of induced subgraphs of the graph of realizations of a degree sequence $\mathbf{d}$ with prescribed chromatic number. We obtained some significant results when $\mathbf{d}$ is a regular degree sequence.

Brooks [4] observed that every graph G may be colored by $\Delta(G)+1$ colors and he characterized the graphs for which $\Delta(G)$ colors are not enough.

Theorem 4.3. *Any graph G satisfies $\chi(G) \leq 1 + \Delta(G)$, with equality holds if and only if either of the following holds:*
(1) *some component of G is the complete graph $K_{\Delta+1}$, where $\Delta = \Delta(G)$;*
(2) *some component of G is an odd cycle, and $\Delta(G) = 2$.*

Note that Brooks' theorem implies $b = \max(\chi, \mathbf{d}) \leq 1+\Delta$, where $\Delta = \max\{d_1, d_2, \ldots, d_n\}$. Thus for a regular degree sequence r^n, we have $b = \max(\chi, r^n) \leq 1 + r$. We found in [25] the corresponding value of b for all values of r and n except the cases when r and n are even and $r + 4 \leq n \leq 2r - 2$. In general, the Brooks bound may be very far from the actual value.

For each $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\ \chi = c)$ denote the subgraph of $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number c. Similarly for any $c \in \chi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\ \chi \leq c)$ denote the subgraph of $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with chromatic number $p \leq c$. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d};\ \chi = c)$ and $\mathcal{R}(\mathbf{d};\ \chi \leq c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\ \chi = c)$ and of $\mathcal{R}(\mathbf{d};\ \chi \leq c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\ \chi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with chromatic number c by beginning with one such realization and applying a suitable sequence of switchings producing only graphs with chromatic number c. The same applies if $\mathcal{R}(\mathbf{d};\ \chi \leq c)$ is connected.

Note that $\chi(0^n) = \{1\}$, $\chi(1^n) = \{2\}$ (n is even), $\chi(2^4) = \{2\}$, $\chi(2^n) = \{2,3\}$ if n is even and $n \geq 6$, and $\chi(2^n) = \{3\}$ if n is odd and $n \geq 3$. For $\chi(3^n)$, we have $\chi(3^4) = \{4\}$, $\chi(3^6) = \{2,3\}$, and $\chi(3^n) = \{2,3,4\}$ if n is even and $n \geq 8$.

Let G be a graph with $\chi(G) = k$ and let σ be a switching on G. σ is called a *k-safe switching* if $\chi(G^\sigma) = k$. A sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of switchings is called a sequence of k-safe switchings if for each i, $i = 1,2,\ldots,t$, $\chi(G^{\sigma_1\sigma_2\cdots\sigma_i}) = k$.

Theorem 4.4 [35]. *If $r \geq 3$ and $\max(\chi, r^n) = r + 1$, then the graphs $\mathcal{R}(r^n;\ \chi = r + 1)$ and $\mathcal{R}(r^n; p \leq r)$ are connected.*

Proof. Note that if $r \geq 3$, then $\max(\chi, r^n) = r + 1$ if and only if $n = r + 1$ or $n \geq 2r + 2$. Moreover, if G is a realization of r^n and $\chi(G) = r + 1$ if and only if G has K_{r+1} as a component, the theorem is true for $n = r + 1$. For $n \geq 2r + 2$, let G_1 and G_2 be any two realizations of r^n such that $\chi(G_1) = \chi(G_2) = r + 1$. Thus $G_1 = K_{r+1} \cup H_1$ and $G_2 = K_{r+1} \cup H_2$. Since H_1 and H_2 are r-regular graphs of order $n - r - 1$, H_2 can be obtained from H_1 by a finite number of switchings. Thus G_2 can as well be obtained by those switchings. Furthermore, it is easy to observe that those switchings are $(r + 1)$-safe switchings.

