COMPACT SPACE-LIKE HYPERSURFACES IN DE SITTER SPACE

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We present some integral formulas for compact space-like hypersurfaces in de Sitter space and some equivalent characterizations for totally umbilical compact space-like hypersurfaces in de Sitter space in terms of mean curvature and higher-order mean curvatures.

1. Introduction

It is well known that the semi-Riemannian (pseudo-Riemannian) manifolds (M,g) of Lorentzian signature play a special role in geometry and physics, and that they are models of space time of general relativity. Let $M_p^{n+1}(c)$ be an $(n+1)$-dimensional complete connected semi-Riemannian manifold with constant sectional curvature c and index p (see [13, page 227]). It is called an *indefinite space form of index p* and simply a *space form* when $p=0$. According to $c>0$, $c=0$, and $c<0$, $M_1^{n+1}(c)$ is called *de Sitter space*, *Minkowski space*, and *anti-de Sitter space*, and is denoted by $S_1^{n+1}(c)$, $\mathbb{R}_1^{n+1}$, and $H_1^{n+1}(c)$, respectively. In spite of the fact that the geometry of de Sitter space is the simplest model of space time of general relativity, this geometry was not studied thoroughly. Let $\phi : M^n \to S_1^{n+1}(c)$ be a smooth immersion of an n-dimensional connected manifold into $S_1^{n+1}(c)$. If the semi-Riemannian metric of $S_1^{n+1}(c)$ induces a Riemannian metric on M^n via ϕ, M^n is called a space-like hypersurface in de Sitter space.

The study of space-like hypersurfaces in de Sitter space $S_1^{n+1}(c)$ has been of increasing interest in the last years, because of their nice Bernstein-type properties. Since Goddard [7] conjectured in 1977 that complete space-like hyperspaces in $S_1^{n+1}(c)$ with constant mean curvature H must be totally umbilical, which turned out to be false in this original statement, an important number of authors have considered the problem of characterizing the totally umbilical space-like hypersurfaces in de Sitter space in terms of some appropriate geometric assumptions. Actually, Akutagawa [1] proved that Goddard's conjecture is true when $H^2 \le c$ if $n=2$, and $H^2 < (4(n-1)/n^2)c$ if $n \ge 3$. On the other hand, Montiel [11] proved that Goddard's conjecture is also true under the additional hypothesis of the compactness of the hypersurfaces. We also refer to [14] for an alternative proof of both facts given by Ramanathan in the 2-dimensional case. More recently, Cheng and Ishikawa [5] have shown that compact space-like hyperspaces in $S_1^{n+1}(c)$ with constant

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scalar curvature $S < n(n-1)c$ must be totally umbilical. Aledo el al. [3] have recently found some other characterizations of the totally umbilical compact space-like hypersurfaces in de Sitter space with constant higher-order mean curvatures, under appropriate hypothesis.

In this paper, we will study various equivalent characterizations of totally umbilical compact space-like hypersurfaces in de Sitter space in terms of mean curvature and higher-order mean curvatures. The whole paper is organized as follows. Section 2 gives some preliminaries, Section 3 gives some inequalities on the normalized symmetric functions, and Section 4 reviews some selfadjoint second-order differential operator. The main results of this paper are contained in Section 5, which gives us a more specific and complete picture of totally umbilical compact space-like hypersurfaces in de Sitter space. For simplicity, we omit the volume form dV in all integrals.

2. Preliminaries

We consider Minkowski space $\mathbb{R}_1^{n+2}$ as the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y\rangle = -x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i, \tag{2.1}$$

for $x, y \in \mathbb{R}^{n+2}$. Then de Sitter space $S_1^{n+1}(c)$ can be defined as the following hyperquadric of $\mathbb{R}_1^{n+2}$:

$$S_1^{n+1}(c) = \left\{ x \in \mathbb{R}_1^{n+2} \mid |x|^2 = \frac{1}{c} \right\}. \tag{2.2}$$

The induced metric from $\langle \cdot, \cdot \rangle$ makes $S_1^{n+1}(c)$ into a Lorentzian manifold with constant sectional curvature c. Moreover, if $x \in S_1^{n+1}(c)$, we can put

$$T_x S_1^{n+1}(c) = \{ v \in \mathbb{R}^{n+2} \mid \langle v, x\rangle = 0 \}. \tag{2.3}$$

We denote by ∇^L and $\overline{\nabla}$ the metric connections of $\mathbb{R}_1^{n+2}$ and $S_1^{n+1}(c)$, respectively. Then, we have

$$\nabla_v^L w - \overline{\nabla}_v w = -c\langle v, w\rangle x \tag{2.4}$$

for all $v, w \in T_x S_1^{n+1}(c)$. Let

$$\phi : M^n \longrightarrow S_1^{n+1}(c) \tag{2.5}$$

be a space-like hypersurface in $S_1^{n+1}(c)$ defined above. First, we want to know whether a compact one is orientable. The following proposition gives us the affirmative answer (see [11] or [2] for a proof).

Proposition 2.1. *Let $\phi : M^n \to S_1^{n+1}(c)$ be a space-like hypersurface in $S_1^{n+1}(c)$, $n \geq 2$. If M^n is compact, then M^n is diffeomorphic to S^n. In particular, compact totally umbilical space-like hypersurfaces in $S_1^{n+1}(c)$, $n \geq 2$, are round n-spheres.*

Throughout the following, we will exclusively deal with compact space-like hypersurfaces in $S_1^{n+1}(c)$, $n \geq 2$. The above proposition ensures that M^n is orientable. Let N be a time-like unit normal vector field for the immersion ϕ. The field N can be viewed as the Gauss map of M^n into hyperbolic space:

$$N : M^n \longrightarrow H^{n+1}, \tag{2.6}$$

where $H^{n+1} = \{x \in \mathbb{R}^{n+2} \mid |x|^2 = -1,\ x_0 \geq 1\}$. We will say that M^n is oriented by N. A well-known result is that the Gauss map N is harmonic if and only if the mean curvature H is constant. For a proof, one can refer to [4].

