ANNIHILATORS ON WEAKLY STANDARD BCC-ALGEBRAS

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In a recent paper the authors presented a new construction of BCC-algebras derived from posets with the top element 1. Resulting BCC-algebras, called weakly standard, are those for which every 4-element subset containing 1 is a subalgebra. In this paper we continue our investigations focusing on the properties of their lattices of congruence kernels.

1. Introduction

BCK-algebras, introduced in the 1960s by Imai and Iséki [3], form a well-known class of algebras intensively studied in algebraic logic during the last decades. The reason of their introduction was practically twofold: they describe general properties of algebras of sets with the set subtraction as a binary operation, and secondly, they form a natural generalization of algebraic counterparts of implicational reducts of several kinds of logics. Namely, the class of implication algebras, introduced by Abbott when describing algebraic properties of the logical connective implication in a classical propositional logic, is a proper subclass of BCK-algebras. Bounded commutative BCK-algebras are known to be equivalent to a class of MV-algebras, arising in a fuzzy logic.

In connection with a problem whether the class of BCK-algebras forms a variety, Komori [4] introduced and studied a wider class of BCC-algebras.

We start with the following axiomatic system.

Definition 1.1. An algebra $(A,\bullet,1)$ of type $(2,0)$ is a *BCC-algebra* if it satisfies the following identities:

(BCC1) $(x\bullet y)\bullet[(z\bullet x)\bullet(z\bullet y)]=1$,

(BCC2) $x\bullet x=1$,

(BCC3) $x\bullet 1=1$,

(BCC4) $1\bullet x=x$,

(BCC5) $(x\bullet y=1 \;\&\; y\bullet x=1)\Rightarrow x=y$.

BCK-algebras are just those BCC-algebras which satisfy the axiom of exchange

$$x\bullet(y\bullet z)=y\bullet(x\bullet z). \tag{E}$$

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Table 1.1

$\bullet$	1	x	y	z
1	1	x	y	z
x	1	1	z	1
y	1	z	1	1
z	1	x	y	1

It is well known that the axioms of BCC-algebras allow to define a *natural ordering* on the base set as follows:

$$x \leq y \quad \text{iff } x \bullet y = 1. \tag{1.1}$$

Moreover, substituting $x = 1$ into (BCC1), one gets

$$y \leq z \bullet y \tag{1.2}$$

for each $y, z \in A$.

BCC-algebras $(A, \bullet, 1)$, in which every subset containing the element 1 is a subalgebra, are called *standard* [1]. A representation theorem for standard BCC-algebras was presented in [1].

In [2], a condition on subalgebras of a BCC-algebra was weakened as follows: we considered those BCC-algebras $\mathcal{A} = (A, \bullet, 1)$ having the property

$$x \bullet y = z \Longrightarrow \{1, x, y, z\} \text{ is a subalgebra of } \mathcal{A}. \tag{W}$$

Such BCC-algebras are called *weakly standard*.

It is immediately clear from the definition that every standard BCC-algebra is weakly standard, but not conversely.

Example 1.2. Let us consider an algebra $(A, \bullet, 1)$ given by Table 1.1.

It is easy to verify that $(A, \bullet, 1)$ is a weakly standard BCC-algebra, but

$$1 = z \bullet (x \bullet y) \neq x \bullet (z \bullet y) = x \bullet y = z \tag{1.3}$$

shows that it is neither a BCK-algebra nor a standard BCC-algebra.

To recall the representation theorem for weakly standard BCC-algebras, we need the following notions.

Given a weakly standard BCC-algebra $\mathcal{A} = (A, \bullet, 1)$, $x, y, z \in A$, and $x \bullet y = z \notin \{1, x, y\}$, it has been shown in [2] that necessarily $y \bullet x = z$ holds and only the following two possibilities, presented in Tables 1.2a and 1.2b, for products of elements from $\{1, x, y, z\}$ can occur.

Table 1.2

(a)

$\bullet$	1	x	y	z
1	1	x	y	z
x	1	1	z	1
y	1	z	1	1
z	1	x	y	1

(b)

$\bullet$	1	x	y	z
1	1	x	y	z
x	1	1	z	1
y	1	z	1	1
z	1	z	z	1

This leads us to the following definitions.

Definition 1.3. Let $\mathcal{A} = (A, \bullet, 1)$ be a weakly standard BCC-algebra, $x, y, z \in A$, $x \neq y$. A pair (x, y), for which $x \bullet y$, $y \bullet x \in \{1, x, y\}$ holds, is called *regular*. If moreover $x \bullet y = x$, then the ordered pair (x, y) is called *nonnormal*, otherwise, it is said to be *normal*. In case $x \bullet y \notin \{1, x, y\}$, the ordered triple $(x, y, x \bullet y)$ is called *singular*. A singular triple (x, y, z) is said to be of *type A* if for $\{1, x, y, z\}$, Table 1.2a holds, otherwise, it is of *type B* with Table 1.2b.

Now we are ready to recall the main representation theorem for weakly standard BCC-algebras [2].

