A NOTE ON THE FLOW INDUCED BY A CONSTANTLY ACCELERATING EDGE IN AN OLDROYD-B FLUID

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Analytical expressions for the velocity field and the tangential stresses that are induced due to a constantly accelerating edge in an Oldroyd-B fluid have been established for all values of material constants. The solutions that have been obtained satisfy the governing differential equations and all the imposed initial and boundary conditions. These solutions reduce to those for the Maxwell, second-grade, and Navier-Stokes fluid as limiting cases. Exact solutions such as those determined here for an unsteady problem serve a dual purpose. They have relevance to an interesting physical problem and the solutions can also be used to check the efficacy of the flows of such fluids in more complicated flow domains.

1. Introduction

Numerous models have been proposed to describe response characteristics of fluids that cannot be described sufficiently well by the classical Navier-Stokes fluid model. These models that have been proposed to describe the departure can be classified as fluids of the differential type, rate type and integral type. A rate-type model for the viscoelastic response of fluids was first proposed by Maxwell [7]. This seminal work was followed by important studies concerning rate-type non-Newtonian fluids by Förhlick and Sack [5], Burgers [2], Jeffreys [6], and others. Building on the work of Förhlick and Sack [5], Oldroyd [8] developed a systematic framework for developing models for non-Newtonian fluids. One of them corresponds to the Oldroyd-B model whose constitutive equation appends an additional term to that for the Maxwell model. The Cauchy stress $\mathbf{T}$ in such a fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.1}$$

where $-p\mathbf{I}$ is the constraint response due to the requirement that the fluid be incompressible and the extra-stress tensor $\mathbf{S}$ in such a model is given by [8, 9]

$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T)], \tag{1.2}$$

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where λ and λ_r are the relaxation and retardation times, μ is the dynamic viscosity, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, and the superposed dot indicates the material time derivative.

The above model has become very popular amongst rheologists modeling the response of dilute polymeric solutions. While the model can describe many of the non-Newtonian characteristics exhibited by polymeric materials such as stress relaxation, normal stress differences in simple shear flows and nonlinear creep, it is incapable of describing shear thinning and shear thickening that are observed during the flows of many fluids. This, notwithstanding, model is able to capture qualitatively the response of many dilute polymeric liquids.

While the flow of the Oldroyd-B fluid has been studied in much detail, more than most other non-Newtonian fluid models, and in complicated flow geometries, new exact solutions for the flows of such fluids are most welcome provided they correspond to physically realistic situations, as they serve a dual purpose. First, they provide a solution to a flow that has technical relevance. Second, such solutions can be used as checks against complicated numerical codes that have been developed for much more complex flows. The problem considered in this note corresponds to a meaningful physical problem, that of the flow induced by a flat edge in an Oldroyd-B fluid. The flow in such a geometry has been studied in detail for the Navier-Stokes, second-grade, and Maxwell fluid. The solution that we establish here contains these previous solutions as a special case. One further remark concerning the problem is warranted. While many steady flows concerning the Oldroyd-B model have been carried out, there have been far fewer studies that are concerned with unsteady flows.

2. Formulation of the problem

Let us consider an Oldroyd-B fluid, at rest, occupying the space of the first quadrant of a rectangular edge ($-\infty < x < \infty;\ y, z > 0$). At time zero, the infinitely extended edge is subject to a constant acceleration A. Owing to the shear, the fluid is moved and its velocity field is of the form

$$\mathbf{v} = v(y,z,t)\mathbf{i}, \tag{2.1}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. Since the velocity field is independent of x, we expect that the stress field will also be independent of x.

Equations (1.2) and (2.1) together with the natural condition (the fluid being at rest up to the moment $t = 0$)

$$\mathbf{S}(y,z,0) = \mathbf{0} \tag{2.2}$$

lead to $S_{yy} = S_{yz} = S_{zz} = 0$, for all time and

$$\begin{gathered}(1+\lambda\partial_t)\tau_1 = \mu(1+\lambda_r\partial_t)\partial_y v, \qquad (1+\lambda\partial_t)\tau_2 = \mu(1+\lambda_r\partial_t)\partial_z v, \\ (1+\lambda\partial_t)\sigma = 2\lambda(\tau_1\partial_y v + \tau_2\partial_z v) - 2\mu\lambda_r\left[(\partial_y v)^2 + (\partial_z v)^2\right],\end{gathered} \tag{2.3}$$

where $\tau_1 = S_{xy}$, $\tau_2 = S_{xz}$ are tangential stresses and $\sigma = S_{xx}$ is a normal stress. The equations of motion, in the absence of body forces and a pressure gradient in the x-direction, reduce to

$$\partial_y \tau_1 + \partial_z \tau_2 = \rho \partial_t v, \tag{2.4}$$

where ρ is the constant density of the fluid.