Let ∇ be the Levi-Civita connection associated to the Riemannian metric on M^n induced from $\langle \cdot, \cdot \rangle$. Then, we have

$$\begin{aligned} h(v,w) &= \overline{\nabla}_v w - \nabla_v w = -\langle \mathcal{A} v, w\rangle N, \\ \mathcal{A} v &= -\overline{\nabla}_v N = -\nabla_v^L N, \end{aligned} \tag{2.7}$$

where $\mathcal{A}$ stands for the shape operator of the immersion ϕ with respect to N and v, w are vector fields tangent to M^n. The operator $L = -\mathcal{A}$ is the Weingarten endomorphism. The eigenvalues of the operator L are called the principal curvatures and will be denoted by $\lambda_1, \ldots, \lambda_n$. The Codazzi equation is expressed by

$$(\nabla_v \mathcal{A}) w = (\nabla_w \mathcal{A}) v. \tag{2.8}$$

For a suitably chosen local field of orthonormal frames $e_1, \ldots, e_n$ on M^n, we have

$$\mathcal{A} e_i = -\lambda_i e_i. \tag{2.9}$$

The kth mean curvature of the space-like hypersurface M^n is defined by

$$H_k = \frac{1}{\binom{n}{k}} \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2.10}$$

Note that when $k = 1$, H_1 is the mean curvature H, and when $k = n$, H_n is the Gauss-Kronecker curvature. We can easily see that the scalar curvature

$$S = n(n-1)c - \left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2 = n(n-1)(c - H_2) \tag{2.11}$$

and the characteristic polynomial of $\mathcal{A}$ can be written in terms of the H_k's as

$$\det(tI - \mathcal{A}) = \sum_{k=0}^{n} \binom{n}{k} H_k t^{n-k}, \tag{2.12}$$

where $H_0 = 1$.

Minkowski formulas provide us with a convenient tool in the study of hypersurfaces. One can refer to [12] for the well-known version for space forms. Many interesting results have been got in the study of hypersurfaces by means of Minkowski formulas, for example, [9, 10, 12, 16, 17], and so forth. The proof in [12] followed the idea in [15]. Similar to it, one can easily give the proof of Minkowski formulas for compact space-like hypersurfaces in de Sitter space (see [3]). The following proposition is Minkowski formulas for compact space-like hypersurfaces in de Sitter space.

Proposition 2.2. *Let $\phi : M^n \to S_1^{n+1}(c)$ be a compact space-like hypersurface in $S_1^{n+1}(c)$, $n \geq 2$, then*

$$\int_{M^n} cH_k\langle \phi, a\rangle - H_{k+1}\langle N, a\rangle = 0, \quad k = 0, 1, \ldots, n-1, \tag{2.13}$$

for any $a \in \mathbb{R}^{n+2}$.

3. Inequalities on the normalized symmetric functions

Let $x_1, \ldots, x_n \in \mathbb{R}$. The elementary symmetric functions of n variables $x_1, \ldots, x_n$ are defined by

$$\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad k = 0, 1, \ldots, n, \tag{3.1}$$

where $\sigma_0 = 1$. For our purpose, it is useful to consider the normalized symmetric functions by dividing each σ_k by the number of its summands. We denote the normalized symmetric function by

$$E_k = \frac{1}{\binom{n}{k}} \sigma_k, \quad k = 0, 1, \ldots, n, \tag{3.2}$$

where $E_0 = 1$. Since

$$(x - x_1) \cdots (x - x_n) = \sum_{i=0}^{n} (-1)^i \sigma_i x^{n-i} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} E_i x^{n-i}, \tag{3.3}$$

we see that at least r of x_i's are zero if and only if $E_{n-r+1} = \cdots = E_n = 0$.

Proposition 3.1. *All $x_i \geq 0$ if and only if all $E_i \geq 0$, and all $x_i > 0$ if and only if all $E_i > 0$.*

Proof. We prove it by induction on n. For $n = 1$, the proposition holds clearly. Now assume that $n > 1$ and the proposition holds for $n - 1$. Let $P(x) = (x - x_1) \cdots (x - x_n)$ and $Q(x) = (1/n)P'(x) = (x - y_1) \cdots (x - y_n)$. By Rolle's theorem, $y_1, \ldots, y_{n-1}$ are all real and $x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq y_{n-1} \leq x_n$. Clearly, the inductive assumption applies to $y_1, \ldots, y_{n-1}$. Thus, it follows easily that the proposition holds for n. □

There are some well-known inequalities on the normalized symmetric functions, for example, Newton-Maclaurin inequalities. One can refer to [8] for the case of n positive numbers. For the sake of completeness, we include here a proof of Newton's inequalities for the general case.

PROPOSITION 3.2.

$$E_k^2 \geq E_{k-1}E_{k+1}, \quad k = 1,\ldots,n-1, \tag{3.4}$$

and each equality holds if and only if $x_1 = \cdots = x_n$, *or* $E_k = 0 = E_{k-1}E_{k+1}$.