Let G be a realization of r^n such that $\chi(G) \leq r$. If G is disconnected, then G does not contain K_{r+1} as its component. Thus there exists a suitable sequence of switchings which transforms G to a connected realization of r^n such that each resulting graph obtained in this transformation will have chromatic number less than or equal to r. By using the result by Taylor [38], the proof is complete. □

Let C_m be the cycle of order m. Thus C_m exists for all integer $m \geq 3$, we call C_m *odd cycle* or *even cycle* according to whether m is odd or even. A realization of 2^n can be written as $\bigcup_{i=1}^t C_{n_i}$, where $\sum_{i=1}^t n_i = n$. It is well known that if $G = \bigcup_{i=1}^t C_{n_i}$, then $\chi(G) = 2$ if and only if for all i, n_i is even. It is clear that for any two even cycles C_r and C_s, there is a 2-safe switching which transforms these two cycles to C_{r+s}. Thus $\mathcal{R}(2^n;\ \chi = 2)$ is connected. The corresponding result can be obtained for the graph $\mathcal{R}(2^n;\ 3)$ with only one exception.

Theorem 4.5 [35]. *If $3 \in \chi(2^n)$, then the graph $\mathcal{R}(2^n;\ \chi = 3)$ is connected if and only if $n \neq 10$.*

Proof. Observe that if n is odd and $n \geq 3$, then $\chi(2^n) = \{3\}$. Thus the graph $\mathcal{R}(2^n;\ \chi = 3)$ is connected. If $n = 10$, we cannot transform $2C_5$ to $2C_3 \cup C_4$ without passing C_{10}. Thus the graph $\mathcal{R}(2^10;\ \chi = 3)$ is not connected. For $n \geq 12$, let $\bigcup_{i=1}^{t} C_{n_i}$ be a realization of 2^n and suppose that n_1 is odd. Then there is a sequence of 3-safe switchings which transforms $\bigcup_{i=1}^{t} C_{n_i}$ to $C_{n_1} \cup C_{n-n_1}$. Since $n \geq 12$, $C_5 \cup C_{n-5}$ can be transformed to $C_5 \cup C_3 \cup C_{n-8}$ and to $C_3 \cup C_{n-3}$. If m is odd and $m \geq 7$, then $C_m \cup C_{n-m}$ can be transformed to $C_3 \cup C_{m-3} \cup C_{n-m}$ and then to $C_3 \cup C_{n-3}$. The proof is complete. □

Theorem 4.6 [42, page 53]. *Let G and H be bipartite graphs with bipartition (X, Y). If $d_G(v) = d_H(v)$ for all $v \in X \cup Y$, then there is a sequence of 2-safe switchings that transforms G into H.*

The following theorem is a consequence of Theorem 4.6.

Theorem 4.7 [35]. *If $2 \in \chi(r^n)$, then the graph $\mathcal{R}(r^n;\ \chi = 2)$ is connected.*

Theorem 4.8 [35]. *If $c \in \chi(3^n)$, then the graph $\mathcal{R}(3^n;\ \chi = c)$ is connected.*

Proof. We have already proved when $c = 2,4$. Note that $G \in \mathcal{R}(3^n)$ with $\chi(G) = 3$ if and only if G does not contain K_4 as its component and G has an odd cycle as its subgraph. It is easy to check that the theorem is true if $n \leq 8$. Let G be a cubic graph of order $n \geq 10$ with $\chi(G) = 3$. We first suppose that G has a triangle T with $V(T) = \{x, y, z\}$ and $E(T) = \{xy, yz, xz\}$. If x and y have a common neighbor $v \in G - T$, then there exists an edge ab in $G - T$ independent of the edge xv. Thus there is a 3-safe switching σ such that G^σ contains a triangle T where any two vertices of T have no common neighbor in $G - T$.