Eliminating τ_1 and τ_2 between (2.3) and (2.4), we obtain the following third-order linear partial differential equation:

$$\lambda \partial_t^2 v(y,z,t) + \partial_t v(y,z,t) = \nu (1 + \lambda_r \partial_t)(\partial_y^2 + \partial_z^2) v(y,z,t), \quad y,z,t > 0, \tag{2.5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

Since the fluid has been at rest, for all $t \leq 0$, we have

$$v(y,z,0) = 0, \quad y,z > 0, \tag{2.6}$$

$$v(0,z,t) = v(y,0,t) = At, \quad t > 0. \tag{2.7}$$

Furthermore, the appropriate boundary conditions are (see [12])

$$v(y,z,t), \partial_y v(y,z,t), \partial_z v(y,z,t) \longrightarrow 0 \quad \text{as } y^2 + z^2 \longrightarrow \infty,\ t > 0, \tag{2.8}$$

and (see, e.g., [11])

$$\partial_t v(y,z,t) \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \tag{2.9}$$

3. The solution of the problem

Multiplying both sides of (2.5) by $(2/\pi)\sin(y\xi)\sin(z\eta)$, integrating them with respect to y and z from 0 to ∞, and having in mind the boundary conditions (2.7) and (2.8), we find that

$$\lambda \partial_t^2 v_s(\xi,\eta,t) + [1 + \alpha(\xi^2 + \eta^2)] \partial_t v_s(\xi,\eta,t) + \nu(\xi^2 + \eta^2) v_s(\xi,\eta,t) = \frac{2}{\pi} A \frac{\xi^2 + \eta^2}{\xi\eta} (\nu t + \alpha), \tag{3.1}$$

where $\alpha = \nu\lambda_r$ and $v_s(\xi,\eta,t)$ is the double Fourier sine transform of $v(y,z,t)$. In view of (2.6) and (2.9), $v_s(\xi,\eta,t)$ has to satisfy the initial conditions

$$v_s(\xi,\eta,0) = \partial_t v_s(\xi,\eta,0) = 0, \quad \xi,\eta > 0. \tag{3.2}$$

The solution of the ordinary differential equation (3.1), subject to the initial conditions (3.2), has one of the following three forms:

$$\begin{aligned} v_s(\xi,\eta,t) &= \frac{2}{\pi} \frac{A}{\xi\eta} \left[\frac{r_2 r_3 e^{r_1 t} - r_1 r_4 e^{r_2 t}}{\nu(\xi^2 + \eta^2)(r_2 - r_1)} \lambda + t - \frac{1}{\nu(\xi^2 + \eta^2)} \right], \quad \text{if } \lambda < \lambda_r, \\ v_s(\xi,\eta,t) &= \frac{2}{\pi} \frac{A}{\xi\eta} \left[\frac{e^{-\nu(\xi^2+\eta^2)t}}{\nu(\xi^2 + \eta^2)} + t - \frac{1}{\nu(\xi^2 + \eta^2)} \right], \quad \text{if } \lambda = \lambda_r, \end{aligned} \tag{3.3}$$

or

$$
v_s(\xi,\eta,t) = \begin{cases} \dfrac{2}{\pi}\dfrac{A}{\xi\eta}\left[\dfrac{r_2r_3e^{r_1t}-r_1r_4e^{r_2t}}{\nu(\xi^2+\eta^2)(r_2-r_1)}\lambda+t-\dfrac{1}{\nu(\xi^2+\eta^2)}\right], & \text{in } \mathcal{D}_1, \\ \dfrac{2}{\pi}\dfrac{A}{\xi\eta}\left\{\dfrac{1}{\nu(\xi^2+\eta^2)}e^{-[(1+\alpha(\xi^2+\eta^2))/(2\lambda)]t}\right. & \\ \quad\times\left[\dfrac{1+\nu(\xi^2+\eta^2)(\lambda_r-2\lambda)}{\beta}\sin\left(\dfrac{\beta t}{2\lambda}\right)+\cos\left(\dfrac{\beta t}{2\lambda}\right)\right]\left.+t-\dfrac{1}{\nu(\xi^2+\eta^2)}\right\}, & \text{in } \mathcal{D}_2, \end{cases} \tag{3.4}
$$

if $\lambda > \lambda_r$.

