STONELEY WAVES IN A NON-HOMOGENEOUS ORTHOTROPIC GRANULAR MEDIUM UNDER THE INFLUENCE OF GRAVITY

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The aim of this paper is to investigate the Stoneley waves in a non-homogeneous orthotropic granular medium under the influence of a gravity field. The frequency equation obtained, in the form of a sixth-order determinantal expression, is in agreement with the corresponding result when both media are elastic. The frequency equation when the gravity field is neglected has been deduced as a particular case.

1. Introduction

Problem of Stoneley waves play an important role in the earthquake science, optics, geophysics, and plasma physics. Many authors such as Abd-Alla and Ahmed [1, 2], El-Naggar et al. [8], Das et al. [6], and others studied the effect of gravity of the propagation of surface waves (Stoneley waves, Rayleigh waves, and Love waves) in an elastic solid medium. Goda [9] studied the effect of inhomogeneity and anisotropy on Stoneley waves.

The study of granular medium has been necessiated by its possible application in soil mechanics, geophysical prospecting, mining engineering, and so forth. The theoretical outline of the development of the subject from the mid-1930s was given by Paria [13]. The present paper, however, is based on the dynamics of granular media as propounded by Oshima [11, 12].

The medium under consideration is discontinuous such as one composed numerous large or small grains. Unlike a continuous body, each element or grain cannot only translate but also rotate about its centre of gravity. This motion is the characteristic of the medium and has an important effect upon the equation of motion to produce internal friction. It is assumed that the medium contains so many grains that they will never be separated from each other during the deformation and that the grain has perfect elasticity. The propagation of Rayleigh waves in granular medium was given by many authors such as Bhattacharyya [5], El-Naggar [7], Ahmed [4], and others. In [3], Ahmed discussed the influence of gravity on the propagation of Rayleigh waves in granular medium.

This paper is devoted to the study of the effect of granular body and also of the gravity field in the propagation of Stoneley waves. The wave velocity equation has been derived in the form of a sixth-order determinant. The roots of this equation are in general complex

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and the imaginary part of an appropriate root measures the attenuation of the waves. It is shown that the frequency of Stoneley waves contains terms involving the acceleration due to gravity and so the phase velocity changes with respect to this acceleration due to gravity. When the gravity field is neglected, the frequency equation has been deduced as a particular case. Also when both media are elastic, the frequency equation reduces to the corresponding result obtained by Abd-Alla and Ahmed [2] in the form of a fourth-order determinant.

2. Formulation of the problem

Let M_1 and M_2 be two non-homogeneous orthotropic granular media. They are perfectly welded in contact and are under the influence of gravity. These two media extend to infinitely great distance from the origin and are separated by a plane horizontal boundary and M_2 is to be taken above M_1. Let $Ox_1x_2x_3$ be a set of orthogonal Cartesian coordinates, the origin O being any point on the plane boundary, x_3-axis is vertically downwards into the medium M_1.

We consider the possibility of a type of wave traveling in the direction Ox_1, in such a manner that the disturbance is largely confined to the neighborhood of the boundary which implies that the wave is a surface wave.

Notice that at any instant all particles in any line are parallel to Ox_2, having equal displacement, therefore all partial derivatives with respect to u_2 are zero and there is no propagation of displacement u_2 [2].

The state of deformation in the granular medium is described by the displacement vector $\underline{U}(u_1,0,u_3)$ of the centre of gravity of a grain and the rotation vector $\underline{\xi}(\xi,\eta,\zeta)$ of the grain about its centre of gravity. There exist a stress tensor and a stress couple and are non-symmetric, that is,

$$\tau_{ij} \neq \tau_{ji}, \quad M_{ij} \neq M_{ji}, \quad i = 1,2,3. \tag{2.1}$$

The stress tensor τ_{ij} can be expressed as the sum of symmetric and anti-symmetric tensors

$$\tau_{ij} = \sigma_{ij} + \sigma'_{ij}, \tag{2.2}$$

where

$$\sigma_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ji}), \qquad \sigma'_{ij} = \frac{1}{2}(\tau_{ij} - \tau_{ji}). \tag{2.3}$$

The symmetric tensor $\sigma_{ij} = \sigma_{ji}$ is related to the symmetric strain tensor

$$e_{ij} = e_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

by the Hook's law.