Theorem 1.4. *Let $\mathcal{A} = (A, \leq, 1)$ be a partially ordered set with a greatest element 1, $a, b, c, d, x, y, z, v \in A$ and let $\bullet$ be a binary operation on A such that the following hold:*

(1) $1 \bullet x = x$,

(2) $x \bullet y = 1$ *if* $x \leq y$,

(3) *if* $x > y$, *then either* $x \bullet y = x$ *or* $x \bullet y = y$; *in the first case, moreover,*

  (a) $x \succ y$,

  (b) *if* $z > y$, *then* $z \geq x$, *and for* $z > x$, $z \bullet x = x$ *and* $z \bullet y = y$,

  (c) *if* $z < y$, *then* $x \bullet z = y \bullet z = z$,

  (d) *if* $z < x$ *and* $x \bullet z = z$, *then* $z \leq y$,

  (e) *if* $y \neq z$ *and* $x \bullet z = x$, *then* (z, y, a) *is singular for some* $a \in A$,

(4) *if* $x \parallel y$, *then either* $x \bullet y = y$ *and* $y \bullet x = x$ *or* $x \bullet y = y \bullet x = v$ *for some* $v > x, y$; *in the latter case, two possibilities can happen:*

  (α) $v \bullet x = x$ *and* $v \bullet y = y$ *(i.e.,* (x, y, v) *is singular of type A); then moreover,*

    (A1) *if* $x < a < v$, *then* $y < a$, $a \bullet x = x$ *iff* $a \bullet y = y$ *and* $v \bullet a = a$,

    (A2) *if* $a > x$ *and* $v \parallel a$, *then* $a > y$, $a \bullet x = x$, *and* $a \bullet y = y$,

    (A3) *if* $a > v$, *then* $a \bullet x = x$, $a \bullet y = y$,

    (A4) *if* $a < x$, *then* $a < y$ *and* $x \bullet a = y \bullet a = v \bullet a = a$,

    (A5) *if* $a < v$, $a \parallel x$, $a \parallel y$, *then* $v \bullet a = a$ *and one of the following cases occurs:*

      (i) $x \bullet a = a = y \bullet a$, $a \bullet x = x$, *and* $a \bullet y = y$,

      (ii) (x, a, b), (y, a, b) *are singular of type B for some* $b \in A$, $v > b$, *and* $v \bullet b = b$,

      (iii) (x, a, b), (y, a, c) *are singular of type A for some* $b, c \in A$ *and one of the following cases occurs:*

(a1) *some of the elements* v, b, c *are equal, for example,* $b = v$, *and then*

(a) $v \parallel c$ *implies* (v,c,d) *is singular,*

(b) $v > c$ *implies* $c \bullet x = x$, $v \bullet a = a$, *and* $v \bullet c = v$,

(c) $v < c$ *implies* $c \bullet x = x, v \bullet a = a$,

(d) $v = c$,

(a2) $|\{v,b,c\}| = 3$ *and the set* $\{v,b,c\}$ *has one maximal and two minimal elements, for example,* $b,v < c$ *and* $v \parallel b$. *Then either* (v,b,c) *is singular of type B or* (v,b,d) *is singular of type A, where* $c < d$,

(a3) v, b, c *are pairwise incomparable,* (v,c,p), (v,b,q), (c,b,r) *are singular for some* $p,q,r \in A$, *and for all* $\alpha,\beta \in \{p,q,r\}$, $\alpha \neq \beta$, $\alpha \bullet \beta \geq p$, q,r,

(β) (x,y,v) *is singular of type B, that is,* $v \bullet x = v \bullet y = v$; *then,*

(B1) $v \succ x$, $v \succ y$,

(B2) *if* $a > x$, *then* $a \geq v$,

(B3) *if* $a > v$, *then* $a \bullet x = x$, $a \bullet y = y$, *and* $a \bullet v = v$,

(B4) *if* $a < x$, *then* $a \leq y$ *and* $v \bullet a = x \bullet a = y \bullet a = a$,

(B5) *if* $a < v$, $a \parallel x$, $a \parallel y$, *then there exist* $b,c \in A$ *such that* (a,x,b), (a,y,c) *are singular and one of the following cases occurs:*

(i) *both* (a,x,b), (a,y,c) *are of type B and* $b = c = v$,

(ii) *one of* (a,x,b), (a,y,c) *is of type A, for example,* (a,x,b), *and the second one is of type B. In this case,* $b > v$, $c = v$, $b \bullet x = x$, $b \bullet y = y$, $b \bullet v = v$, *and* $v \bullet a = v$.

Then $(A,\bullet,1)$ *is a weakly standard BCC-algebra and every weakly standard BCC-algebra is of this form.*

Remark 1.5. Some of the conditions of Theorem 1.4 can be partly visualized as in Figures 1.1, 1.2, 1.3, and 1.4.

They show how one can generate a structure of a weakly standard BCC-algebra on a given poset $(P,\bullet,1)$ with a top element 1.

Example 1.6. Let us consider a poset $(P,\leq)$ with the diagram in Figure 1.5.

Describe all the possibilities to get a structure of a weakly standard BCC-algebra $(P,\bullet,$ $1)$ on P.

(1) If there is no singular triple in $(P,\bullet,1)$, then according to (3) we have $\alpha \bullet \beta = \beta$ for all $\alpha,\beta \in P$ with $\alpha \nleq \beta$.

(2) Assume that there is a singular triple in $(P,\bullet,1)$, for example, (x,y,v). Evidently, by (2) and (4), $b \bullet v = v$, $v \bullet b = b$. Since $b \ngeq v$, by (4), (x,y,v) has to be of type A, hence $v \bullet x = x$, $v \bullet y = y$.