In the above relations, $r_{1,2} = (-[1+\alpha(\xi^2+\eta^2)] \pm \sqrt{[1+\alpha(\xi^2+\eta^2)]^2-4\nu\lambda(\xi^2+\eta^2)})/(2\lambda)$, $r_3 = (1+\lambda r_1)/\lambda$, $r_4 = (1+\lambda r_2)/\lambda$,
$\beta = \sqrt{4\nu\lambda(\xi^2+\eta^2)-[1+\alpha(\xi^2+\eta^2)]^2}$,
$\mathcal{D}_1 = \{(\xi,\eta);\ \xi,\eta>0;\ 0<\xi^2+\eta^2\le a^2\}\bigcup\{(\xi,\eta);\ \xi,\eta>0;\ \xi^2+\eta^2>b^2\}$,
and
$\mathcal{D}_2 = \{(\xi,\eta);\ \xi,\eta>0;\ a^2<\xi^2+\eta^2\le b^2\}$,
where $a = 1/(\sqrt{\nu}(\sqrt{\lambda}+\sqrt{\lambda-\lambda_r}))$ and $b = 1/(\sqrt{\nu}(\sqrt{\lambda}-\sqrt{\lambda-\lambda_r}))$.

Inverting (3.3)–(3.4) by means of double Fourier sine formula [10], we find that the velocity field is given by

$$
\begin{aligned} v(y,z,t) &= At-\frac{4A}{\nu\pi^2}\int_0^\infty\int_0^\infty\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta \\ &\quad+\frac{4A\lambda}{\nu\pi^2}\int_0^\infty\int_0^\infty\frac{r_2r_3e^{r_1t}-r_1r_4e^{r_2t}}{r_2-r_1}\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta, \quad \text{if } \lambda<\lambda_r, \end{aligned} \tag{3.5}
$$

$$
v(y,z,t) = At-\frac{4A}{\nu\pi^2}\int_0^\infty\int_0^\infty\left[1-e^{-\nu(\xi^2+\eta^2)t}\right]\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta, \quad \text{if } \lambda=\lambda_r, \tag{3.6}
$$

$$
\begin{aligned} v(y,z,t) &= At-\frac{4A}{\nu\pi^2}\int_0^\infty\int_0^\infty\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta \\ &\quad+\frac{4A\lambda}{\nu\pi^2}\iint_{\mathcal{D}_1}\left[\frac{r_2r_3e^{r_1t}-r_1r_4e^{r_2t}}{r_2-r_1}\right]\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta \\ &\quad+\frac{4A}{\nu\pi^2}e^{-t/(2\lambda)}\iint_{\mathcal{D}_2}e^{-[\alpha(\xi^2+\eta^2)/(2\lambda)]t} \\ &\qquad\times\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1+\nu(\xi^2+\eta^2)(\lambda_r-2\lambda)}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right] \\ &\qquad\times\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\,d\eta, \quad \text{if } \lambda>\lambda_r. \end{aligned} \tag{3.7}
$$

The tangential stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$, corresponding to these velocity fields, are solutions of the ordinary differential equation (2.3) with the initial conditions (2.2).

The expressions for the shear stresses are

$$\begin{aligned} S_{xy}(y,z,t) &= \tau_1(y,z,t)\\ &= -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty\left[\frac{r_4e^{r_2t}-r_3e^{r_1t}}{r_2-r_1}\right]\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda<\lambda_r, \end{aligned} \tag{3.8}$$

$$S_{xy}(y,z,t)=\tau_1(y,z,t)=-\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty\left[1-e^{-\nu(\xi^2+\eta^2)t}\right]\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda=\lambda_r, \tag{3.9}$$

$$\begin{aligned} S_{xy}(y,z,t) &= \tau_1(y,z,t)\\ &= -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}\iint_{\mathcal{D}_1}\left[\frac{r_4e^{r_2t}-r_3e^{r_1t}}{r_2-r_1}\right]\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}e^{-t/2\lambda}\iint_{\mathcal{D}_2}e^{-[\alpha(\xi^2+\eta^2)/(2\lambda)]t}\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1-\alpha(\xi^2+\eta^2)}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\\ &\qquad\times\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda>\lambda_r, \end{aligned} \tag{3.10}$$

respectively, and

$$\begin{aligned} S_{xz}(y,z,t) &= \tau_2(y,z,t)\\ &= -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty\left[\frac{r_4e^{r_2t}-r_3e^{r_1t}}{r_2-r_1}\right]\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda<\lambda_r, \end{aligned} \tag{3.11}$$

$$S_{xz}(y,z,t)=\tau_2(y,z,t)=-\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty\left[1-e^{-\nu(\xi^2+\eta^2)t}\right]\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda=\lambda_r, \tag{3.12}$$

$$\begin{aligned} S_{xz}(y,z,t) &= \tau_2(y,z,t)\\ &= -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}\iint_{\mathcal{D}_1}\left[\frac{r_4e^{r_2t}-r_3e^{r_1t}}{r_2-r_1}\right]\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta\\ &\quad+\frac{4\rho A}{\pi^2}e^{-t/(2\lambda)}\iint_{\mathcal{D}_2}e^{-[\alpha(\xi^2+\eta^2)/(2\lambda)]t}\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1-\alpha(\xi^2+\eta^2)}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\\ &\qquad\times\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta,\quad \text{if } \lambda>\lambda_r. \end{aligned} \tag{3.13}$$

It is worth remarking that the exact solutions satisfy the initial and boundary conditions. As Bandelli et al. [1] observe, other transform techniques such as the Laplace transforms do not lead to solutions that satisfy the initial conditions even though they are enforced, while the solutions are obtained, if the data do not satisfy certain compatability conditions. This fact cannot be overemphasized as Laplace transform techniques are usually employed to solve such problems.