Proof. We prove it by induction on n. For $n = 2$, the inequality holds clearly and the equality holds if and only if $x_1 = x_2$ since $E_1 = 0 = E_0E_2 = E_2$ implies that $x_1 = x_2 = 0$. Now assume that $n > 2$ and the proposition holds for $n-1$. Let $P(x) = (x - x_1)\cdots(x - x_n)$ and $Q(x) = (1/n)P'(x)$. Then

$$\begin{aligned} P(x) &= \sum_{i=0}^{n}(-1)^i\sigma_i x^{n-i} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}E_i x^{n-i}, \\ Q(x) &= \frac{1}{n}P'(x) = \sum_{i=0}^{n-1}(-1)^i\frac{n-i}{n}\binom{n}{i}E_i x^{n-i-1} = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}E_i x^{n-1-i}. \end{aligned} \tag{3.5}$$

On the other hand,

$$Q(x) = (x - y_1)\cdots(x - y_{n-1}) = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}E_i(y_1,\ldots,y_{n-1})x^{n-1-i}, \tag{3.6}$$

where $y_1,\ldots,y_{n-1}$ are $n-1$ roots of the polynomial $Q(x)$. Comparing the coefficients of the powers of x in the above two expressions for $Q(x)$ gives us

$$E_i(y_1,\ldots,y_{n-1}) = E_i(x_1,\ldots,x_n), \quad i = 0,\ldots,n-1. \tag{3.7}$$

By Rolle's theorem $y_1,\ldots,y_{n-1}$ are all real. Clearly, $y_1 = \cdots = y_{n-1}$ if and only if $x_1 = \cdots = x_n$. Thus the inductive assumption applies to $E_i(y_1,\ldots,y_{n-1})$, $i = 0,\ldots,n-1$, and the proposition holds for $k = 1,\ldots,n-2$ by (3.7).

It remains to prove for $k = n-1$, that is,

$$E_{n-1}^2(x_1,\ldots,x_n) \geq E_{n-2}(x_1,\ldots,x_n)E_n(x_1,\ldots,x_n), \tag{3.8}$$

with equality if and only if $x_1 = \cdots = x_n$, or $E_{n-1} = 0 = E_{n-2}E_n$.

Case 1. If some $x_i = 0$, then $E_n(x_1,\ldots,x_n) = x_1\cdots x_n = 0$. Clearly, (3.8) holds with equality if and only if $E_{n-1} = (1/n)\prod_{j\neq i}x_j = 0$, and thus if and only if some $x_j = 0$, $j \neq i$.

Case 2. If all $x_i \neq 0$, let $x_i' = 1/x_i$. Then, we have

$$\frac{E_i(x_1,\ldots,x_n)}{E_n(x_1,\ldots,x_n)} = E_{n-i}(x_1',\ldots,x_n'). \tag{3.9}$$

Since $E_n(x_1,\ldots,x_n) = x_1\cdots x_n \neq 0$, we see that (3.8) is equivalent to

$$E_1^2(x_1',\ldots,x_n') \geq E_2(x_1',\ldots,x_n'), \tag{3.10}$$

which is true since $n > 2$.

This completes the proof. □

Remark 3.3. For our future purpose, we concern most when each of the above equalities holds if and only if $x_1 = \cdots = x_n$, that is, to find some restrictions on x_i's to exclude the possibility of $E_k = 0 = E_{k-1}E_{k+1}$ and x_i's are not all zero. We only know that $E_1^2 = E_2$ holds if and only if $x_1 = \cdots = x_n$ since $E_1 = 0 = E_0E_2 = E_2$ implies that $x_1 = \cdots = x_n = 0$, while we cannot expect it for $k \geq 2$ even if all $x_i \geq 0$, for example, when only one of x_i's is positive. In particular, when all x_i's have the same sign, that is, nonnegative or nonpositive simultaneously, and at least k of x_i's are nonzero (equivalently, $E_1 \cdots E_k \neq 0$) or $E_1 = \cdots = E_n = 0$, we have that $E_k^2 = E_{k-1}E_{k+1}$ holds if and only if $x_1 = \cdots = x_n$.

Newton's inequalities have a very important consequence, Maclaurin's inequalities, by investigating that

$$E_1{}^2E_2{}^4\cdots E_k{}^{2k} \geq (E_0E_2)(E_1E_3)^2\cdots(E_{k-1}E_{k+1})^k, \tag{3.11}$$

where all $x_i \geq 0$. When all $x_i > 0$ and $2 \leq k \leq n-1$, we have

$$E_k{}^{1/k} \geq E_{k+1}{}^{1/(k+1)}, \tag{3.12}$$

with equality if and only if $x_1 = \cdots = x_n$. If some of x_i's are zero and the rest of them are positive, then for $2 \leq k \leq n-1$, we still have

$$E_k{}^{1/k} \geq E_{k+1}{}^{1/(k+1)}, \tag{3.13}$$

with equality if and only if $x_1 = \cdots = x_n$, or at least $n-k+1$ of x_i's are zero.

Corollary 3.4. *If all $x_i \geq 0$, $1 \leq k \leq n-1$, then*

$$E_k{}^{1/k} \geq E_{k+1}{}^{1/(k+1)}, \tag{3.14}$$

with equality if and only if $x_1 = \cdots = x_n$, or $E_{n-k+1} = \cdots = E_n = 0$.

Corollary 3.5. *If $E_1 > 0,\dots,E_k > 0$ and $2 \leq k \leq n$, then*

$$E_1 \geq E_2{}^{1/2} \geq \cdots \geq E_k{}^{1/k} \tag{3.15}$$

with each equality if and only if $x_1 = \cdots = x_n$.

Now we can give a result on the positiveness of mean curvature and higher-order mean curvatures of the compact space-like hypersurfaces in de Sitter space.

