# ON VARIANTS OF A THEOREM OF SCHUR

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We provide improved versions of the statements and proofs of the two main theorems of Hilton (2001), together with the statement and proof of a new theorem on the localization of nilpotent groups.

## 1. Introduction

In a paper [1] published in 2001, we modified a famous theorem of Issai Schur, which asserts that if $G$ is a group with center $Z$, such that $G/Z$ is finite, then the commutator subgroup $G' = [G,G]$ is also finite. Our modification was twofold; in the first place, we confined ourselves to nilpotent groups $G$, so that we could use effective localization methods at an arbitrary family $P$ of primes, and, second, we relativized the situation by replacing $G$ by a pair of groups $(G,N)$, where $N$ is a normal subgroup of $G$. Then $Z$ was replaced by the centralizer $C_G(N)$ of $N$ in $G$, and $[G,G]$ was replaced by $[G,N]$.

We also considered in [1] a partial converse of Schur's theorem and its modification. In this partial converse, we showed that $G/Z$ is finite if $G'$ is finite, provided that $G$ is finitely generated (fg). In talks the author has given on this topic, he has expressed the opinion that the converse would not hold without some supplementary hypothesis. This remark was taken up by Dr. Edwin Clark of the University of South Florida, who raised the question on sci.math.research, and quickly received negative answers from Derek Holt and Andreas Caronti, whose counterexamples were very similar, being infinite extraspecial $p$-groups. Thus Holt considered a group $G$ with generators $x_i$, $y_i$, $i>0$, and $z$, subject to the relations $x_i^p = y_i^p = z^p = 1$, $[x_i,x_j] = [y_i,y_j] = 1$, and $[x_i,y_i] = z$, $[x_i,y_j] = 1$, $i \neq j$, and $[z,x_i] = [z,y_i] = 1$, for all $i$. Then $Z = G' = \langle z\rangle$ is finite, but $G/Z$ is infinite. The present author is very grateful to Edwin Clark for providing this elucidation.

In Section 2, we provide improved versions of the proofs of the two main theorems of [1], namely, if $P$ is a family of primes with complementary family $Q$, then Theorems 2.1 and 2.3 hold.

It is a striking fact, not brought out in [1], that, whereas the proof of Theorem 2.1 leans heavily on the $P$-localization theory of nilpotent groups, the proof of Theorem 2.3 does not use localization methods.


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In Section 3, we prove a localization theorem which is closely related to the arguments involved in proving Theorem 2.1.

We close the introduction by remarking that the assertions of Theorems 2.1 and 2.3 make good sense because, indeed, $C_G(N)$ is a normal subgroup of $G$. To see this, let $x \in C_G(N)$, $y \in G$, $z \in N$. We want to show that $yxy^{-1} \in C_G(N)$. Now

$$yxy^{-1}zyx^{-1}y^{-1} = yx(y^{-1}zy)x^{-1}y^{-1}. \tag{1.1}$$

But $y^{-1}zy \in N$ since $N$ is normal in $G$, so $x(y^{-1}zy)x^{-1} = y^{-1}zy$, and $yxy^{-1}zyx^{-1}y^{-1} = yx(y^{-1}zy)x^{-1}y^{-1} = y(y^{-1}zy)y^{-1} = z$, whence $yxy^{-1} \in C_G(N)$.

## 2. The main theorems

Theorem 2.1. *If $G/C_G(N)$ is a $Q$-group, then $[G,N]$ is a $Q$-group.*

*Proof.* We $P$-localize. Of course $e : G \to G_P$ kills $G/C_G(N)$, a $Q$-group; but localization preserves quotients, so

$$C_G(N)_P = G_P. \tag{2.1}$$

We next prove a general result on the localization of nilpotent groups (which, of course, does not require the hypothesis of Theorem 2.1). We state it as a lemma.