Now suppose that G contains an odd cycle C of smallest order $k \geq 5$. Let x, y be two adjacent vertices in the cycle. Thus x and y have no common neighbor in G. Let z be the neighbor of y in $G - C$. Then there exists a 3-safe switching σ such that G^σ contains a triangle $T = \{x, y, z\}$. Thus for a graph $G \in \mathcal{R}(3^n)$ and $\chi(G) = 3$, there is a sequence of 3-safe switchings which transforms G to G' such that G' contains a triangle $T' = \{x, y, z\}$ and $G' - T'$ is a graph having exactly 3 vertices of degree 2. Since G is an arbitrary graph in $\mathcal{R}(3^n)$, the proof is complete. □

Conjecture 4.9. *If $c \in \chi(r^n)$, then the graph $\mathcal{R}(r^n;\ \chi = c)$ is connected.*

For the graph parameter ϕ, we investigated the values of $\phi(G)$ where G runs over the class of cubic graphs in [34]. As an application, we were able to answer a problem asked by Bau and Beineke [2].

For a graphic degree sequence $\mathbf{d}$, let $\phi(\mathbf{d}) = \{\phi(G) : G \in \mathcal{R}(\mathbf{d})\}$. Thus there exist integers a and b such that $\phi(\mathbf{d}) = \{k \in \mathbb{Z} : a \leq k \leq b\}$. For each $c \in \phi(\mathbf{d})$, let $\mathcal{R}(\mathbf{d};\ \phi = c)$ denote the subgraph of the graph $\mathcal{R}(\mathbf{d})$ induced by the vertices corresponding to graphs with decycling number c. We consider the problem of determining the structure of induced subgraph $\mathcal{R}(\mathbf{d};\ \phi = c)$. In general, what is the structure of $\mathcal{R}(\mathbf{d};\ \phi = c)$? In particular, are these graphs connected? If $\mathcal{R}(\mathbf{d};\ \phi = c)$ is connected, it must be possible to generate all realizations of $\mathbf{d}$ with decycling number c by beginning with one such realization and applying a suitable sequence of switchings to produce only graphs with decycling number c.

Bau and Beineke posed the following problem: Which cubic graphs G with $|G| = 2n$ satisfy $\phi(G) = \lceil (n+1)/2 \rceil$?

We answered the problem by finding all cubic graphs G of order $2n$ with $\phi(G) = \lceil (n+1)/2 \rceil$. Furthermore, we proved that the induced subgraph $\mathcal{R}(3^{2n};\ \phi = \lceil (n+1)/2 \rceil)$ is connected. We proved in the same paper that if $\mathcal{R}(3^{2n})$ is the class of all cubic graphs of order $2n$, then $\min\{\phi(G) : G \in \mathcal{R}(3^{2n})\} = \lceil (n+1)/2 \rceil$. Thus to answer the problem is equivalent to finding all cubic graphs of order $2n$ in $\mathcal{R}(3^{2n})$ having minimum cardinality of decycling set.

Conjecture 4.10. *If $c \in \phi(r^n)$, then the graph $\mathcal{R}(r^n;\ \phi = c)$ is connected.*

5. New approach on graph parameters

We have introduced a new area of research on graph parameters. Let f be a graph parameter and let $f(n,k,r)$ be the set of all connected r-regular graphs G of order n and $f(G) = k$. We consider the problem of determining the set of all integers r for which $f(n,k,r) \neq \emptyset$.

Let $\chi(n,k,r)$ be the set of all connected, r-regular, k-chromatic graphs on n vertices. By using the results in [25], we completely solved the problem for the graph parameter χ, for all n and k with $n \geq 2k$ as stated in the following theorems.

Theorem 5.1 [29]. *Let n, k, r be integers such that $3 \leq k \leq r \leq n-2$. Let $D(n,k) = \max\{r : \chi(n,k,r) \neq \emptyset\}$. Suppose $n = kt + l$ with $t \geq 2$ and $0 \leq l \leq k-1$. Let $\epsilon(n,t) = 1$ or 0 according to whether nt is odd or even. Then*

(1) $D(kt,k) = (k-1)t$,
(2) $D(kt+k-1,k) = (k-1)t + k - 3 - \epsilon(n,t+1)$,
(3) $D(kt+l,k) = (k-1)t + l - 1 - \epsilon(n,t)$, $1 \leq l \leq k-2$.