The anti-symmetric stress σ'_{ij} are given by

$$\sigma'_{23} = -F\frac{\partial \xi}{\partial t}, \qquad \sigma'_{31} = -F\frac{\partial \eta}{\partial t}, \qquad \sigma'_{12} = -F\frac{\partial \zeta}{\partial t}, \\ \sigma'_{11} = \sigma'_{22} = \sigma'_{33} = 0, \tag{2.5}$$

where F is the coefficient of fraction.

The stress couple M_{ij} is given by

$$M_{ij} = M\nu_{ij}, \tag{2.6}$$

where M is the third elastic constant,

$$\nu_{11} = \frac{\partial \xi}{\partial x_1}, \qquad \nu_{22} = 0, \qquad \nu_{33} = \frac{\partial \zeta}{\partial x_3}, \qquad \nu_{23} = 0, \\ \nu_{31} = \frac{\partial \xi}{\partial x_3}, \qquad \nu_{12} = \frac{\partial}{\partial x_1}(\eta + \omega_2), \qquad \nu_{32} = \frac{\partial}{\partial x_3}(\eta + \omega_2), \\ \nu_{13} = \frac{\partial \zeta}{\partial x_1}, \qquad \nu_{21} = 0, \tag{2.7}$$

where $\omega_2 = \partial u_1/\partial x_3 - \partial u_3/\partial x_1$.

If g is the acceleration due to gravity, then the components of body forces are $X = 0$, $Z = g$. Assuming that the initial stress field due to gravity is hydrostatic, the states of initial stress τ_{ij} are [10]

$$\begin{aligned} \tau_{ij} &= \tau, \quad i = j, \\ \tau_{ij} &= 0, \quad i \neq j, \end{aligned} \qquad i, j = 1, 2, 3, \tag{2.8}$$

where τ is a function of depth Ox_3 only.

The equilibrium conditions of the initial stress field are [10]

$$\frac{\partial \tau}{\partial x_1} = \frac{\partial \tau}{\partial x_2} = 0, \qquad \frac{\partial \tau}{\partial x_3} + \rho g = 0, \tag{2.9}$$

where ρ is the density of the material medium.

The six equations of motion are [2, 5]

$$\begin{gathered}
\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{13}}{\partial x_3}+\rho g\frac{\partial u_3}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2},\\
\frac{\partial \tau_{12}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_3}=0,\\
\frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \tau_{33}}{\partial x_3}-\rho g\frac{\partial u_1}{\partial x_1}=\rho\frac{\partial^2 u_3}{\partial t^2},\\
\tau_{23}-\tau_{32}+\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{31}}{\partial x_3}=0,\\
\tau_{31}-\tau_{13}+\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{32}}{\partial x_3}=0,\\
\tau_{12}-\tau_{21}+\frac{\partial M_{13}}{\partial x_1}+\frac{\partial M_{33}}{\partial x_3}=0.
\end{gathered} \tag{2.10}$$

These equations, when the stresses are substituted, take the forms

$$\begin{gathered}
\frac{\partial}{\partial x_1}\left[C_{11}\frac{\partial u_1}{\partial x_1}+C_{13}\frac{\partial u_3}{\partial x_3}\right]+\frac{\partial}{\partial x_3}\left[C_{55}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)-F\frac{\partial \eta}{\partial t}\right]+\rho g\frac{\partial u_3}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2},\\
\frac{\partial}{\partial x_1}\left(-F\frac{\partial \zeta}{\partial t}\right)+\frac{\partial}{\partial x_3}\left(F\frac{\partial \xi}{\partial t}\right)=0,\\
\frac{\partial}{\partial x_1}\left[C_{55}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)+F\frac{\partial \eta}{\partial t}\right]+\frac{\partial}{\partial x_3}\left[C_{13}\frac{\partial u_1}{\partial x_1}+C_{33}\frac{\partial u_3}{\partial x_3}\right]-\rho g\frac{\partial u_1}{\partial x_1}=\rho\frac{\partial^2 u_3}{\partial t^2},\\
-F\frac{\partial \xi}{\partial t}+\nabla^2(M\xi)=0,\\
-F\frac{\partial \eta}{\partial t}+\nabla^2\left[M\left(\eta+\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)\right]=0,\\
-F\frac{\partial \zeta}{\partial t}+\nabla^2(M\zeta)=0,
\end{gathered} \tag{2.11}$$

where C_{ij} are elastic constants.