Lemma 2.2. *$C_{G_P}(N_P)$ is $P$-local, and $C_G(N)_P \subseteq C_{G_P}(N_P) \subseteq G_P$.*

*Proof of Lemma 2.2.* We first prove that, if $x \in C_G(N)$, $y \in N_P$, then $ex$ commutes with $y$. For $y^q = ez$ for some $Q$-number $q$, with $z \in N$. Thus $(ex)y^q(ex^{-1}) = e(xzx^{-1}) = ez = y^q$, since $x$ centralizes $N$. Since $q$th roots are unique in $G_P$, $(ex)y(ex^{-1}) = y$, as claimed.

We next prove that if $u \in C_G(N)_P$, then $y$ commutes with $u$. For there exists a $q$-number $k$ such that $u^k = ex$ for $x \in C_G(N)$. Thus $yu^ky^{-1} = u^k$, whence, by the uniqueness of $k$th roots in $G_P$, $yuy^{-1} = u$, so $y$ commutes with $u$.

Now, of course, $u \in G_P$ and $u$ centralizes $N_P$. Thus

$$C_G(N)_P \subseteq C_{G_P}(N_P) \subseteq G_P. \tag{2.2}$$

It remains to prove that $C_{G_P}(N_P)$ is $P$-local. Let $u \in C_{G_P}(N_P)$ and let $q$ be a $Q$-number. Then $u = v^q$, $v \in G_P$, and, for all $y \in N_P$, $yuy^{-1} = u$. Hence $yv^qy^{-1} = v^q$ and, taking unique $q$th roots in $G_P$, $yvy^{-1} = v$, so that $v \in C_{G_P}(N_P)$, completing the proof of Lemma 2.2. □

We revert to the proof of Theorem 2.1. From (2.1) and (2.2), we know that $C_{G_P}(N_P) = G_P$, so that $[G_P,N_P] = 1$. Thus the $P$-localization of $[G,N]$ is trivial, so that $[G,N]$ is a $Q$-group, as claimed. □

We turn now to Theorem 2.3.

Theorem 2.3. *If $[G,N]$ is a $Q$-group of exponent $m$, then $G/C_G(N)$ is a $Q$-group of exponent dividing $m^{c-1}$, where* $\operatorname{nil} G = c$.

*Proof.* We adopt the proof strategy of [1] up to the conclusion of [1, Lemma 2.3], which asserts that, if $a, b \in G$, where $b^m = 1$, $\operatorname{nil} G = c$, and if $b \in \Gamma^j G$, the $j$th term of the lower central series of $G$, then

$$(ab)^{m^{c-j}} = a^{m^{c-j}}. \tag{2.3}$$

(Here, by contrast with [1], we adopt the convention that $\Gamma^0(G) = G$, $\Gamma^{i+1}(G) = [G, \Gamma^i(G)]$, $i \geq 0$.) To complete the proof of Theorem 2.3, we consider the commutator $[x, y] = x^{-1}y^{-1}xy$; $x \in G$, $y \in N$. By hypothesis, $[x, y]^m = 1$ for some $Q$-number $m$, independent of $x$ and $y$. Then $x[x, y] = y^{-1}xy$, so that by (2.3),

$$\left(y^{-1}xy\right)^{m^{c-1}} = x^{m^{c-1}}, \tag{2.4}$$

since $[x, y] \in \Gamma^1 G$. Thus

$$y^{-1}x^{m^{c-1}}y = x^{m^{c-1}}, \tag{2.5}$$

so that, for all $x \in G$, $x^{m^{c-1}} \in C_G(N)$. This completes the proof that $G/C_G(N)$ has exponent dividing $m^{c-1}$. But $m$ is a $Q$-number so that $G/C_G(N)$ is a $Q$-group. $\square$

Notice that no localization arguments have been used in the proof of Theorem 2.3.