Further, due to (3), $b \bullet y \in \{b,y\}$. The case $b \bullet y = b$ gives by (3b) $v \geq b$, a contradiction. Thus we have $b \bullet y = y$ and analogously $b \bullet x = x$ and $v \bullet a = a$.

If one of the pairs (x,a), (y,a) is normal, then by (A5)(i), $x \bullet a = y \bullet a = a$, $a \bullet x = x$, and $a \bullet y = y$.

If $(x,a,x \bullet a)$ is singular, then either $x \bullet a = b$ or $x \bullet a = v$. By (A5) again, $(y,a,$ $y \bullet a)$ has to be also singular and $(x,a,x \bullet a)$, $(y,a,y \bullet a)$ are both of type A. If (x,a,b), (y,a,b) are singular, then by (1a), $(v,b,v \bullet b)$ is singular, a contradiction.

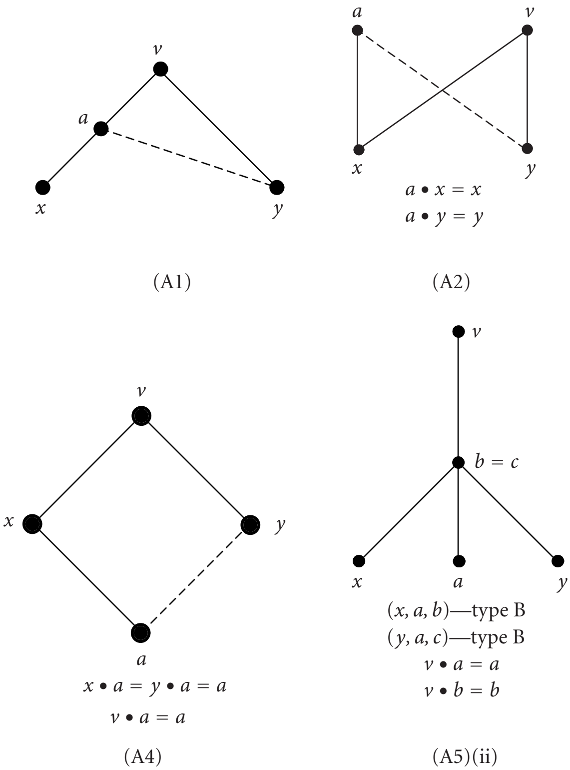

Figure 1.1

Also the case $x \bullet a = b, y \bullet a = v$ leads to a contradiction, thus $x \bullet a = y \bullet a = v$.

2. Congruence kernels, ideals, deductive systems, and annihilators on weakly standard BCC-algebras

The aim of this paper is to describe congruence kernels on weakly standard BCC-algebras as well as properties of the corresponding lattice of all congruence kernels.

Given a poset $(A, \leq, 1)$ with a greatest element 1, $x, y \in A$, $x > y$; we call a pair (x, y) *bridge* if for each $z \in A$ the following (dual) conditions hold:

(b1) $z > y$ implies $z \geq x$,

(b2) $z < x$ implies $z \leq y$.

Definition 2.1. A subset $\varnothing \neq I \subseteq A$ of a weakly standard BCC-algebra $\mathcal{A} = (A, \bullet, 1)$ which satisfies the conditions

(I1) $x \in I$, $y \in A$, and $x \leq y$ imply $y \in I$,

(I2) (x, y) being a bridge and $x \bullet y = x \in I$ imply $y \in I$,

(I3) (x, y, z) being a singular triple and $x \in I$ imply $y \in I$,

is called an *ideal* of $\mathcal{A}$.

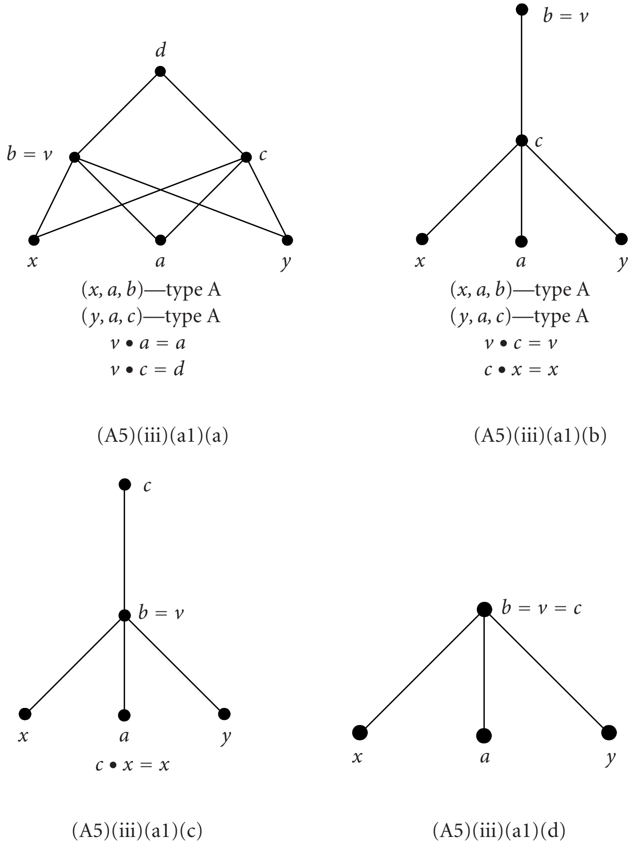

Figure 1.2

The set of all ideals of $\mathscr{A}$ will be denoted by $\mathrm{Id}(\mathscr{A})$. For a congruence θ on $\mathscr{A}$, denote by $[1]_\theta$ its congruence class containing the element 1, the so-called *kernel* of θ.