4. Limiting cases

We will now consider special limiting cases wherein the solution reduces to that for specific fluids such as the second-grade fluid, Maxwell fluid, and the Navier-Stokes fluid. We start by taking the limits of (3.5), (3.8), and (3.11). As $\lambda \to 0$, we find

$$v(y,z,t) = At - \frac{4A}{\nu\pi^2}\int_0^\infty\int_0^\infty \left[1 - e^{-(\nu(\xi^2+\eta^2)/(1+\alpha(\xi^2+\eta^2)))t}\right]\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\, d\eta, \tag{4.1}$$

$$\begin{aligned}\tau_1(y,z,t) = -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty &\left[1 - \frac{1}{1+\alpha(\xi^2+\eta^2)}e^{-(\nu(\xi^2+\eta^2)/(1+\alpha(\xi^2+\eta^2)))t}\right]\\ &\times\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\, d\eta,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\tau_2(y,z,t) = -\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty &\left[1 - \frac{1}{1+\alpha(\xi^2+\eta^2)}e^{-(\nu(\xi^2+\eta^2)/(1+\alpha(\xi^2+\eta^2)))t}\right]\\ &\times\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\, d\eta,\end{aligned} \tag{4.3}$$

which represent the solutions corresponding to a second-grade fluid, the velocity field (4.1) being identical with that resulting from (3.3) of [4] for $V(t) = At$. It is interesting to note that the second-grade fluid model is not obtained by setting $\lambda = 0$ in the model (1.2). However, the solution can be obtained by letting $\lambda \to 0$.

We will next consider the case corresponding to the Maxwell fluid which is obtained by setting the retardation time $\lambda_r \equiv 0$ in the model (1.2). By letting now $\lambda_r \to 0$ in (3.7), (3.10), and (3.13), we obtain the solution corresponding to a Maxwell fluid (cf. [3, (3.4), (3.6), and (3.7)])

$$\begin{aligned}v(y,z,t) = At &- \frac{4A}{\nu\pi^2}\int_0^\infty\int_0^\infty\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\, d\eta\\ &+ \frac{4\lambda A}{\nu\pi^2}\iint_{\mathcal{D}_3}\frac{r_5^2e^{r_6t} - r_6^2e^{r_5t}}{r_6 - r_5}\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\, d\eta\\ &+ \frac{4A}{\nu\pi^2}e^{-t/(2\lambda)}\iint_{\mathcal{D}_4}\left[\cos\left(\frac{\gamma t}{2\lambda}\right) + \frac{1-2\nu\lambda(\xi^2+\eta^2)}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right)\right]\frac{\sin(y\xi)\sin(z\eta)}{\xi\eta(\xi^2+\eta^2)}d\xi\, d\eta,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\tau_1(y,z,t) = &-\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta\\&+\frac{4\rho A}{\pi^2}\iint_{\mathcal{D}_3}\frac{r_6e^{r_5t}-r_5e^{r_6t}}{r_6-r_5}\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta\\&+\frac{4\rho A}{\pi^2}e^{-t/(2\lambda)}\iint_{\mathcal{D}_4}\left[\cos\left(\frac{\gamma t}{2\lambda}\right)+\frac{1}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right)\right]\frac{\cos(y\xi)\sin(z\eta)}{\eta(\xi^2+\eta^2)}d\xi\,d\eta,\end{aligned}\tag{4.5}$$

respectively,

$$\begin{aligned}\tau_2(y,z,t) = &-\frac{4\rho A}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta\\&+\frac{4\rho A}{\pi^2}\iint_{\mathcal{D}_3}\frac{r_6e^{r_5t}-r_5e^{r_6t}}{r_6-r_5}\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta\\&+\frac{4\rho A}{\pi^2}e^{-t/(2\lambda)}\iint_{\mathcal{D}_4}\left[\cos\left(\frac{\gamma t}{2\lambda}\right)+\frac{1}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right)\right]\frac{\sin(y\xi)\cos(z\eta)}{\xi(\xi^2+\eta^2)}d\xi\,d\eta,\end{aligned}\tag{4.6}$$

in which $r_{5,6} = (-1 \pm \sqrt{1-4\nu\lambda(\xi^2+\eta^2)})/(2\lambda)$, $\gamma = \sqrt{4\nu\lambda(\xi^2+\eta^2)-1}$,

$$\begin{aligned}\mathcal{D}_3 &= \left\{(\xi,\eta);\ \xi,\eta>0;\ \xi^2+\eta^2 \le \frac{1}{4\nu\lambda}\right\},\\ \mathcal{D}_4 &= \left\{(\xi,\eta);\ \xi,\eta>0;\ \xi^2+\eta^2 > \frac{1}{4\nu\lambda}\right\}.\end{aligned}\tag{4.7}$$