Under the conditions of Theorem 5.1, we get the following theorem.

Theorem 5.2 [29]. *$\chi(n,k,r) \neq \emptyset$ if $k \leq r \leq D(n,k)$.*

It is interesting to solve the problem by removing the condition $n \geq 2k$. It is also interesting to solve the problem for other graph parameters such as ω, ϕ, α_0, α_1, and γ.

6. Applications

In 1963, Erdős and Gallai [11] proved that any regular graph on n vertices has chromatic number $k \leq 3n/5$ unless the graph is complete. Commenting on their result in a personal communication, Erdős wrote,"probably such a graph exists for every $k \leq 3n/5$, except possibly for trivial exceptional cases."

Caccetta and Pullman [6] confirmed and strengthened their conjecture by showing that if $k > 1$, then for every $n \geq 5k/3$, there exists a connected, regular, k-chromatic graph on n vertices. By using the interpolation results on graph parameter χ in [25], we are able to provide an alternate proof of Erdős' question.

In [24] we generalized the result of Erdős and Gallai by introducing the concept of $F(j)$-graphs and showed that their result is a special case when $j = 3$.

Let n be an integer with $n \geq 4$, $r = n - 3$ and $n = 5p + i$, $0 \leq i \leq 4$. Then an explicit formula for $\max(\chi, r^n)$ can be obtained as follows:

$$\max(\chi, r^n) = \begin{cases} 3p & \text{if } i = 0,1, \\ 3p+1 & \text{if } i = 2,3, \\ 3p+2 & \text{if } i = 4. \end{cases} \tag{6.1}$$

For an integer $n \geq 4$, let $n = 5p + i$, $0 \leq i \leq 4$, and $n = 3q + t$, $0 \leq t \leq 2$.

If n is even and $n = 2m$, then

$$\chi(m^n) = [2, m], \qquad \chi((n-3)^n) = \left[q + t, 3p + \left\lfloor \frac{i}{2} \right\rfloor\right]. \tag{6.2}$$

If n is odd, $n \geq 7$, and $n = 2m + 1$, then

$$\chi(r^n) = [3, r], \qquad \chi((n-3)^n) = \left[q + t, 3p + \left\lfloor \frac{i}{2} \right\rfloor\right], \tag{6.3}$$

where r is an even integer and either $r = m - 1$ or $r = m$.

It is easy to check that

$$\chi(m^n) \cap \chi((n-3)^n) \neq \varnothing, \qquad \chi(r^n) \cap \chi((n-3)^n) \neq \varnothing. \tag{6.4}$$

Thus we have the following interpolation theorem.

Theorem 6.1 [30]. (1) *If n is even and $n \geq 6$, then there exists a connected noncomplete regular graph G of order n with $\chi(G) = k$ if and only if k is an integer satisfying* $2 \leq k \leq 3n/5$.

(2) *If n is odd and $n \geq 7$, then there exists a connected noncomplete regular graph G of order n with $\chi(G) = k$ if and only if k is an integer satisfying* $3 \leq k \leq 3n/5$.

7. Suggested problems

This final section is to give some problems related to interpolation graph parameters. We first consider the graph parameter ϕ. We have already mentioned that ϕ is an interpolation graph parameter with respect to $\mathscr{CR}(\mathbf{d})$. The values of $\text{Min}(\phi, r^n)$ and $\text{Max}(\phi, r^n)$ have been obtained in [36]. Let G be a connected r-regular graph and let S be a minimum decycling set of G. Since for any $v \in S$ there is a connected component C of $G-S$ such that v is adjacent to at least two vertices of C, there exists $u \in G-S$ such that $vu = e \in E(G)$ and $G-e$ is connected. Thus for two disjoint connected r-regular graphs G and H with minimum decycling sets S and T of G and H, respectively, there exist $u \in S$, $v \in G-S$, $x \in T$, $y \in H-T$ such that $uv = e \in E(G)$, $xy = f \in E(H)$, and $G-e$, $H-f$ are connected. A connected r-regular graph $K = ((G-e) \cup (H-f)) + \{ux, vy\}$ satisfies

$$\phi(K) \leq \phi(G \cup H) = \phi(G) + \phi(H), \tag{7.1}$$

and the following theorem holds.