Theorem 3.6. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_k > 0$ and $2 \leq k \leq n$. If there exists a point of M^n, where $H_1,\dots,H_{k-1}$ are positive, then $H_1,\dots,H_{k-1}$ are positive everywhere on M^n, that is, $H_1 > 0,\dots,H_{k-1} > 0$.*

Proof. We prove it by an open-closed argument. Let

$$U = \{x \in M^n \mid H_1(x) > 0, \ldots, H_{k-1}(x) > 0\}. \tag{3.16}$$

Clearly U is open, and it is nonempty by the assumption. To prove that $U = M^n$, we only need to prove that U is also closed by the connectedness of M^n. Since $H_k > 0$ and M^n is compact, we have

$$a = \min_{x \in M^n} H_k(x) > 0. \tag{3.17}$$

For any $x \in U$, we have

$$H_1(x) \geq H_2(x)^{1/2} \geq \cdots \geq H_{k-1}(x)^{1/(k-1)} \geq H_k(x)^{1/k} \geq a^{1/k} > 0, \tag{3.18}$$

by Corollary 3.5. Thus U is closed. This completes the proof. □

Finally, we give another two sets of important inequalities by investigating that

$$\begin{aligned} &E_k^2 E_{k+1}^2 \cdots E_l{}^2 \geq (E_{k-1}E_{k+1})(E_k E_{k+2}) \cdots (E_{l-1}E_{l+1}), \\ &E_k{}^{1/k} \cdots E_{l-1}{}^{1/(l-1)} E_l{}^{(l+1)/l} \geq E_{k+1}{}^{1/(k+1)} \cdots E_l{}^{1/l} E_{l+1}, \end{aligned} \tag{3.19}$$

where all $x_i \geq 0$ and $1 \leq k < l \leq n-1$. Using the argument above leading to Corollary 3.4, we can get the following important inequalities.

THEOREM 3.7. *If all $x_i \geq 0$ and $1 \leq k < l \leq n-1$, then*

$$E_k E_l \geq E_{k-1} E_{l+1}, \tag{3.20}$$

with equality if and only if $x_1 = \cdots = x_n$, or $E_{n-l+1} = \cdots = E_n = 0$.

THEOREM 3.8. *If $E_{k-1} > 0, \ldots, E_{l+1} > 0$ and $1 \leq k < l \leq n-1$, then*

$$E_k E_l \geq E_{k-1} E_{l+1}, \tag{3.21}$$

with equality if and only if $x_1 = \cdots = x_n$.

THEOREM 3.9. *If all $x_i \geq 0$ and $1 \leq k < l \leq n-1$, then*

$$E_k^{1/k} E_l \geq E_{l+1}, \tag{3.22}$$

with equality if and only if $x_1 = \cdots = x_n$, or $E_{n-l+1} = \cdots = E_n = 0$.

THEOREM 3.10. *If $E_1 > 0, \ldots, E_{l+1} > 0$ and $1 \leq k < l \leq n-1$, then*

$$E_k^{1/k} E_l \geq E_{l+1}, \tag{3.23}$$

with equality if and only if $x_1 = \cdots = x_n$.

4. Some selfadjoint second-order differential operators

First, we introduce two known selfadjoint second-order differential operators, the Laplace operator $\triangle$ and the Cheng-Yau operator $\square$. For any C^2-function f defined on M^n, we consider the symmetric bilinear form

$$(\nabla^2 f)(w,v) = v(wf) - (\nabla_v w)f. \tag{4.1}$$

The Laplace operator $\triangle$ acting on any C^2-function f defined on M^n is given by

$$\triangle f = \sum_i (\nabla^2 f)(e_i,e_i). \tag{4.2}$$

Since M^n is compact and oriented, the Laplace operator $\triangle$ is selfadjoint relative to the L^2-inner product of M^n, that is,

$$\int_{M^n} f(\triangle g) = \int_{M^n} (\triangle f)g. \tag{4.3}$$

Following Cheng and Yau [6], we introduce an operator $\square$ acting on any C^2-function f defined on M^n by

$$\square f = \sum_{i,j} [nH\langle e_i,e_j\rangle + \langle \mathcal{A}e_i,e_j\rangle](\nabla^2 f)(e_i,e_j) = \sum_i (nH - \lambda_i)(\nabla^2 f)(e_i,e_i). \tag{4.4}$$

Note that the following holds at umbilical points:

$$\square f = \sum_i (n-1)H(\nabla^2 f)(e_i,e_i) = (n-1)H\triangle f. \tag{4.5}$$

By the Codazzi equation and [6, Proposition 1], we can prove that the operator $\square$ is selfadjoint relative to the L^2-inner product of M^n, that is,

$$\int_{M^n} f(\square g) = \int_{M^n} (\square f)g. \tag{4.6}$$

Naturally, we may ask the following question.

Question 4.1. Can we find other selfadjoint second-order differential operators in terms of the shape operator $\mathcal{A}$, mean curvature, and higher-order mean curvatures?

Fortunately, we do have such a selfadjoint second-order differential operator $\mathcal{L}_k$ for each $k = 0,1,\dots,n-1$. The idea is contained in [15, 17]. Following [3], we introduce the kth Newton transformation T_k associated to the shape operator $\mathcal{A}$:

$$T_k = \sum_{i=0}^{k} \binom{n}{i} H_i \mathcal{A}^{k-i}, \tag{4.7}$$

or inductively,

$$T_0 = I, \qquad T_k = \binom{n}{k} H_k I + \mathscr{A} T_{k-1}. \tag{4.8}$$

It follows from (2.12) that $T_n = 0$. Since the shape operator $\mathscr{A}$ is selfadjoint, it follows easily that the Newton transformations T_k's are selfadjoint. Clearly, the orthonormal basis $\{e_1,\dots,e_n\}$ diagonalizes the Newton transformations T_k's since it diagonalizes the shape operator $\mathscr{A}$.