We finally consider the case of the Navier-Stokes fluid which can be obtained by setting $\lambda = \lambda_r = 0$. In the special case, when both λ_r and $\lambda \to 0$ in any one of (3.5) or (3.7), (3.8) or (3.10), and (3.11) or (3.13), or only $\lambda_r \to 0$ in (4.1), (4.2), and (4.3), respectively, $\lambda \to 0$ in (4.4), (4.5), and (4.6) we recover the solutions (3.6), (3.9), and (3.12) corresponding to a Navier-Stokes fluid.

5. Conclusions and numerical results

In this note, the velocity fields and the associated tangential stresses corresponding to the flow induced by a constantly accelerating edge in an Oldroyd-B fluid have been determined by means of the double Fourier sine transform. The solutions in the form of (3.5), (3.6), and (3.7) for the velocity field $v(y,z,t)$, and (3.8), (3.9), (3.10), (3.11), (3.12), and (3.13) for the shear stresses do not provide a feel for the nature of the solutions. In order to obtain a sense of the solutions, we will provide figures that depict their structure. When $\lambda_r = \lambda = 0$, as it was to be expected, these solutions reduce to those corresponding to a Navier-Stokes fluid. Direct computations show that $v(y,z,t)$, $\tau_1(y,z,t)$, and $\tau_2(y,z,t)$, given by (3.5), (3.7), (3.8), (3.10), (3.11), and (3.13), satisfy both the associated partial differential equations (2.3), (2.4), and (2.5) and all imposed initial and boundary conditions. In the special cases, when λ_r or $\lambda \to 0$ these solutions reduce to those corresponding to a Maxwell or a second-grade fluid. If both λ_r and $\lambda \to 0$, the solutions corresponding to a Navier-Stokes fluid are obtained. In the case $\lambda > \lambda_r$, the corresponding solutions (3.7), (3.10), and (3.13), as well as those corresponding to a Maxwell fluid (4.4), (4.5), and (4.6), contain sine and cosine terms. This indicates that in contrast with the Newtonian