3. Solution of the problem

We assume that the non-homogeneities are of the form

$$C_{ij}=a_{ij}e^{mx_3},\qquad \rho=\rho_0 e^{mx_3},\qquad F=F_0 e^{mx_3},\qquad M=M_0 e^{mx_3}, \tag{3.1}$$

where a_{ij}, ρ_0, F_0, M_0, and m are constants.

Substituting from (3.1) into (2.11), we get

$$\begin{aligned}
&a_{11}\frac{\partial^2 u_1}{\partial x_1^2}+a_{13}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+a_{55}\left(\frac{\partial^2 u_3}{\partial x_1\partial x_3}+\frac{\partial^2 u_1}{\partial x_3^2}\right)\\
&\qquad+m\left[a_{55}\left(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3}\right)-F_0\frac{\partial \eta}{\partial t}\right]-F_0\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x_3}\right)+\rho_0 g\frac{\partial u_3}{\partial x_1}=\rho_0\frac{\partial^2 u_1}{\partial t^2},\\
&\frac{\partial}{\partial t}\left(m\xi+\frac{\partial \xi}{\partial x_3}-\frac{\partial \zeta}{\partial x_1}\right)=0,\\
&a_{55}\left(\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_1\partial x_3}\right)+a_{13}\frac{\partial^2 u_1}{\partial x_1\partial x_3}+a_{33}\frac{\partial^2 u_3}{\partial x_3^2}\\
&\qquad+m\left(a_{31}\frac{\partial u_1}{\partial x_1}+a_{33}\frac{\partial u_3}{\partial x_3}\right)+F_0\frac{\partial}{\partial t}\left(\frac{\partial n}{\partial x_1}\right)-\rho_0 g\frac{\partial u_1}{\partial x_1}=\rho_0\frac{\partial^2 u_3}{\partial t^2},\\
&-F_0\frac{\partial \xi}{\partial t}+M_0\nabla^2\xi+mM_0\frac{\partial \xi}{\partial x_3}=0,\\
&-F_0\frac{\partial \eta}{\partial t}+M_0\nabla^2\left(\eta+\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)+mM_0\frac{\partial}{\partial x_3}\left(\eta+\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)=0,\\
&-F_0\frac{\partial \zeta}{\partial t}+M_0\nabla^2\zeta+mM_0\frac{\partial \zeta}{\partial x_3}=0.
\end{aligned}\tag{3.2}$$

We assume that the displacements u_1 and u_3 are derivable from the displacement potentials $\phi(x_1, x_3, t)$, $\psi(x_1,x_3,t)$ by the relations

$$u_1=\frac{\partial \phi}{\partial x_1}-\frac{\partial \psi}{\partial x_3},\qquad u_3=\frac{\partial \phi}{\partial x_3}+\frac{\partial \psi}{\partial x_1}.\tag{3.3}$$

Substituting from (3.3) into (3.2), we get the following wave equations satisfied by ϕ, ψ, ξ, η, and ζ:

$$a_{11}\frac{\partial^2 \phi}{\partial x_1^2}+(a_{13}+2a_{55})\frac{\partial^2 \phi}{\partial x_3^2}+2ma_{55}\frac{\partial \phi}{\partial x_3}+(ma_{55}+\rho_0 g)\frac{\partial \psi}{\partial x_1}=\rho_0\frac{\partial^2 \phi}{\partial t^2},\tag{3.4}$$

$$\frac{\partial}{\partial t}\left(m\xi+\frac{\partial \xi}{\partial x_3}-\frac{\partial \zeta}{\partial x_1}\right)=0,\tag{3.5}$$

$$a_{55}\frac{\partial^2 \psi}{\partial x_1^2}+(a_{33}-a_{31}-a_{55})\frac{\partial^2 \psi}{\partial x_3^2}+ma_{33}\frac{\partial \psi}{\partial x_3}+(ma_{31}-\rho_0 g)\frac{\partial \phi}{\partial x_1}+F_0\frac{\partial \eta}{\partial t}=\rho_0\frac{\partial^2 \psi}{\partial t^2},\tag{3.6}$$