Proposition 4.2. *If the shape operator $\mathscr{A}$ is negative definite, the Newton transformations T_k's, $k = 0,1,\dots,n-1$, are positive definite.*

Proof. Since the shape operator $\mathscr{A}$ is negative definite, all $\lambda_i > 0$. Without loss of generality, to prove that T_k is positive definite, we only need to prove that $\langle T_k e_1, e_1\rangle > 0$. Let $\lambda_i' = \lambda_i/\lambda_1$, $i = 1,\dots,n$, then we have

$$\begin{aligned} \langle T_k e_1, e_1\rangle &= \sum_{i=0}^{k} \binom{n}{i} H_i(-\lambda_1)^{k-i} \\ &= \sum_{i=0}^{k} \sigma_i(\lambda_1,\dots,\lambda_n)(-\lambda_1)^{k-i} \\ &= \lambda_1^k \sum_{i=0}^{k} (-1)^{k-i}\sigma_i(1,\lambda_2',\dots,\lambda_n'). \end{aligned} \tag{4.9}$$

Now we prove that

$$\sum_{i=0}^{k}(-1)^{k-i}\sigma_i(1,x_2,\dots,x_n) > 0, \quad k = 0,1,\dots,n-1, \tag{4.10}$$

by induction on n, where $x_2,\dots,x_n > 0$. Clearly, (4.10) holds for $k = 0$ or $n = 1$. Now assume that $m > 1$, $0 < l \le m-1$, and (4.10) holds for all $n < m$ and all $k < l$ for $n = m$. Let $n = m$ and $k = l$, then we have

$$\begin{aligned} \sum_{i=0}^{k}(-1)^{k-i}\sigma_i(1,x_2,\dots,x_n) &= \sum_{i=0}^{k}(-1)^{k-i}\sigma_i(1,x_2,\dots,x_{n-1}) \\ &\quad + x_n \sum_{i=1}^{k}(-1)^{k-i}\sigma_{i-1}(1,x_2,\dots,x_n) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\sigma_i(1,x_2,\dots,x_{n-1}) \\ &\quad + x_n \sum_{i=0}^{k-1}(-1)^{k-1-i}\sigma_i(1,x_2,\dots,x_n) > 0 \end{aligned} \tag{4.11}$$

by the inductive assumption and the fact that $\sum_{i=0}^{k}(-1)^{k-i}\sigma_i(1,x_2,\dots,x_{n-1}) = 0$ for $k = n-1$. This completes the proof. □

The following algebraic properties of T_k can be easily established from the definitions.

$$\operatorname{tr} T_k = (n-k)\binom{n}{k} H_k = n\binom{n-1}{k} H_k, \tag{4.12}$$

$$\operatorname{tr}(T_k\mathcal{A}) = -(k+1)\binom{n}{k+1} H_{k+1} = -n\binom{n-1}{k} H_{k+1}, \tag{4.13}$$

$$\begin{aligned}\operatorname{tr}\left(T_k\mathcal{A}^2\right) &= n\binom{n}{k+1} HH_{k+1} - (k+2)\binom{n}{k+2} H_{k+2}\\ &= n\binom{n}{k+1} HH_{k+1} - n\binom{n-1}{k+1} H_{k+2}.\end{aligned} \tag{4.14}$$

One can also easily derive the identities

$$\operatorname{tr}\left(T_k \nabla_v \mathcal{A}\right) = -\binom{n}{k+1}\langle \nabla H_{k+1}, v\rangle, \tag{4.15}$$

where v is any vector field tangent to M^n. Now for each $k = 0,1,\dots,n-1$, we can define a second-order differential operator $\mathcal{L}_k$ acting on any C^2-function f defined on M^n by

$$\mathcal{L}_k f = \operatorname{div}\left(T_k \nabla f\right). \tag{4.16}$$

It can be easily seen that the operators $\mathcal{L}_k$'s are selfadjoint. Clearly when $k=0$, the operator $\mathcal{L}_0$ is the Laplace operator $\triangle = \operatorname{div}\circ\nabla$. Later, we will see that when $k=1$, the operator $\mathcal{L}_1$ is the Cheng-Yau operator $\square$.

Finally, we can easily derive the following useful expression for $\mathcal{L}_k$ (see [3]):

$$\mathcal{L}_k f = \sum_i \langle T_k \nabla_{e_i}\nabla f, e_i\rangle = \sum_i \langle T_k e_i, e_i\rangle \nabla^2 f(e_i, e_i) \tag{4.17}$$

for any C^2-function f defined on M^n.

Remark 4.3. More specifically,

$$\mathcal{L}_k f = \sum_i \sum_{j=0}^{k} \binom{n}{j} H_j\left(-\lambda_i\right)^{k-j}\nabla^2 f(e_i, e_i). \tag{4.18}$$

Clearly when $k=1$, the operator $\mathcal{L}_1$ is the Cheng-Yau operator $\square = \sum_i (nH-\lambda_i)\nabla^2$. Note that the following holds at umbilical points:

$$\mathcal{L}_k f = \sum_i \sum_{j=0}^{k} \binom{n}{j} H_j\left(-\lambda_i\right)^{k-j}\nabla^2 f(e_i, e_i) = \sum_{i=0}^{k}(-1)^{k-i}\binom{n}{i}\cdot H_k \triangle f. \tag{4.19}$$

Remark 4.4. When T_k is positive definite, the operator $\mathscr{L}_k$ is elliptic. In particular, when the shape operator $\mathscr{A}$ is negative definite, the operator $\mathscr{L}_k$ is elliptic by proposition 4.2.