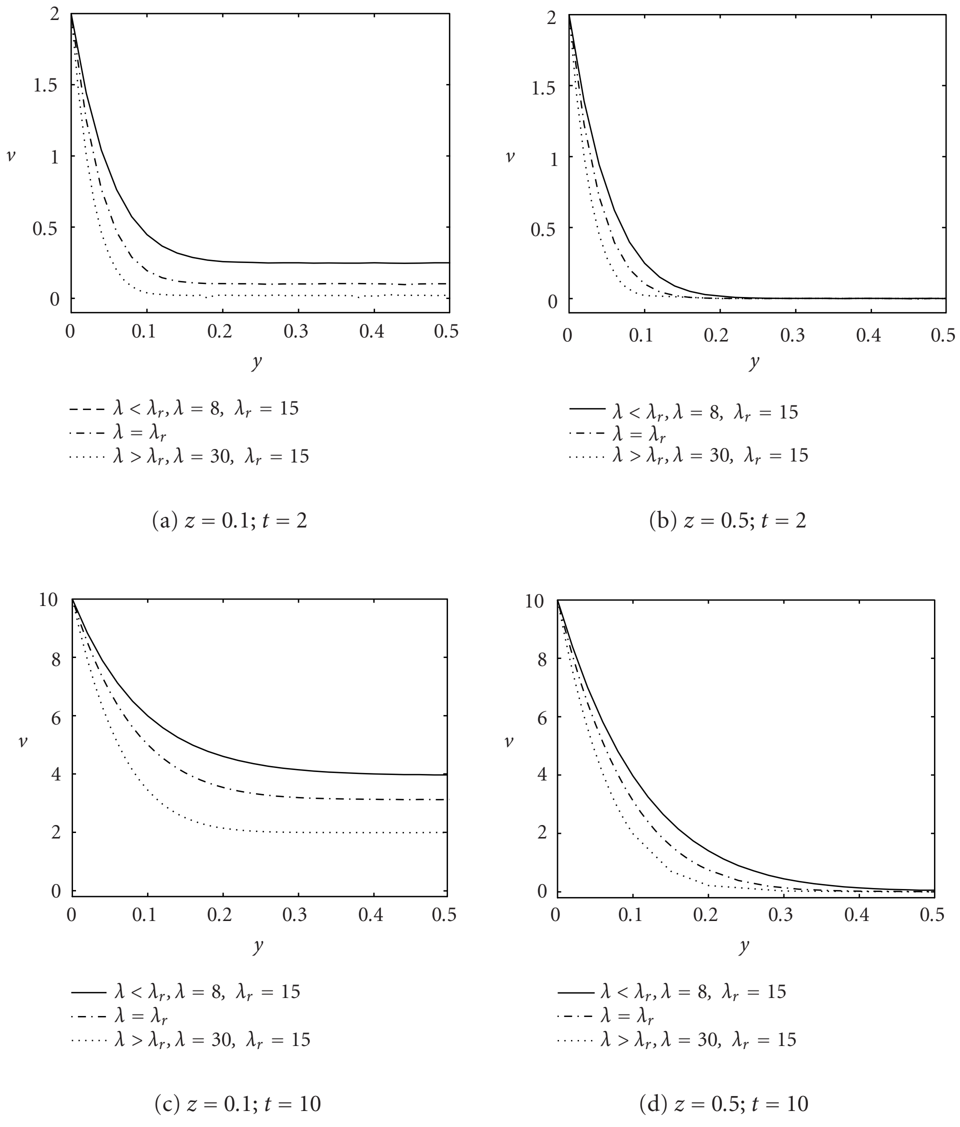


Figure 5.1. Velocity profiles v corresponding to $\lambda < \lambda_r$, $\lambda = \lambda_r$, and $\lambda > \lambda_r$ for $A = 1.0$, $\nu = 0.0011746$, for $z = 0.1$ and $z = 0.5$ cm, and $t = 2$ and 10 s, respectively.

and second grade fluids, whose solutions (3.6), (3.9), (3.12), and (4.1), (4.2), and (4.3) do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

In Figures 5.1 and 5.3, the profiles of the velocity fields $v(y,z,t)$ and the associated tangential stress $\tau_1(y,z,t)$, corresponding to an Oldroyd-B fluid, are plotted as functions of