Definition 2.2. A subset $D \subseteq A$ of a weakly standard BCC-algebra $\mathscr{A} = (A, \bullet, 1)$ satisfying the conditions
(D1) $1 \in D$,
(D2) $x \bullet (y \bullet z) \in D$ and $y \in D$ imply $x \bullet z \in D$,
is called a *deductive system* of $\mathscr{A}$.

Denote by $\mathrm{Ck}(\mathscr{A})$ or $\mathrm{Ded}(\mathscr{A})$ the set of all congruence kernels of $\mathscr{A}$ or the set of all deductive systems of $\mathscr{A}$, respectively.

Proposition 2.3. *For an arbitrary weakly standard BCC-algebra $\mathscr{A}$, it holds that* $\mathrm{Id}(\mathscr{A}) = \mathrm{Ded}(\mathscr{A})$.

Proof. Let $I \in \mathrm{Id}(\mathscr{A})$. Since $\varnothing \neq I$, (I1) immediately gives $1 \in I$. Further, suppose $x \bullet (y \bullet z) \in I$ and $y \in I$. For y, z the following cases can occur.

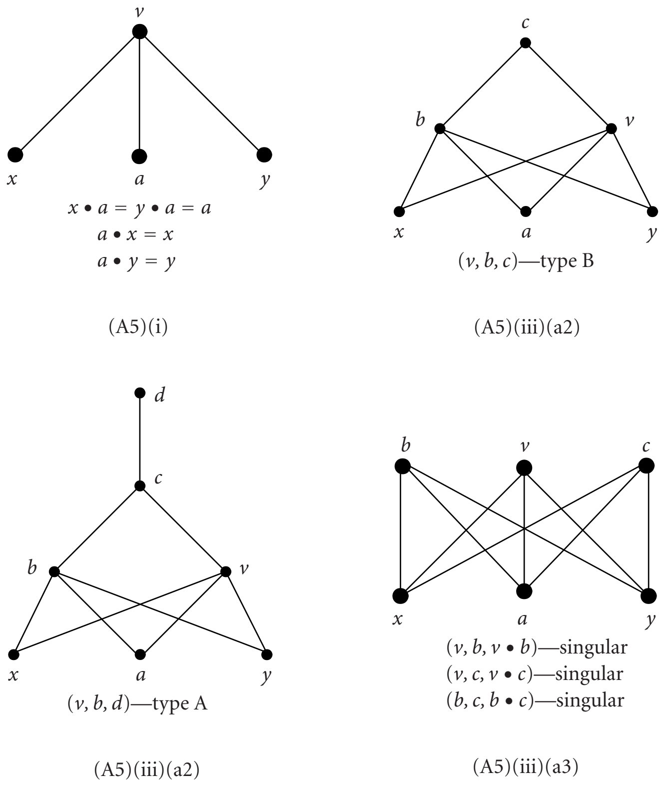

Figure 1.3

Case 1. (y,z,a) is singular for some $a \in A$. Then by (I3), we have $z \in I$ and since $z \leq x \bullet z$, also $x \bullet z \in I$ by (I1).
Case 2. Let $y \bullet z = 1$, that is, $y \leq z$. Then by (I1), $z \in I$ and as in the previous case, $x \bullet z \in I$.
Case 3. Assume $y \bullet z = y$; this yields by (I2) $z \in I$ and again $x \bullet z \in I$.
Case 4. If $y \bullet z = z$, we obtain $x \bullet z = x \bullet (y \bullet z) \in I$. We have shown that $I \in \mathrm{Ded}(\mathscr{A})$ and $\mathrm{Id}(\mathscr{A}) \subseteq \mathrm{Ded}(\mathscr{A})$.

Conversely, suppose $D \in \mathrm{Ded}(\mathscr{A})$. Let $x \in D$, $y \in A$, and $x \leq y$. Then $1 \bullet (x \bullet y) = 1 \in D$ and by (D2), $y \in D$, which shows the validity of (I1).

Further, let $x \bullet y = x \in D$. Then since $1 \bullet (x \bullet y) = x \in D$, by (D1) and (D2), $y \in D$, so condition (I2) holds.

Finally, let (x,y,z) be a singular triple and $x \in D$. If (x,y,z) is of type B, then in view of (I1), we have $z \in D$. This together with $z \bullet y = z$ gives by (I2) $y \in D$.

If (x,y,z) is of type A, then by (I1), $z \in D$. Since $z \bullet (x \bullet y) = 1 \in D$, applying (D2) twice, we get $z \bullet y \in D$ and $y \in D$. We have proved $D \in \mathrm{Id}(\mathscr{A})$ and $\mathrm{Ded}(\mathscr{A}) \subseteq \mathrm{Id}(\mathscr{A})$, finishing the proof. $\square$

Note that the set $\mathrm{Id}(\mathscr{A})$ of all ideals of $\mathscr{A}$ forms a lattice with respect to set inclusion.