$$\nabla^2\xi + m\frac{\partial\xi}{\partial x_3} - S_0\frac{\partial\xi}{\partial t} = 0, \tag{3.7}$$

$$\nabla^2\eta + m\frac{\partial\eta}{\partial x_3} - S_0\frac{\partial\eta}{\partial t} - \nabla^4\psi - m\frac{\partial}{\partial x_3}(\nabla^2\psi) = 0, \tag{3.8}$$

$$\nabla^2\zeta + m\frac{\partial\zeta}{\partial x_3} - S_0\frac{\partial\zeta}{\partial t} = 0, \tag{3.9}$$

where $S_0 = F_0/M_0$.

Eliminating η from (3.6) and (3.8), we get

$$\begin{aligned}
&F_0\nabla^4\left(\frac{\partial\psi}{\partial t}\right) + mF_0\frac{\partial^2}{\partial x_3\partial t}(\nabla\psi) \\
&\quad + \left(\nabla^2 + m\frac{\partial}{\partial x_3} - S_0\frac{\partial}{\partial t}\right)\left[a_{55}\frac{\partial^2\psi}{\partial x_1^2} + (a_{33} - a_{31} - a_{55})\frac{\partial^2\psi}{\partial x_3^2}\right. \\
&\qquad\qquad \left. + ma_{33}\frac{\partial\psi}{\partial x_3} + (ma_{31} - \rho_0 g)\frac{\partial\phi}{\partial x_1} - \rho_0\frac{\partial^2\psi}{\partial t^2}\right] = 0.
\end{aligned} \tag{3.10}$$

Assuming that

$$(\phi,\psi) = \{\phi_1(x_3),\psi_1(x_3)\}\exp\{i(Lx_1 - bt)\}, \tag{3.11}$$

$$(\xi,\eta,\zeta) = \{\xi_1(x_3),\eta_1(x_3),\zeta(x_3)\}\exp\{i(Lx_1 - bt)\}. \tag{3.12}$$

Substituting from (3.11) into (3.4) and (3.10), we get

$$((a_{13} + 2a_{55})D^2 + 2ma_{55}D - a_{11}L^2 + \rho_0 b^2)\phi_1 + iL(ma_{55} + \rho_0 g)\psi_1 = 0, \tag{3.13}$$

$$\begin{aligned}
&\{[(a_{33} - a_{31} - a_{55}) - ibF_0]D^4 + m[(2a_{33} - a_{31} - a_{55}) - ibF_0]D^3 \\
&\quad - [(L^2a_{55} - \rho_0 b^2 + m^2 a_{33} - 2ibLF_0) + (L^2 - ibs_0)(a_{33} - a_{31} - a_{55})]D^2 \\
&\quad - m((L^2a_{55} - \rho_0 b^2) + a_{33}(L^2 - ibS_0) - ibF_0L^2)D \\
&\quad + ((L^2 - ibS_0)(L^2a_{55} - \rho_0 b^2) - ibF_0L^4)\}\psi_1 \\
&\quad - iL(\rho_0 g - ma_{31})(D^2 + mD - (L^2 - ibS_0))\phi_1 = 0,
\end{aligned} \tag{3.14}$$

where $D \equiv d/dx_3$.

Equations (3.13) and (3.14) must have exponential solutions in order that ϕ_1, ψ_1 will describe surface waves; they must become vanishingly small as $x_3 \to \infty$. Hence, for the medium M_1,

$$\phi_1 = A_j e^{-\lambda_j x_3}, \tag{3.15}$$

$$\psi_1 = B_j e^{-\lambda_j x_3}, \quad (j = 3,4,5), \tag{3.16}$$

where the constants A_j are related with the constants B_j, respectively, by means of (3.13).