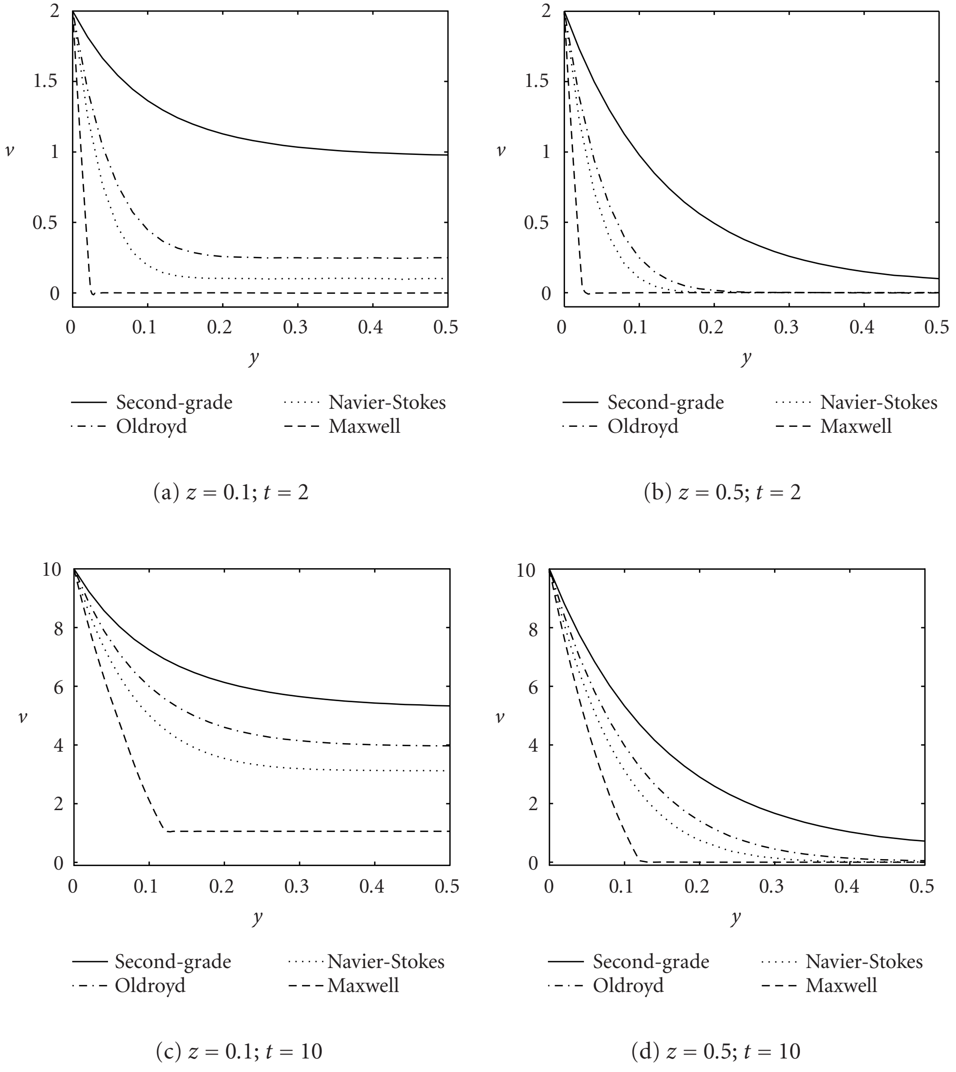


Figure 5.2. Velocity profiles v corresponding to a second-grade, Oldroyd, Navier-Stokes, and Maxwell fluid for $A = 1.0$, $\nu = 0.0011746$, $\lambda_r = 15$ and $\lambda = 8$, for $z = 0.1$ and $z = 0.5$ cm, and $t = 2$ and 10 s, respectively.

y, for $A = 1.0$, $\nu = 0.0011746$, $z = 0.1$, and $z = 0.5$ cm and different values of t and the material constants. It can be seen from the figures that an Oldroyd-B fluid, in such motions, flows faster if the relaxation time λ is smaller than the retardation time λ_r, the corresponding stress being smaller. Figures 5.2 and 5.4 present, for comparison, the variations of the velocity fields $v(y,z,t)$ and the associated tangential stress $\tau_1(y,z,t)$ corresponding to a second-grade, Maxwell, Oldroyd-B, and Navier-Stokes fluid. It is clearly seen from

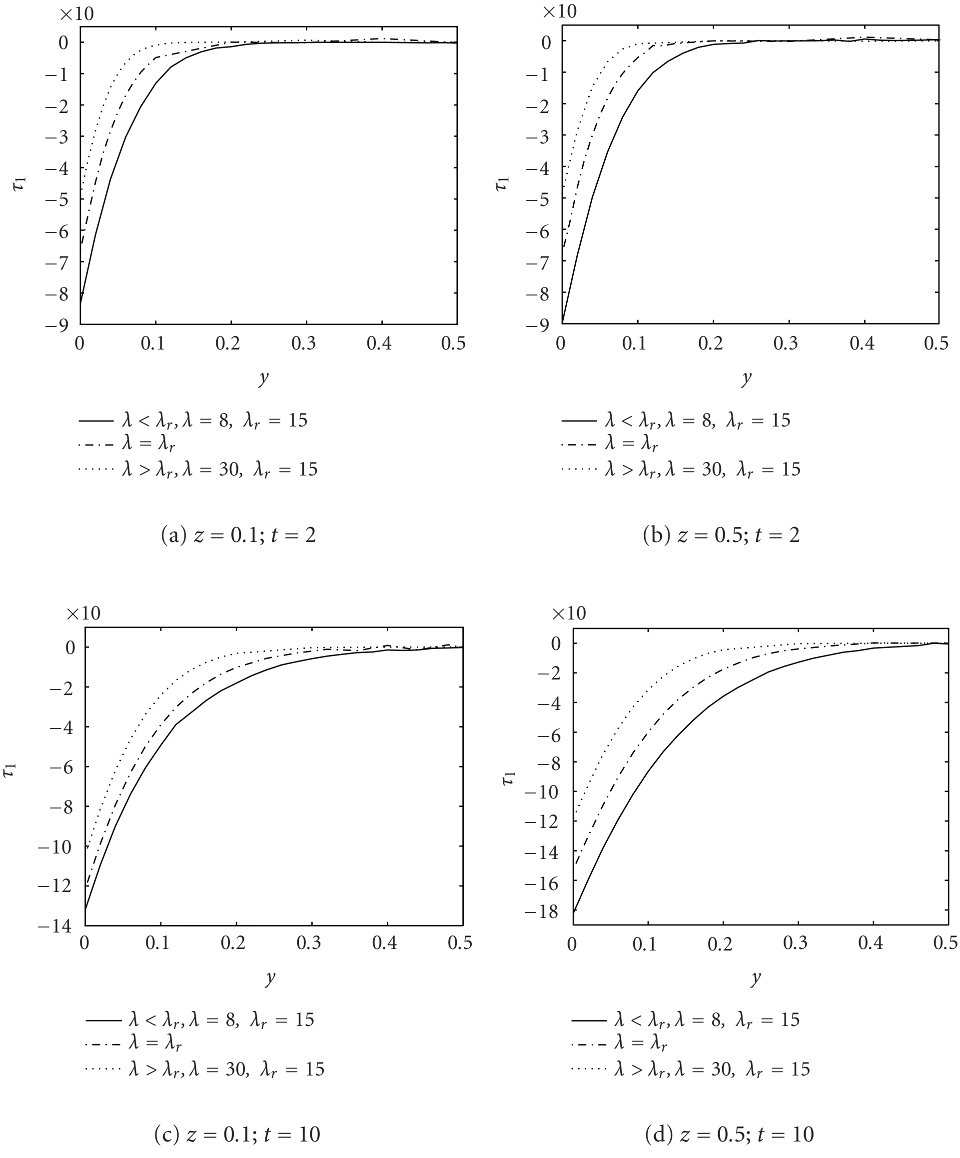
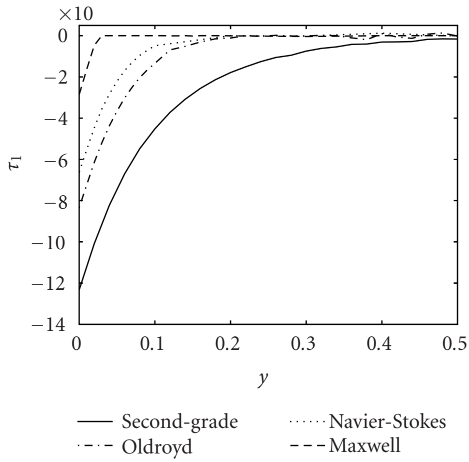
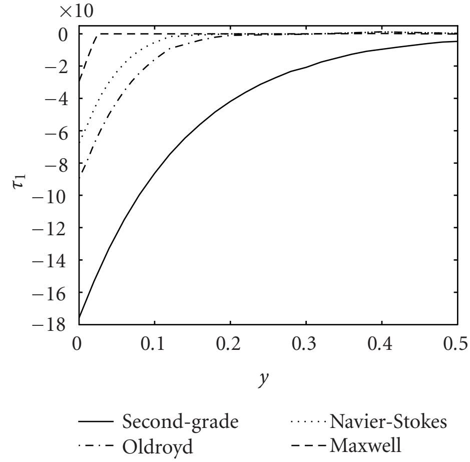