5. Main results

Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space, N a time-like unit normal vector field for ϕ, and $a \in \mathbb{R}_1^{n+2}$ arbitrary. We consider the height function $\langle \phi, a\rangle$ and the function $\langle N, a\rangle$ on M^n. Using (2.4), (2.7), we can get the following expressions for the gradient and Hessian of the above two functions:

$$\begin{aligned}
\langle \nabla \langle \phi, a\rangle, v\rangle &= \langle v, a\rangle, \qquad \langle \nabla \langle N, a\rangle, v\rangle = -\langle \mathscr{A} v, a\rangle, \\
(\nabla^2 \langle \phi, a\rangle)(v,w) &= wv\langle \phi, a\rangle - (\nabla_w v)\langle \phi, a\rangle \\
&= -c\langle v, w\rangle \langle \phi, a\rangle - \langle \mathscr{A} v, w\rangle \langle N, a\rangle, \\
(\nabla^2 \langle N, a\rangle)(v,w) &= wv\langle N, a\rangle - (\nabla_w v)\langle N, a\rangle \\
&= c\langle \mathscr{A} v, w\rangle \langle \phi, a\rangle + \langle \mathscr{A} v, \mathscr{A} w\rangle \langle N, a\rangle - \langle (\nabla_w \mathscr{A}) v, a\rangle,
\end{aligned} \tag{5.1}$$

where v, w are vector fields tangent to M^n. Thus, we have

$$\begin{aligned}
\mathscr{L}_k \langle \phi, a\rangle &= \sum_i \sum_{j=0}^{k} \binom{n}{j} H_j (-\lambda_i)^{k-j} \nabla^2 \langle \phi, a\rangle (e_i, e_i) \\
&= \sum_i \sum_{j=0}^{k} \binom{n}{j} H_j (-\lambda_i)^{k-j} [-c\langle \phi, a\rangle + \lambda_i \langle N, a\rangle] \\
&= -c \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{j} H_j \sum_i \lambda_i^{k-j} \cdot \langle \phi, a\rangle \\
&\quad + \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{j} H_j \sum_i \lambda_i^{k+1-j} \cdot \langle N, a\rangle \\
&= -c(n-k) \binom{n}{k} H_k \langle \phi, a\rangle + (k+1) \binom{n}{k+1} H_{k+1} \langle N, a\rangle \\
&= n \binom{n-1}{k} [-cH_k \langle \phi, a\rangle + H_{k+1} \langle N, a\rangle].
\end{aligned} \tag{5.2}$$

Note that the Minkowski formulas in Proposition 2.2 are regained by the selfadjointness of the operators $\mathscr{L}_k$'s.

For any vector field v tangent to M^n, we have

$$\nabla_v \nabla \langle N,a\rangle = c\mathcal{A}v\langle \phi,a\rangle + \mathcal{A}^2 v\langle N,a\rangle - (\nabla_v\mathcal{A})a^T, \tag{5.3}$$

by the selfadjointness of the operator $\nabla_v\mathcal{A}$, where a^T is the tangent component of a to M^n. Thus by (2.8), (4.13), (4.14), and (4.15), we have

$$\begin{aligned}
\mathcal{L}_k\langle N,a\rangle &= \sum_i \langle T_k\nabla_{e_i}\nabla\langle N,a\rangle, e_i\rangle \\
&= \sum_i \langle cT_k\mathcal{A}e_i\langle\phi,a\rangle + T_k\mathcal{A}^2 e_i\langle N,a\rangle - T_k(\nabla_{e_i}\mathcal{A})a^T, e_i\rangle \\
&= c\operatorname{tr}(T_k\mathcal{A})\langle\phi,a\rangle + \operatorname{tr}(T_k\mathcal{A}^2)\langle N,a\rangle - \sum_i\langle T_k(\nabla_{e_i}\mathcal{A})a^T, e_i\rangle \\
&= c\operatorname{tr}(T_k\mathcal{A})\langle\phi,a\rangle + \operatorname{tr}(T_k\mathcal{A}^2)\langle N,a\rangle - \sum_i\langle T_k(\nabla_{a^T}\mathcal{A})e_i, e_i\rangle \\
&= c\operatorname{tr}(T_k\mathcal{A})\langle\phi,a\rangle + \operatorname{tr}(T_k\mathcal{A}^2)\langle N,a\rangle - \operatorname{tr}(T_k\nabla_{a^T}\mathcal{A}) \\
&= -cn\binom{n-1}{k}H_{k+1}\langle\phi,a\rangle + n\left[\binom{n}{k+1}HH_{k+1} - \binom{n-1}{k+1}H_{k+2}\right]\langle N,a\rangle \\
&\quad + \binom{n}{k+1}\langle\nabla H_{k+1}, a^T\rangle \\
&= -cn\binom{n-1}{k}H_{k+1}\langle\phi,a\rangle + n\left[\binom{n}{k+1}HH_{k+1} - \binom{n-1}{k+1}H_{k+2}\right]\langle N,a\rangle \\
&\quad + \binom{n}{k+1}\langle\nabla H_{k+1}, a\rangle.
\end{aligned} \tag{5.4}$$

Remark 5.1. In particular, when $k = 0$, we have

$$\triangle\langle N,a\rangle = \mathcal{L}_0\langle N,a\rangle = -cnH\langle\phi,a\rangle + [n^2H^2 - n(n-1)H_2]\langle N,a\rangle + n\langle\nabla H,a\rangle. \tag{5.5}$$

Proposition 5.2.

$$\mathcal{L}_k\langle N,a\rangle = -\mathcal{L}_{k+1}\langle\phi,a\rangle + \binom{n}{k+1}[H_{k+1}\triangle\langle\phi,a\rangle + \langle\nabla H_{k+1},a\rangle] \tag{5.6}$$

for $k = 0,1,\dots,n-1$.