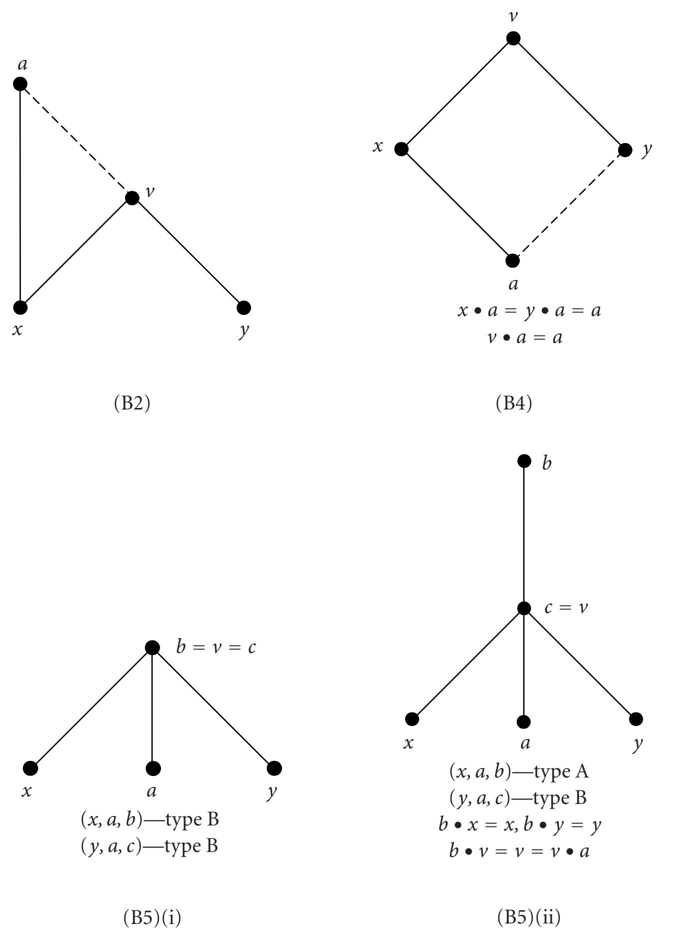

Figure 1.4

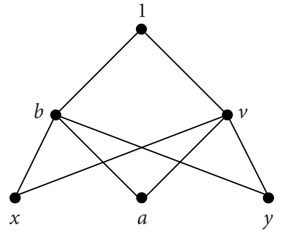

Figure 1.5

PROPOSITION 2.4. *Let $\mathcal{A} = (A, \bullet, 1)$ be a weakly standard BCC-algebra, $D \in \mathrm{Ded}(\mathcal{A})$, and $\theta \in \mathrm{Con}(\mathcal{A})$. Then,*

(a) $[1]_\theta \in \mathrm{Id}(\mathcal{A})$,

(b) *the relation θ_D defined by*

$$\langle x, y\rangle \in \theta_D \quad \textit{iff}\; x \bullet y, y \bullet x \in D, \tag{2.1}$$

is a congruence on $\mathcal{A}$ with $[1]_{\theta_D} = D$.

Proof. (a) We are going to show that $[1]_\theta$ satisfies (I1), (I2), and (I3). Let $x \in [1]_\theta$, that is, $\langle x, 1\rangle \in \theta$. If $y \geq x$, then $\langle x \bullet y, 1 \bullet y\rangle = \langle 1, y\rangle \in \theta$, that is, $y \in [1]_\theta$ and (I1) holds.

Suppose further $x \bullet y = x \in [1]_\theta$. This yields $\langle x \bullet y, 1 \bullet y\rangle = \langle x, y\rangle \in \theta$ and $[x]_\theta = [y]_\theta = [1]_\theta$ verifying (I2). Finally, let (x, y, z) be a singular triple and $x \in [1]_\theta$. Then using (I1), we get $z \in [1]_\theta$, and since $\langle x \bullet y, 1 \bullet y\rangle = \langle z, y\rangle \in \theta$, also $y \in [1]_\theta$; thus (I3) is satisfied.

(b) Reflexivity and symmetry of θ_D are clear. To prove its transitivity, assume that $\langle x, y\rangle, \langle y, z\rangle \in \theta_D$, that is, $x \bullet y, y \bullet x, y \bullet z, z \bullet y \in D$. Then by (D1) and (BCC1), we have $1 = (y \bullet z) \bullet [(x \bullet y) \bullet (x \bullet z)] \in D$, and since $x \bullet y \in D$, (D2) yields $(y \bullet z) \bullet (x \bullet z) \in D$. Applying (D2) once more to $1 \bullet [(y \bullet z) \bullet (x \bullet z)] \in D$ with $y \bullet z \in D$, we obtain $1 \bullet (x \bullet z) = x \bullet z \in D$. The validity of $z \bullet x \in D$ can be proved analogously, and so θ_D is transitive.

Further, let $\langle x, y\rangle \in \theta_D$ and $u \in A$ be an arbitrary element. Then $x \bullet y, y \bullet x \in D$. By (BCC1) and (D1), we have

$$1 = (x \bullet y) \bullet [(u \bullet x) \bullet (u \bullet y)] \in D. \tag{2.2}$$

Since $x \bullet y \in D$, using (D2), we obtain $(u \bullet x) \bullet (u \bullet y) \in D$. Analogously one can prove $(u \bullet y) \bullet (u \bullet x) \in D$ and $\langle u \bullet x, u \bullet y\rangle \in \theta_D$. Applying (D2), again for $1 = (x \bullet u) \bullet [(y \bullet x) \bullet (y \bullet u)] \in D$ and $y \bullet x \in D$, one gets $(x \bullet u) \bullet (y \bullet u) \in D$. Interchanging x and y, we also have $(y \bullet u) \bullet (x \bullet u) \in D$ and $\langle x \bullet u, y \bullet u\rangle \in \theta_D$. Finally, due to the transitivity of θ_D, this gives $\theta_D \in \mathrm{Con}(\mathcal{A})$ with $[1]_{\theta_D} = D$. □

Definition 2.5. Let $\mathcal{A} = (A, \bullet, 1)$ be a weakly standard BCC-algebra and B, C be nonvoid subsets of A. The set

$$\langle C\rangle = \{x \in A \mid x \bullet c = c \text{ for each } c \in C\} \tag{2.3}$$

is called the *annihilator* of C. The set

$$\langle C, B\rangle = \{x \in A \mid (x \bullet c) \bullet c \in B \text{ for each } c \in C\} \tag{2.4}$$

is called the *relative annihilator* of C with respect to B.