Equating the coefficients of the exponentials $e^{-\lambda_j x_3}$ $(j = 3,4,5)$ to zero and using (3.13) and (3.14), we have

$$A_j = n_j B_j, \tag{3.17}$$

where

$$n_j = \frac{-iL(\rho_0 g + m a_{55})}{(a_{13}+2a_{55})\lambda_j^2 - 2m a_{55}\lambda_j + \rho_0 b^2 - a_{11}L^2}, \quad (j = 3,4,5), \tag{3.18}$$

λ_3, λ_4, λ_5 are the roots which have a positive real part of the equation

$$\begin{aligned}
&k_0\lambda^6 + k_1\lambda^5 + k_2\lambda^4 + k_3\lambda^3 + k_4\lambda^2 + k_5\lambda + k6 = 0,\\
&\quad k_0 = (a_{13}+2a_{55})\left[(a_{33}-a_{31}-a_{55}) - ibF_0\right],\\
&\quad k_1 = -m\left\{(a_{13}+4a_{55})\left[(a_{33}-a_{31}-a_{55}) - ibF_0\right] + a_{33}(a_{13}+2a_{55})\right\},\\
&\quad k_2 = \left[(a_{33}-a_{31}-a_{55}) - ibF_0\right](\rho_0 b^2 - a_{11}L^2 + 2m^2 a_{55}) + 2m^2 a_{33}a_{55}\\
&\qquad\quad - (a_{13}+2a_{55})\left(L^2 a_{55} - \rho_0 b^2 + m^2 a_{33} - 2ibL^2F_0 + (L^2 - ibS_0)(a_{33}-a_{31}-a_{55})\right),\\
&\quad k_3 = -m\left\{(\rho_0 b^2 - a_{11}L^2)\left[(2a_{33}-a_{31}-a_{55}) - ibF_0\right]\right.\\
&\qquad\qquad - 2a_{55}\left(L^2 a_{55} - \rho_0 b^2 + m^2 a_{33} - 2ibL^2F_0 + (L^2 - ibS_0)(a_{33}-a_{31}-a_{55})\right)\\
&\qquad\qquad \left.- (a_{13}+2a_{55})\left(L^2 a_{55} - \rho_0 b^2 - ibL^2F_0 + a_{33}(L^2 - ibS_0)\right)\right\},\\
&\quad k_4 = (a_{11}L^2 - \rho_0 b^2)\left(L^2 a_{55} - \rho_0 b^2 + m^2 a_{33} - 2ibL^2F_0 + (L^2 - ibS_0)(a_{33}-a_{31}-a_{55})\right)\\
&\qquad\quad - 2m^2 a_{55}\left[(L^2 a_{55} - \rho_0 b^2) + a_{33}(L^2 - ibS_0) - ibF_0L^2\right]\\
&\qquad\quad - (a_{13}+2a_{55})\left[(L^2 - ibS_0)(\rho_0 b^2 - L^2 a_{55}) + ibF_0L^4\right]\\
&\qquad\quad + L^2(m a_{55} + \rho_0 g)(m a_{31} - \rho_0 g),\\
&\quad k_5 = m\{L^2 a_{55} - \rho_0 b^2 + a_{33}(L^2 - ibS_0) - ibF_0L^2\\
&\qquad\qquad + 2a_{55}\left((L^2 - ibS_0)(\rho_0 b^2 - L^2 a_{55}) + ibF_0L^4\right) - L^2(m a_{55} + \rho_0 g)(m a_{31} - \rho_0 g)\},\\
&\quad k_6 = (a_{11}L^2 - \rho_0 b^2)\left((L^2 - ibS_0)(\rho_0 b^2 - a_{55}L^2) + ibF_0L^4\right)\\
&\qquad\quad - L^2(m a_{55} + \rho_0 g)(m a_{31} - \rho_0 g)(L^2 - ibS_0).
\end{aligned} \tag{3.19}$$

Using (3.8), (3.11), (3.12), and (3.16), one gets

$$\eta_1 = \Omega_j\left(B_j e^{-\lambda_j x_3}\right), \tag{3.20}$$

where

$$\Omega_j = \frac{\lambda_j^4 - m\lambda_j^3 - 2L^2\lambda_j^4 + mL^2\lambda_i + L^4}{\lambda_j^2 - m\lambda_i + ibS_0 - L^2}. \tag{3.21}$$

Also, substituting from (3.12) into (3.5), (3.7), and (3.9), we get

$$(D+m)\xi_1 - iL\zeta_1 = 0, \tag{3.22}$$

$$(D^2 + mD + h^2)\xi_1 = 0, \tag{3.23}$$

$$(D^2 + mD + h^2)\zeta_1 = 0, \tag{3.24}$$

where $h^2 = ibS_0 - L^2$.