Proof. By (5.2), we have

$$\begin{aligned}
\frac{1}{n}H_{k+1}\triangle\langle\phi,a\rangle &- \frac{1}{n\binom{n-1}{k+1}}\mathcal{L}_{k+1}\langle\phi,a\rangle \\
&= (H_1H_{k+1} - H_{k+2})\langle N,a\rangle.
\end{aligned} \tag{5.7}$$

Thus by (5.2) and (5.4), we have

$$\begin{aligned}\mathscr{L}_k\langle N,a\rangle &= \frac{\binom{n-1}{k}}{\binom{n-1}{k+1}}\mathscr{L}_{k+1}\langle\phi,a\rangle + \binom{n}{k+1}\left[H_{k+1}\triangle\langle\phi,a\rangle - \frac{1}{\binom{n-1}{k+1}}\mathscr{L}_{k+1}\langle\phi,a\rangle\right]\\ &\quad + \binom{n}{k+1}\langle\nabla H_{k+1},a\rangle \\ &= -\mathscr{L}_{k+1}\langle\phi,a\rangle + \binom{n}{k+1}\left[H_{k+1}\triangle\langle\phi,a\rangle + \langle\nabla H_{k+1},a\rangle\right].\end{aligned} \tag{5.8}$$

□

Theorem 5.3. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space and $0 \leq i < j \leq n-1$, then*

$$\int_{M^n}\left[\frac{1}{n\binom{n-1}{j}}\mathscr{L}_j H_i - \frac{1}{n\binom{n-1}{i}}\mathscr{L}_i H_j\right]\langle\phi,a\rangle + \left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle = 0, \tag{5.9}$$

or equivalently,

$$\int_{M^n}\frac{1}{n}\left\langle\frac{1}{\binom{n-1}{i}}T_i\nabla H_j - \frac{1}{\binom{n-1}{j}}T_j\nabla H_i, a\right\rangle + \left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle = 0, \tag{5.10}$$

for any vector $a \in \mathbb{R}_1^{n+2}$.

Proof. By (5.2), we have

$$\frac{1}{n\binom{n-1}{i}}H_j\mathscr{L}_i\langle\phi,a\rangle - \frac{1}{n\binom{n-1}{j}}H_i\mathscr{L}_j\langle\phi,a\rangle = \left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle. \tag{5.11}$$

Thus,

$$\int_{M^n}\frac{1}{n\binom{n-1}{i}}H_j\mathscr{L}_i\langle\phi,a\rangle - \frac{1}{n\binom{n-1}{j}}H_i\mathscr{L}_j\langle\phi,a\rangle = \int_{M^n}\left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle. \tag{5.12}$$

Since the operators $\mathscr{L}_k$'s are selfadjoint, we have

$$\int_{M^n}\left[\frac{1}{n\binom{n-1}{j}}\mathscr{L}_j H_i - \frac{1}{n\binom{n-1}{i}}\mathscr{L}_i H_j\right]\langle\phi,a\rangle + \left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle = 0, \tag{5.13}$$

or equivalently,

$$\int_{M^n}\frac{1}{n}\left\langle\frac{1}{\binom{n-1}{i}}T_i\nabla H_j - \frac{1}{\binom{n-1}{j}}T_j\nabla H_i, a\right\rangle + \left(H_{i+1}H_j - H_i H_{j+1}\right)\langle N,a\rangle = 0 \tag{5.14}$$

since the operators $\mathscr{L}_k = \operatorname{div}\circ T_k\nabla$, for any vector $a \in \mathbb{R}_1^{n+2}$. □

Theorem 5.4. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space, $a \in \mathbb{R}_1^{n+2}$ any unit time-like vector with the same time-orientation as N, and $0 \leq k \leq n-2$, then*

$$\int_{M^n} \left\langle \binom{n-1}{k+1} T_k \nabla H_{k+1} - \binom{n-1}{k} T_{k+1} \nabla H_k, a \right\rangle \geq 0, \tag{5.15}$$

and the equality holds if and only if M^n is totally umbilical when $k = 0$, or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$.

Proof. For any unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N, that is, $|x|^2 = -1$ and $x_0 \geq 1$, we have $\langle N, a \rangle \leq -1$. Thus by taking $i = k$, $j = k+1$ in Theorem 5.3 and Proposition 3.2, we can deduce that

$$\int_{M^n} \left\langle \binom{n-1}{k+1} T_k \nabla H_{k+1} - \binom{n-1}{k} T_{k+1} \nabla H_k, a \right\rangle \geq 0, \tag{5.16}$$

and the equality holds if and only if M^n is totally umbilical when $k = 0$ or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$. □

Remark 5.5. In particular, when $k = 0$, we have

$$\int_{M^n} \langle \nabla H, a \rangle \geq 0, \tag{5.17}$$

and the equality holds if and only if M^n is totally umbilical for any unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N.

Remark 5.6. In particular, if H_k and H_{k+1} are constant, $0 \leq k \leq n-2$, then M^n is totally umbilical when $k = 0$, or additionally if $H_{k+1}^2 + H_k^2 H_{k+2}^2 \neq 0$ when $1 \leq k \leq n-2$. See also [3].

Theorem 5.7. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_1 \geq 0, \ldots, H_n \geq 0$, $a \in \mathbb{R}_1^{n+2}$ any unit time-like vector with the same time orientation as N, and $0 \leq i < j \leq n-1$, $j \geq i+2$, then*

$$\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a \right\rangle \geq 0. \tag{5.18}$$

Moreover, if $\sum_{k=n-j+1}^{n} H_k^2 \neq 0$, then the equality holds if and only if M^n is totally umbilical.