THEOREM 2.6. *Let $\mathcal{A} = (A, \bullet, 1)$ be a weakly standard BCC-algebra and let I be an ideal of $\mathcal{A}$. Then $\langle I\rangle$ is also an ideal and the pseudocomplement of I in the lattice $\mathrm{Id}(\mathcal{A})$. Moreover,*

$$\langle I\rangle = \{x \in A \mid x \parallel i \text{ for each } i \in I \setminus \{1\}\} \cup \{1\}. \tag{2.5}$$

Proof. First we prove that for each $c \in A$, $\langle c\rangle \in \mathrm{Id}(\mathscr{A})$.

Let $x \in \langle c\rangle$, that is, $x \bullet c = c$, and let $x \leq y$. Then by (1), $c \leq y \bullet c \leq x \bullet c = c$, which yields $y \bullet c = c$ and proves the validity of (I1).

Further, assume $x \bullet c = c$ and $x \bullet y = x$. By (1) and (BCC1), $c \leq y \bullet c \leq (x \bullet y) \bullet (x \bullet c) = x \bullet c = c$, and as before, $y \in \langle c\rangle$, so (I2) holds.

Finally, let (x,y,z) be a singular triple and let $x \bullet c = c$. In view of $x \leq z$, we have by (I1) $z \bullet c = c$, and consequently $c \leq y \bullet c \leq (x \bullet y) \bullet (x \bullet c) = z \bullet c = c$. This gives $y \bullet c = c$ and proves (I3).

Since evidently $\langle I\rangle = \cap\{\langle i\rangle \mid i \in I\}$, we get $\langle I\rangle \in \mathrm{Id}(\mathscr{A})$. Clearly $I \cap \langle I\rangle = \{1\}$. Suppose that J is any ideal of $\mathscr{A}$ with $I \cap J = \{1\}$. If $j \in J\setminus\{1\}$ and $i \in I\setminus\{1\}$, then $i \parallel j$; otherwise, either $i \leq j \in I \cap J$ or $j \leq i \in I \cap J$, a contradiction. Further, if $j \bullet i = j$ or $j \bullet i = z$ for $z \neq i, j$, then by (I2) or (I3), we have $i \in J$, a contradiction. Hence necessarily $j \bullet i = i$ and since $i \in I$ was arbitrary, we get $j \in \langle I\rangle$ and $J \subseteq \langle I\rangle$.

Now we will show that $\langle I\rangle = \{x \in A \mid x \parallel i \text{ for each } i \in I\setminus\{1\}\} \cup \{1\}$. Take $a \in \langle I\rangle$, that is, $a \bullet i = i$, for each $i \in I$. Evidently, for each $i \in I \setminus \{1\}$, either $a \parallel i$ or $a \geq i$. If there exists an element $i \in I \setminus \{1\}$ with $a \geq i$, then we have by (I1) $a \in I$ and $1 = a \bullet a = a$, proving that $a \in \{x \in A \mid x \parallel i \text{ for each } i \in I \setminus \{1\}\} \cup \{1\}$. Conversely, assume that there exists $a \in \{x \in A \mid x \parallel i \text{ for each } i \in I \setminus \{1\}\}$ such that $(a,i,a \bullet i)$ is singular. Then by (I3), $a \in I$ and $1 = a \bullet a = a$, a contradiction. The rest is obvious. □

Theorem 2.7. *Let B, C be ideals of a weakly standard BCC-algebra $\mathscr{A} = (A,\bullet,1)$. Then $\langle C,B\rangle$ is the relative pseudocomplement of C with respect to B in the lattice* $\mathrm{Id}(\mathscr{A})$. *Moreover,*

$$\langle C,B\rangle = \{x \in A \mid x \parallel i \text{ for each } i \in C \setminus B\} \cup B. \tag{2.6}$$

Proof. At first we show that

$$\langle C,B\rangle = \{x \in A \mid x \parallel i \text{ for each } i \in C \setminus B\} \cup B. \tag{2.7}$$

It is easily seen that $B \subseteq \langle C,B\rangle$. Suppose $x \parallel c$ for each $c \in C \setminus B$ and $x \notin B$. Then either $x \bullet c = c$ and $(x \bullet c) \bullet c = c \bullet c = 1 \in B$ for each $c \in C \setminus B$ or (x,c,d) is a singular triple for some $d \in A$. Let us show that the latter case leads to a contradiction. Indeed, $c \in C$ gives by (I3) $x \in C \setminus B$, which yields a contradiction $x \parallel x$. In the remaining case we have $c \leq (x \bullet c) \bullet c \in B$ whenever $c \in C \cap B$, and altogether, $(x \bullet c) \bullet c \in B$ for each $c \in C$, that is, $x \in \langle C,B\rangle$.