The solutions of (3.23) and (3.24) are

$$\xi_1 = A_2 e^{-ih_2x_3}, \qquad \zeta_1 = B_2 e^{-ih_2x_3}, \tag{3.25}$$

where $h_2 = (-m + \sqrt{m^2 - 4h^2})/2$.

From (3.25) and (3.22), one can obtain

$$A_2 = \frac{-L}{h_2 + im} B_2. \tag{3.26}$$

We use the symbols with a bar for the upper medium (except x_3, L, b, g) and the functions $\bar{\xi}_1$, $\bar{\zeta}_1$, $\bar{\eta}_1$, $\bar{\phi}_1$, and $\bar{\psi}_1$ must vanish as $x \to -\infty$.

For the upper medium M_2, we have

$$\begin{gathered} \bar{\xi}_1 = \bar{A}_2 e^{i\bar{h}_2x_3}, \qquad \bar{\zeta}_1 = \bar{B}_2 e^{i\bar{h}_2x_3}, \\ \bar{\eta}_1 = \bar{\Omega}_j \bar{B}_j e^{\bar{\lambda}_j x_3}, \quad \bar{\phi}_1 = \bar{A}_j e^{\bar{\lambda}_j x_3}, \quad \bar{\psi}_1 = \bar{B}_j e^{\bar{\lambda}_j x_3} \quad (j = 3,4,5). \end{gathered} \tag{3.27}$$

4. Boundary conditions and frequency equation

The boundary conditions on the interface $x_3 = 0$ are

(i) $u_1 = \bar{u}_1$,
(ii) $u_3 = \bar{u}_3$,
(iii) $\xi = \bar{\xi}$,
(iv) $\eta = \bar{\eta}$,
(v) $\zeta = \bar{\zeta}$,
(vi) $M_{33} = \bar{M}_{33}$,
(vii) $M_{31} = \bar{M}_{31}$,
(viii) $M_{32} = \bar{M}_{32}$,
(ix) $\tau_{33} = \bar{\tau}_{33}$,
(x) $\tau_{31} = \bar{\tau}_{31}$,
(xi) $\tau_{32} = \bar{\tau}_{32}$,

where

$$M_{33}=M\frac{\partial\zeta}{\partial x_3},\qquad M_{32}=M\frac{\partial}{\partial x_3}(\eta-\nabla^2\psi),\qquad M_{31}=M\frac{\partial\xi}{\partial x_3},$$
$$\tau_{33}=C_{13}\frac{\partial^2\phi}{\partial x_1^2}+C_{33}\frac{\partial^2\phi}{\partial x_3^2}+(C_{33}-C_{13})\frac{\partial^2\psi}{\partial x_1\partial x_3},\tag{4.1}$$
$$\tau_{32}=-F\frac{\partial\xi}{\partial t},\qquad \tau_{31}=C_{55}\Big(\frac{\partial^2\psi}{\partial x_1^2}-\frac{\partial^2\psi}{\partial x_3^2}+2\frac{\partial^2\phi}{\partial x_1\partial x_3}\Big)-F\frac{\partial\eta}{\partial t}.$$

From the boundary conditions (iii), (v), (vi), and (vii), we get

$$A_2=\bar{A}_2,\qquad B_2=\bar{B}_2,\qquad h_2M_0=-\bar{h}_2\bar{M}_0,\tag{4.2}$$

whence $A_2=\bar{A}_2=B_2=\bar{B}_2=0$, $\xi=\zeta=\bar{\xi}=\bar{\zeta}=0$.