Proof. For any unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N, we have $\langle N, a \rangle \leq -1$. Thus by Theorems 5.3 and 3.7, we can deduce that

$$\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a \right\rangle \geq 0, \tag{5.19}$$

and when $\sum_{k=n-j+1}^{n} H_k^2 \neq 0$, the equality holds if and only if M^n is totally umbilical. □

Corollary 5.8. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_1 \geq 0,\dots,H_n \geq 0$ and constant $\sum_{i=1}^{k-1} a_iH_i + H_k$, $a_i \geq 0$, $2 \leq k \leq n-1$. If $\sum_{i=n-k+1}^{n} H_i^2 \neq 0$, then M^n is totally umbilical.*

Proof. Fix a unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N. By Theorems 5.4 and 5.7, we have

$$\int_{M^n} \langle \nabla H_i, a\rangle \geq 0, \quad i = 1,\dots,k. \tag{5.20}$$

Since

$$0 = \int_{M^n} \left\langle \nabla\left(\sum_{i=1}^{k-1} a_iH_i + H_k\right), a\right\rangle = \sum_{i=1}^{k-1} a_i \int_{M^n} \langle \nabla H_i, a\rangle + \int_{M^n} \langle \nabla H_k, a\rangle \geq 0, \tag{5.21}$$

we have

$$\int_{M^n} \langle \nabla H_k, a\rangle = 0. \tag{5.22}$$

Thus, M^n is totally umbilical by Theorem 5.7. □

Theorem 5.9. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space with $H_{k+1} > 0$, $a \in \mathbb{R}_1^{n+2}$ any unit time-like vector with the same time orientation as N, and $0 \leq i < j \leq k \leq n-1$, $j \geq i+2$. If there exists a point of M^n, where $H_1,\dots,H_k$ are positive, then*

$$\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a\right\rangle \geq 0, \tag{5.23}$$

with equality if and only if M^n is totally umbilical.

Proof. For any unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N, we have $\langle N, a\rangle \leq -1$. Thus by Theorems 5.3, 3.6, and 3.8, we can deduce that

$$\int_{M^n} \left\langle \binom{n-1}{j} T_i \nabla H_j - \binom{n-1}{i} T_j \nabla H_i, a\right\rangle \geq 0, \tag{5.24}$$

and the equality holds if and only if M^n is totally umbilical. □

Let $a \in \mathbb{R}_1^{n+2}$ be a unit time-like vector. The intersection of $S_1^{n+1}(c) \subset \mathbb{R}_1^{n+2}$ and the space-like hyperplane $\{x \in \mathbb{R}_1^{n+2} \mid \langle x, a\rangle = 0\}$ defines an n-sphere which is a totally geodesic hypersurface in $S_1^{n+1}(c)$. We will refer to that sphere as the equator of $S_1^{n+1}(c)$ determined by a. This equator divides the de Sitter space into two connected components; the future which is given by

$$\{x \in \mathbb{R}_1^{n+2} \mid \langle x, a\rangle < 0\}, \tag{5.25}$$

and the past given by

$$\{x \in \mathbb{R}_1^{n+2} \mid \langle x, a\rangle > 0\}. \tag{5.26}$$

Following [3], we can easily get the following corollary.

Corollary 5.10. *Let $\phi : M^n \to S_1^{n+1}(c)$, $n \geq 2$, be a compact space-like hypersurface in de Sitter space and $2 \leq k \leq n-1$. If M^n is contained in the chronological future (or past) relative to the equator of $S_1^{n+1}(c)$ determined by a unit time-like vector $a \in \mathbb{R}_1^{n+2}$ with the same time orientation as N and $H_{k+1} > 0$ (or $(-1)^{k+1}H_{k+1} > 0$), then*

$$\int_{M^n} \langle \nabla H_i, a\rangle \geq 0 \quad \left(\text{or } (-1)^{i+1}\int_{M^n} \langle \nabla H_i, a\rangle \geq 0\right), \quad 2 \leq i \leq k, \tag{5.27}$$

with each equality if and only if M^n is totally umbilical.

Proof. First we prove the future case. By Theorem 5.9, it is sufficient to prove that there exists a point of M^n, where all $H_i > 0$. Since M^n is contained in the chronological future relative to the equator determined by a and M^n is compact, there exists a point $x_0 \in M^n$ such that

$$\max_{x \in M^n} \langle \phi(x), a\rangle = \langle \phi(x_0), a\rangle < 0. \tag{5.28}$$

Thus by maximum principle, we have

$$\begin{aligned} -c\langle \phi(x_0), a\rangle + \lambda_i\langle N(x_0), a\rangle &= -c\langle e_i, e_i\rangle\langle \phi(x_0), a\rangle - \langle \mathcal{A}e_i, e_i\rangle\langle N(x_0), a\rangle \\ &= \nabla^2\langle \phi, a\rangle(e_i, e_i) \leq 0. \end{aligned} \tag{5.29}$$

Since $a \in \mathbb{R}_1^{n+2}$ is a unit time-like vector with the same time orientation as N, we have $\langle N, a\rangle \leq -1$. So

$$\lambda_i \geq c\frac{\langle \phi(x_0), a\rangle}{\langle N(x_0), a\rangle} > 0, \quad i = 1, \ldots, n. \tag{5.30}$$

Thus all $H_i > 0$. For the past case, we only need to replace N and a by $-N$ and $-a$, respectively, and the proof for the future case applies. This completes the proof. □

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