Conversely, suppose $y \in \langle C,B\rangle \setminus B$ and let $y \nparallel c$ for some $c \in C \setminus B$. If $y \leq c$, then $(y \bullet c) \bullet c = 1 \bullet c = c \in B$, a contradiction. In the case $y \geq c$, we conclude $y \in C$ and, moreover, $y = 1 \bullet y = (y \bullet y) \bullet y \in B$, which is also a contradiction. This proves $y \parallel c$ for each $c \in C \setminus B$.

Now we show that $\langle C,B\rangle$ is an ideal of $\mathscr{A}$. Let $x \in \langle C,B\rangle$ and $x \leq y$. We have $y \in B$ whenever $x \in B$. Suppose further $x \parallel c$ for each $c \in C \setminus B$. It is clear that $y \nleq c$ for each $c \in C \setminus B$, otherwise $x \leq c$. So let $y \geq c$ for some $c \in C \setminus B$. Then by (I1), also $y \in C$. Moreover, $y \in B \subseteq \langle C,B\rangle$ since in the opposite case we would have $x \parallel y$. In the remaining case, $y \parallel c$ for each $c \in C \setminus B$, hence also $y \in \langle C,B\rangle$.

Prove that $\langle C,B\rangle$ is bridge-closed. Let $x \bullet y = x \in \langle C,B\rangle$ for some bridge (x,y). Then $y \in B$ whenever $x \in B$. Suppose further that $x \parallel c$ for each $c \in C \setminus B$ and let $y \nparallel c$ for some $c \in C \setminus B$. Then $y \leq c$ gives $x \leq c$ (by Theorem 1.4(3b)), a contradiction. Similarly, $c \leq y < x$ contradicts $c \parallel x$, so (I2) holds.

Let now $(x, y, x \bullet y)$ be a singular triple. Then $y \in B$ whenever $x \in B$. Since by (A1), (A2), or (B2), $y \parallel c$ if and only if $x \parallel c$, $x \parallel c$ for each $c \in C \setminus B$ implies $y \parallel c$ for each $c \in C \setminus B$, and, finally, $\langle C, B\rangle$ is an ideal of P.

It is clear that $C \cap \langle C, B\rangle \subseteq B$. Let J be an ideal of $\mathcal{A}$ with the property $C \cap J \subseteq B$ and assume $j \in J \setminus B$. Suppose further $j \nparallel c$ for some $c \in C \setminus B$. If $c \leq j$, then $j \in C \cap J \subseteq B$, a contradiction. The case $j \leq c$ leads to a contradiction $c \in J \cap C \subseteq B$. This means that $J \subseteq \langle C, B\rangle$ and $\langle C, B\rangle$ is the relative pseudocomplement of C with respect to B. □

Definition 2.8. Consider a poset P with a greatest element 1, where $P \setminus \{1\}$ is composed of pairwise incomparable blocks B_i, $i \in \Omega$, being either singletons or two-element chains $B_i = \{a_i, b_i\}$, $a_i < b_i$, with $b_i \bullet a_i = b_i$, or $B_i = S_i \cup \{v_i\}$, $S_i = \{a_i^\alpha \mid \alpha \in \Lambda\}$, where $(a_i^\alpha, a_i^\beta, v_i)$ is a singular triple of type B for all $a_i^\alpha, a_i^\beta \in S_i$, $a_i^\alpha \neq a_i^\beta$. Such a weakly standard BCC-algebra will be called *a weak semi-implication algebra.*

Theorem 2.9. *Let $\mathcal{A} = (A, \bullet, 1)$ be a weakly standard BCC-algebra and $c \in A$. For $I(c)$, the principal ideal generated by $\{c\}$, it holds that*

$$I(c) = U(c) \cup B(c) \cup S(c), \tag{2.8}$$

where $U(c) = \{x \in A \mid c \leq x\}$ is the upper cone of $\{c\}$ in A, $B(c) = \{d \in A \mid (c, d)$ is non-normal$\}$, and $S(c) = \{d \in A \mid (c, d, c \bullet d)$ is a singular triple$\}$.

Proof. First we prove that $P = U(c) \cup B(c) \cup S(c)$ is an ideal of $\mathcal{A}$. Let $x \in P$, $x < y$. If $x \in U(c)$, then evidently $y \in U(c)$. Further, assuming $x \in B(c)$, that is, $c \bullet x = c$, we obtain by Theorem 1.4(3b) $y \geq c$ and $y \in U(c)$. Finally, let $(c, x, c \bullet x)$ be a singular triple. Then due to (A1), (A2), or (B2), we have $c \leq y$. Altogether we proved that $y \in P$, so (I1) holds.

To prove (I2), assume $x \bullet y = x$ and $x \in P$. If $x \in U(c)$, then either $x \bullet c = c$ and, in view of Theorem 1.4(3d), $c < y$ and $y \in U(c)$, or $x \bullet c = x$ and according to Theorem 1.4(3e), (c, y, a) is singular for some $a \in A$, that is, $y \in S(c) \subseteq P$. Suppose further $x \in B(c) \cup S(c)$, that is, $c \bullet x = c$ or $(c, x, c \bullet x)$ is singular. In both cases we get a contradiction: in the former one with Theorem 1.4(3c) and in the latter one with (A4) or (B4). So we have proved the validity of (I2).