The other significant boundary conditions are responsible for the following relations:

(i) $(iLn_j+\lambda_j)B_j=(iL\bar{n}_j-\bar{\lambda}_j)\bar{B}_j$,

(ii) $(iL-n_j\lambda_j)B_j=(iL+\bar{n}_j\bar{\lambda}_j)\bar{B}_j$,

(iv) $\Omega_jB_j=\bar{\Omega}_j\bar{B}_j$,

(viii) $M_0[(L^2+\Omega_j)\lambda_j-\lambda_j^3]B_j=\bar{M}_0[-(L^2+\bar{\Omega}_j)\bar{\lambda}_j+\bar{\lambda}_j^3]\bar{B}_j$,

(ix) $[(a_{33}\lambda_j^2-a_{13}L^2)n_j-iL(a_{33}-a_{13})\lambda_i]B_j=[(\bar{a}_{33}\bar{\lambda}_j^2-\bar{a}_{13}L^2)\bar{n}_j+iL(\bar{a}_{33}-\bar{a}_{13})\bar{\lambda}_i]\bar{B}_j$,

(x) $[a_{55}(L^2+\lambda_j^2+2iLn_j\lambda_j)-ibF_0\Omega_j]B_j=[\bar{a}_{55}(L^2+\bar{\lambda}_j^2-2iL\bar{n}_j\bar{\lambda}_j)-ib\bar{F}_0\bar{\Omega}_j]\bar{B}_j$.

Eliminating the constants B_j, $\bar{B}_j$ $(j=3,4,5)$, we obtain the wave velocity equation in the form of a sixth-order determinantal equation,

$$\begin{vmatrix} iLn_3+\lambda_3 & iLn_4+\lambda_4 & iLn_5+\lambda_5 & iL\bar{n}_3-\bar{\lambda}_3 & iL\bar{n}_4-\bar{\lambda}_4 & iL\bar{n}_5-\bar{\lambda}_5\\ iL-n_3\lambda_3 & iL-n_4\lambda_4 & iL-n_5\lambda_5 & iL+\bar{n}_3\bar{\lambda}_3 & iL+\bar{n}_4\bar{\lambda}_4 & iL+\bar{n}_5\bar{\lambda}_5\\ \Omega_3 & \Omega_4 & \Omega_5 & \bar{\Omega}_3 & \bar{\Omega}_4 & \bar{\Omega}_5\\ Q_{13} & Q_{14} & Q_{15} & \bar{Q}_{13} & \bar{Q}_{14} & \bar{Q}_{15}\\ Q_{23} & Q_{24} & Q_{25} & \bar{Q}_{23} & \bar{Q}_{24} & \bar{Q}_{25}\\ Q_{33} & Q_{34} & Q_{35} & \bar{Q}_{33} & \bar{Q}_{34} & \bar{Q}_{35}\end{vmatrix}=0,\tag{4.3}$$

where

$$\begin{aligned} Q_{1j}&=M_0(L^2+\Omega_j-\lambda_j^2)\lambda_j,\\ Q_{2j}&=n_j(a_{33}\lambda_j^2-a_{13}L^2)-iL(a_{33}-a_{13})\lambda_j,\\ Q_{3j}&=a_{55}(L^2+\lambda_j^2+2iLn_j\lambda_j)-ibF_0\Omega_j,\\ \bar{Q}_{1j}&=-\bar{M}_0(L^2+\bar{\Omega}_j-\bar{\lambda}_j^2)\bar{\lambda}_j,\\ \bar{Q}_{2j}&=iL(a_{33}-a_{13})\bar{\lambda}_j-\bar{n}_j(a_{13}L^2-a_{33}\bar{\lambda}_j^2),\\ \bar{Q}_{3j}&=\bar{a}_{55}(L^2-2iL\bar{n}_j\bar{\lambda}_j+\bar{\lambda}_j^2)-ib\bar{F}_0\bar{\Omega}_j,\quad j=3,4,5.\end{aligned}\tag{4.4}$$

Equation (4.3) is the frequency equation of Stoneley waves in a non-homogeneous orthotropic granular medium under the influence of gravity, this equation depends on the particular values of λ_j and $\bar{\lambda}_j$ creating a dispersion of the general wave form. Moreover, the wave velocity C $(= b/L)$ depends on the gravity field, the non-homogeneous of the material medium and the granular rotations.

From (3.18), (3.19), and (4.3), we can assert that when L is Large, so that the length of the wave is small, the effect of gravity is sufficiently small, that is, the wave length of the wave is large, the effect of gravity is no longer negligible and plays an important role on the determination of the wave velocity C.