Theorem 7.1. *Let $r \geq 3$ and nr be even. Then*

$$\operatorname{Min}(\phi, r^n) = \begin{cases} r-1 & \text{if } r+1 \leq n \leq 2r-1, \\ \left\lceil \dfrac{nr-2n+2}{2(r-1)} \right\rceil & \text{if } n \geq 2r. \end{cases} \tag{7.2}$$

Thus the values of $\operatorname{Min}(\phi, r^n)$, for all r and n, are already obtained. Moreover, $\operatorname{Min}(\phi, r^n) = \min(\phi, r^n)$.

It is clear that $\operatorname{Max}(\phi, r^n) = \max(\phi, r^n)$ for all r and n with $r+1 \leq n \leq 2r+1$.

Let G be a K_5-free graph of order n, $\Delta(G) = 4$. Let F be a maximal induced forest of G. We denote by $c(F)$ the number of cycles in $G-F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G-F$, is an *interchangeable pair of vertices with respect to F* if $(F-X) \cup Y$ is a forest, $|(F-X) \cup Y| \geq |F|$, and $c((F-X) \cup Y) < c(F)$. In general, we can define an interchangeable pair of vertices for a graph G with $\Delta(G) > 4$ as follows. Let G be a graph of order n, $\Delta(G) = \Delta > 4$, and G contains no $K_{\Delta+1}$ as its component. Let F be a maximal induced forest of G. We denote by $k(F)$ the number of $K_{\Delta-1}$ in $G-F$. A pair (X, Y), where $X \subseteq F$ and $Y \subseteq G-F$, is an *interchangeable pair of vertices with respect to F* if $(F-X) \cup Y$ is a forest, $|(F-X) \cup Y| \geq |F|$, and $k((F-X) \cup Y) < k(F)$.

By using the notion of interchangeable pair of vertices, we obtained in [36] the following results.

Theorem 7.2. *Let n be an even integer where $n \geq 12$. Then*

$$\operatorname{Max}(\phi, 3^n) = \begin{cases} \dfrac{3}{8}n + \dfrac{1}{4} & \text{if } n \equiv 2 \pmod 8, \\ \left\lfloor \dfrac{3}{8}n \right\rfloor & \text{otherwise.} \end{cases} \tag{7.3}$$

Theorem 7.3. *Let G be a connected r-regular graph of order $n \geq 2r+2$. Then $\phi(G) \leq n(r-2)/r$ for all $r \geq 4$.*

Theorem 7.4. *Let $n = rq + t$. Then* $\operatorname{Max}(\phi, r^n) = n - 2q$ *if $t = 0$,* $\operatorname{Max}(\phi, r^n) = n - 2q - 1$ *if $t = 1, 2$,* $\operatorname{Max}(\phi, r^n) = n - 2q - 2$ *if $2t > r$, and* $\operatorname{Max}(\phi, r^n) \in \{n-2q-2, n-2q-1\}$ *if $3 \leq t \leq r/2$.*

Conjecture 7.5. $\operatorname{Max}(\phi, r^n) = n - 2q - 2$ *if $3 \leq t \leq r/2$, for all $r \geq 6$.*

It is interesting to determine the values of $\operatorname{Min}(f, r^n)$ and $\operatorname{Max}(f, r^n)$ for other graph parameter f.

It is also interesting to consider the following problems.

Problem 7.6. Find all values of positive integers $n \geq 10$ such that the graph $\mathcal{R}(2^n)$ contains a Hamiltonian cycle.

Problem 7.7. Find the longest length of induced path in the graph $\mathcal{R}(2^n)$.

Problem 7.8. Find the longest length of induced path in the subgraph of $\mathcal{R}(3^{2n})$ induced by the set of 2-connected realizations of 3^{2n}.

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