## 3. A related theorem

In this section, we prove a theorem which is, in fact, a stronger form of [1, Theorem 2.4]. We consider the following commutative diagram of nilpotent groups and homomorphisms:

$$\begin{tikzcd} \overline{G} \arrow[r, "\overline{\varphi}"] \arrow[d, "\alpha"] & \overline{H} \arrow[d, "\beta"] \\ G \arrow[r, "\varphi"] & \overline{G} \end{tikzcd} \tag{3.1}$$

Theorem 3.1. *Suppose, in (3.1), that $\varphi$ is $P$-bijective and $\alpha$, $\beta$ are $P$-injective. Then $\overline{\varphi}$ is $P$-bijective if and only if $\varphi$ has the following property $\Pi$: for all $x \in G$ such that $\varphi x \in \operatorname{im}\beta$, there exists a $Q$-number $n$ such that $x^n \in \operatorname{im}\alpha$.*

*Proof.* We first remark that since $\varphi\alpha$ is $P$-injective, $\beta\overline{\varphi}$ is $P$-injective, so $\overline{\varphi}$ is $P$-injective, with no reference to property $\Pi$. Thus we only have to prove that $\overline{\varphi}$ is $P$-surjective if and only if $\varphi$ has property $\Pi$.

Suppose that $\overline{\varphi}$ is $P$-surjective, and let $x \in G$ with $\varphi x = \beta y$, $y \in \overline{H}$. Then there exist a $Q$-number $m$ and an element $z \in \overline{G}$ with $\overline{\varphi} z = y^m$. Then

$$\varphi x^m = \beta y^m = \beta\overline{\varphi} z = \varphi\alpha z. \tag{3.2}$$

But $\varphi$ is $P$-injective, so $x^m = (\alpha z)u$, where $u^\ell = 1$ for some $Q$-number $\ell$. Suppose that $\operatorname{nil} G = c$. Then (see [1, Lemma 2.2]) $x^{m\ell^c} = \alpha z^{\ell^c}$. However, $m\ell^c$ is a $Q$-number, so $\varphi$ has property $\Pi$.

Suppose now, conversely, that $\varphi$ has property $\Pi$ and let $y \in \overline{H}$. Then, since $\varphi$ is $P$-surjective, there exist a $Q$-number $m$ and an element $x \in G$ such that

$$\varphi x = \beta y^n. \tag{3.3}$$

Since $\varphi$ has property $\Pi$, we infer that there is a $Q$-number $\ell$ such that $x^\ell \in \operatorname{im}\alpha$; that is to say, $x^\ell = \alpha z$, for some $z \in \overline{G}$. Then

$$\beta y^{n\ell} = \varphi x^\ell = \varphi\alpha z = \beta\overline{\varphi}z. \tag{3.4}$$

But $\beta$ is $P$-injective, so $y^{n\ell} = (\overline{\varphi}z)v$, where $v^k = 1$, for some $Q$-number $k$. Suppose that $\operatorname{nil}\overline{H} = \overline{c}$. Then (again by [1, Lemma 2.2]) $y^{n\ell k^{\overline{c}}} = \overline{\varphi}z^{k^{\overline{c}}}$. But $n\ell k^{\overline{c}}$ is a $Q$-number, so $\overline{\varphi}$ is $P$-surjective. □

Corollary 3.2. *Under the hypotheses of Theorem 3.1, if $\alpha$ is $P$-surjective, then $\overline{\varphi}$ is $P$-bijective.*

*Proof.* For $\varphi$ certainly has property $\Pi$ since, for *all* $x \in G$, there exists a $Q$-number $n$ such that $x^n \in \operatorname{im}\alpha$. □

Note, finally, that under the hypotheses of Theorem 2.1 (or, of course, Theorem 2.3), the diagram

$$\begin{tikzcd} C_G(N) \arrow[r, hook] \arrow[d, "e"] & G \arrow[d, "e"] \\ C_G(N)_P \arrow[r, equal] & G_P \end{tikzcd} \tag{3.5}$$

is a very special case of diagram (3.1).

## References

[1] P. Hilton, *On a theorem of Schur*, Int. J. Math. Math. Sci. **28** (2001), no. 8, 455–460.

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