If we neglect the gravity field, we obtain the wave velocity equation for Stoneley waves in a non-homogeneous orthotropic granular medium which is the same equation as (4.3) with

$$n_j = \frac{-imLa_{55}}{(a_{13}+2a_{55})\lambda_i^2 - 2ma_{55}\lambda_i + \rho_0 b^2 - a_{11}L^2}, \quad (j = 3,4,5), \tag{4.5}$$

where λ_j are the roots of the equation

$$\begin{aligned}
&k_0\lambda^6 + k_1\lambda^5 + k_2\lambda^4 + k_3\lambda^3 + k_4'\lambda^2 + k_5'\lambda + k_6' = 0,\\
&\quad k_4' = (a_{11}L^2 - \rho_0 b^2)\left[L^2 a_{55} - \rho_0 b^2 + m^2 a_{33} - 2ibL^2F_0 + (L^2 - ibs_0)(a_{33} - a_{31} - a_{55})\right]\\
&\qquad\quad - 2m^2 a_{55}\left[(L^2 a_{55} - \rho_0 b^2) + a_{33}(L^2 - ibs_0) - ibF_0L^2\right]\\
&\qquad\quad - (a_{13}+2a_{55})\left[(L^2 - ibs_0)(\rho_0 b^2 - L^2 a_{55}) + ibF_0L^4\right] + m^2L^2a_{55}a_{31},\\
&\quad k_5' = m\{L^2 a_{55} - \rho_0 b^2 + a_{33}(L^2 - ibs_0) - ibF_0L^2\\
&\qquad\qquad + 2a_{55}\left[(L^2 - ibs_0)(\rho_0 b^2 - L^2 a_{55}) + ibF_0L^4\right] - m^2L^2a_{55}a_{31}\},\\
&\quad k_6' = (a_{11}L^2 - \rho_0 b^2)\left[(L^2 - ibs_0)(\rho_0 b^2 - a_{55}L^2) + ibF_0L^4\right] - m^2L^2a_{55}a_{31}(L^2 - ibs_0).
\end{aligned} \tag{4.6}$$

When both media are elastic ($M_0 = 0$, $F_0 = 0$), by using (3.4) and (3.6); (3.18) becomes

$$n_j = \frac{-iL(ma_{55} + \rho_0 g)}{(a_{13}+2a_{55})\lambda_j^2 - 2ma_{55}\lambda_j - a_{11}L^2 + \rho_0 b^2}, \quad (j = 3,4), \tag{4.7}$$

λ_j are the real roots of the equation

$$\begin{aligned}
&(a_{13}+2a_{55})(a_{33} - a_{31} - a_{55})\lambda^4\\
&\quad - m\left[2a_{55}(a_{33} - a_{31} - a_{55}) + a_{33}(a_{13}+2a_{55})\right]\lambda^3\\
&\quad + \left[(\rho_0 b^2 - a_{11}L^2)(a_{33} - a_{31} - a_{55}) + (a_{13}+2a_{55})(\rho_0 b^2 - a_{55}L^2) + 2m^2a_{55}a_{33}\right]\lambda^2\\
&\quad + mL^2\left[a_{33}(a_{11} - \rho_0 b^2) + (a_{13}+2a_{55})(a_{55} - \rho_0 b^2)\right]\lambda\\
&\quad + L^2\left[(\rho_0 b^2 - a_{11})(\rho_0 b^2 - a_{55}) + (ma_{31} - \rho_0 g)(ma_{55} + \rho_0 g)\right] = 0,
\end{aligned} \tag{4.8}$$

and the frequency equation (4.3) takes the form

$$\begin{vmatrix} iLn_3+\lambda_3 & iLn_4+\lambda_4 & iL\bar{n}_3-\bar{\lambda}_3 & iL\bar{n}_4-\bar{\lambda}_4 \\ iL-n_3\lambda_3 & iL-n_4\lambda_4 & iL+\bar{n}_3\bar{\lambda}_3 & iL+\bar{n}_4\bar{\lambda}_4 \\ Q_{23} & Q_{24} & \bar{Q}_{23} & \bar{Q}_{24} \\ Q_{33} & Q_{34} & \bar{Q}_{33} & \bar{Q}_{34} \end{vmatrix} = 0. \tag{4.9}$$

Equation (4.9) determines the wave velocity equation for Stoneley wave in a non-homogeneous orthotropic elastic medium under the influence of gravity and is in complete agreement with that obtained by Abd-Alla and Ahmed [2].

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