Figure 5.3. Tangential stress τ_1 corresponding to $\lambda < \lambda_r$, $\lambda = \lambda_r$, and $\lambda > \lambda_r$, for $A = 1.0$, $\nu = 0.0011746$, for $z = 0.1$ and $z = 0.5$ cm, and $t = 2$ and 10 s, respectively.

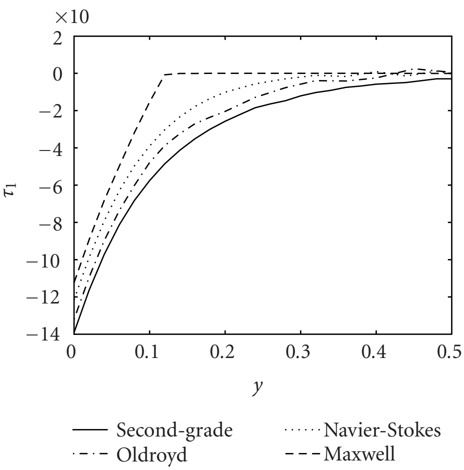
the figures that the second-grade fluid is the swiftest while the Maxwell fluid is the slowest, for the same initial and boundary conditions. For large values of t, the non-Newtonian effects become weak and the profiles for the velocity fields for the four fluids as well as those for the associated stresses are nearly identical.



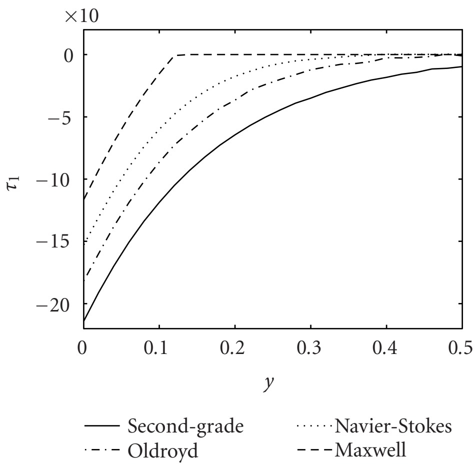
(a) $z = 0.1$; $t = 2$



(b) $z = 0.5$; $t = 2$



(c) $z = 0.1$; $t = 10$



(d) $z = 0.5$; $t = 10$

Figure 5.4. Tangential stress τ_1 corresponding to a second grade, Oldroyd, Navier-Stokes, and Maxwell fluid for $A = 1.0$, $\nu = 0.0011746$, $\lambda_r = 15$ and $\lambda = 8$, for $z = 0.1$ and $z = 0.5$ cm, and $t = 2$ and 10 s, respectively.

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