To prove (I3) assume $x \in P, (x, y, x \bullet y)$ is singular. If $x \in U(c)$, then by (A1), (A2), or (B2), $c < y$ and $y \in U(c)$. The case $x \bullet c = x$ cannot occur due to (A4) and (B4). Finally, the singularity of $(c, x, c \bullet x)$ leads by (A5) or (B5) to the singularity of $(c, y, c \bullet y)$, that is, $y \in S(c)$. Altogether P is an ideal of $\mathcal{A}$ and since it is evident that $c \in P \subseteq I(c)$, we get $P = I(c)$. □

There is a natural question to find conditions under which an annihilator of every subset M of A is equal to the annihilator of $I(M)$, the ideal generated by M. We will show that the answer is closely connected with weak semi-implication algebras.

Theorem 2.10. *For a weakly standard BCC-algebra $\mathcal{A} = (A, \bullet, 1)$, the following conditions are equivalent:*

(a) *for each $\emptyset \neq M \subseteq A$, it holds that $\langle M\rangle = \langle I(M)\rangle$,*

(b) *$\mathcal{A}$ is a weak semi-implication algebra.*

Proof. (a)⇒(b). Take $M = \{c\}$ for $c \in P \setminus \{1\}$. By Theorem 2.9, $I(c) = U(c) \cup B(c) \cup S(c)$. We will show that

$$\langle I(c)\rangle = \{x \in P \mid U(x,c) = \{1\}\}. \tag{2.9}$$

Suppose $x \in \langle I(c)\rangle$ and let $y \in U(x,c)$ be an arbitrary element. Then $y \in I(c)$, hence $1 = x \bullet y = y$ proving that $U(x,c) = \{1\}$.

Suppose conversely that $U(x,c) = \{1\}$ for some $x \in A$ and let $y \in U(c)$. If $y = 1$, then clearly $x \bullet y = y$. Otherwise we have $x \not\leq y$, that is, either $x \parallel y$ or $y < x$. In the former case, $x \bullet y = y$ (if $(x, y, x \bullet y)$ is singular, then by (A1), (A2), or (B2), and $c \leq y$ would lead to a contradiction $c \leq x$). In the latter case, $x \in U(x,c) = \{1\}$ and $x \bullet y = 1 \bullet y = y$. Altogether we proved that $x \in \langle U(c)\rangle$. Further let $y \in B(c)$. If $x = 1$, then $x \bullet y = 1 \bullet y = y$. Supposing $x \parallel c$, clearly $x \bullet c = c$ and since $c \bullet y = c$, necessarily $x \parallel y$ and $x \bullet y = y$ (if $(x, y, x \bullet y)$ was singular, then $y < c$ would lead to a contradiction $x < c$). Hence $x \in \langle B(c)\rangle$. Finally, let $y \in S(c)$. If $x = 1$, then $x \bullet y = 1 \bullet y = y$. In case $x \parallel c$, necessarily $x \bullet c = c$ and we get $x \parallel y$, which by (A5)(a1), gives $x \bullet y = y$. We proved that $x \in \langle S(c)\rangle$. Altogether, the equality (2.9) holds.

Assume now $b > c$ for some $b \in A$. If the pair (b,c) is normal, then $b \bullet c = c$ and $b \in \langle c\rangle = \langle I(c)\rangle$ which, by (2.9), gives $U(b,c) = U(b) = \{1\}$ and $b = 1$. This means that A contains at most three-element chains, otherwise, it would contain a normal pair (x, y), with $1 \neq x > y$. If $1 > b > c$ is a three-element chain of A, then the pair (b,c) is nonnormal. If (c,d,e) is a singular triple, then necessarily $e = b$ and the triple (c,d,b) is of type B. Altogether it gives that $\mathcal{A}$ is a weak semi-implication algebra.

(b)⇒(a). Suppose that $\mathcal{A}$ is a weak semi-implication algebra. Then if $\{c\}$ is a one-element block in $A \setminus \{1\}$, since $c = I(c)$, we have $\langle c\rangle = \langle I(c)\rangle$.

Let $B = \{b,c\}$ with $b > c$ and $b \bullet c = b$ be a two-element block of $A \setminus \{1\}$. Then $I(b) = I(c) = U(c)$, hence $\langle I(b)\rangle = \langle I(c)\rangle = A \setminus B = \langle b\rangle = \langle c\rangle$.

Let $B = S \cup \{v\}$ for $S = \{a^\alpha \mid \alpha \in \Lambda\}$ and (a^α, a^β, v) be a singular triple of type B for all $a^\alpha, a^\beta \in S$, $a^\alpha \neq a^\beta$. Then $I(v) = I(a^\alpha)$ for each $\alpha \in \Lambda$, hence $\langle I(a^\alpha)\rangle = \langle I(v)\rangle = A \setminus (S \cup \{v\}) = \langle v\rangle = \langle a^\alpha\rangle$. Since the join of ideals is their set-theoretical union and $\langle C_1 \cup \cdots \cup C_n\rangle = \langle C_1\rangle \cap \cdots \cap \langle C_n\rangle$ for all $C_1, \ldots, C_n \subseteq A$, we are done. □

Summing up the results of the paper, we characterized the congruences on weakly standard BCC-algebras by means of ideals (a purely algebraic notion) as well as by means of deductive systems (a logical notion). Moreover, these lattices are shown to be relatively pseudocomplemented and their relative pseudocomplements are described.

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