GENERALIZED $(\mathcal{F}, b, \phi, \rho, \theta)$-UNIVEX n-SET FUNCTIONS AND GLOBAL SEMIPARAMETRIC SUFFICIENT EFFICIENCY CONDITIONS IN MULTIOBJECTIVE FRACTIONAL SUBSET PROGRAMMING

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A fairly large number of global semiparametric sufficient efficiency results are established under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions for a multiobjective fractional subset programming problem.

1. Introduction

In this paper, we will present a multitude of global semiparametric sufficient efficiency conditions under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity hypotheses for the following multiobjective fractional subset programming problem:

(P) Minimize$(F_1(S)/G_1(S), F_2(S)/G_2(S), \dots, F_p(S)/G_p(S))$ subject to $H_j(S) \leqq 0$, $j \in \underline{q}$, $S \in \mathbb{A}^n$,

where $\mathbb{A}^n$ is the n-fold product of the σ-algebra $\mathbb{A}$ of subsets of a given set X, F_i, G_i, $i \in \underline{p} \equiv \{1, 2, \dots, p\}$, and H_j, $j \in \underline{q}$, are real-valued functions defined on $\mathbb{A}^n$, and for each $i \in \underline{p}$, $G_i(S) > 0$ for all $S \in \mathbb{A}^n$ such that $H_j(S) \leqq 0$, $j \in \underline{q}$.

The point-function counterparts of (P) are known in the area of mathematical programming as *multiobjective fractional programming problems*. These problems have been the focus of intense interest in the past few years, which has resulted in numerous publications. A fairly extensive list of references concerning various aspects of these problems is given in [43]. For more information about general multiobjective problems with point functions, the reader may consult the recently published monograph by Miettinen [30].

In the area of subset programming, multiobjective problems have been investigated in [2, 4, 7, 14, 15, 16, 18, 23, 24, 26, 27, 29, 38, 42], and multiobjective *fractional* problems in [1, 5, 6, 19, 21, 22, 24, 36, 44, 46]. We next give a brief overview of the available results pertaining to the latter class of problems.

A parametric dual problem for (P) was constructed in [6] and a number of weak and strong duality theorems involving generalized ρ-convexity assumptions were proved. In [18], two parametric dual problems, which are slightly different from the one considered in [6], were formulated and some weak, strong, and strict converse duality results were established using generalized ρ-convexity hypotheses. Some of these results are further extended in [22] by using generalized $\mathcal{F}$-convex n-set functions. A multiobjective fractional

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problem like (P) in which the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are assumed to be convex was considered in [2] where parametric, semiparametric, and Lagrangian-type dual problems were formulated and weak, strong, and strict converse duality theorems were proved; in addition, a set of sufficient conditions characterizing properly efficient solutions of the problem under consideration was given. A problem similar to the one studied in [2], but with one additional restriction, was discussed in [21]. In this paper, it was assumed that the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are convex and that the denominators of the objective functions are equal. With these assumptions, the authors established necessary and sufficient proper efficiency results, formulated a dual problem that has the same objective function as the primal problem, and proved weak and strong duality theorems. In [36], Preda defined a (ρ, b)-vex n-set function, discussed some of its properties, and then established weak, strong, and converse duality results for a parametric dual problem for (P) under appropriate (ρ, b)-vexity conditions. B-vex n-set functions were utilized in [4] for obtaining sufficient proper efficiency criteria and some duality relations for a nonfractional multiobjective subset programming problem. The relevance and applicability of these results to a problem like (P) in which the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are convex, and for each $i \in \underline{p}$, $F_i(S) \geq 0$ and $G_i(S) > 0$ for all $S \in \mathbb{A}^n$ were also discussed. Recently, saddle-point-type proper efficiency conditions and Lagrangian-type duality results were obtained in [5] under cone-convexity assumptions for a cone-constrained multiobjective subset programming problem. In [44], a number of sufficient efficiency criteria and duality theorems were established for (P) under various $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity assumptions.

For brief surveys and additional references dealing with different aspects of subset programming problems, including areas of applications, optimality conditions, and duality models, the reader is referred to [4, 8, 24, 33, 37, 40, 41].

The rest of this paper is organized as follows. In Section 2, we recall the definitions of differentiability, convexity, and certain types of generalized convexity for n-set functions, which will be used frequently throughout the sequel. We begin our discussion of sufficient efficiency criteria for (P) in Section 3 where we state and prove a number of sufficiency results. More general sets of sufficiency conditions are formulated and discussed in Section 4 with the help of two partitioning schemes. The first of these schemes was originally used in [32] for constructing generalized dual problems for nonlinear programs with point functions, whereas the second was utilized in [39] for formulating a dual problem for a multiobjective fractional program involving point functions.

Evidently, all these efficiency results are also applicable, when appropriately specialized, to the following three classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

(P1) $\text{Minimize}_{S\in\mathbb{F}}(F_1(S), F_2(S), \ldots, F_p(S))$;

(P2) $\text{Minimize}_{S\in\mathbb{F}} F_1(S)/G_1(S)$;

(P3) $\text{Minimize}_{S\in\mathbb{F}} F_1(S)$,

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{S \in \mathbb{A}^n : H_j(S) \leqq 0,\ j \in \underline{q}\}. \tag{1.1}$$

Since in most cases, the efficiency results established for (P) can easily be modified and restated for each one of the above problems, we will not explicitly state these results.

2. Preliminaries

In this section, we gather, for convenience of reference, a number of basic definitions along with a few auxiliary results, which will be used often throughout the sequel.

Let $(X,\mathbb{A},\mu)$ be a finite atomless measure space with $L_1(X,\mathbb{A},\mu)$ separable, and let d be the pseudometric on $\mathbb{A}^n$ defined by

$$d(R,S)=\left[\sum_{i=1}^{n}\mu^2(R_i\triangle S_i)\right]^{1/2},\quad R=(R_1,\dots,R_n),\ S=(S_1,\dots,S_n)\in\mathbb{A}^n, \tag{2.1}$$

where $\triangle$ denotes symmetric difference; thus $(\mathbb{A}^n,d)$ is a pseudometric space. For $h\in L_1(X,\mathbb{A},\mu)$ and $T\in\mathbb{A}$ with characteristic function $\chi_T\in L_\infty(X,\mathbb{A},\mu)$, the integral $\int_T h\,d\mu$ will be denoted by $\langle h,\chi_T\rangle$.

We next define the notions of differentiability and convexity for n-set functions. They were originally introduced by Morris [33] for set functions, and subsequently extended by Corley [8] for n-set functions.

Definition 2.1. A function $F:\mathbb{A}\to\mathbb{R}$ is said to be *differentiable* at S^* if there exists $DF(S^*)\in L_1(X,\mathbb{A},\mu)$, called the *derivative* of F at S^*, such that for each $S\in\mathbb{A}$,

$$F(S)=F(S^*)+\langle DF(S^*),\chi_S-\chi_{S^*}\rangle+V_F(S,S^*), \tag{2.2}$$

where $V_F(S,S^*)$ is $o(d(S,S^*))$, that is, $\lim_{d(S,S^*)\to 0}V_F(S,S^*)/d(S,S^*)=0$.

Definition 2.2. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to have a *partial derivative* at $S^*=(S_1^*,\dots,S_n^*)\in\mathbb{A}^n$ with respect to its ith argument if the function $F(S_i)=G(S_1^*,\dots,S_{i-1}^*,S_i,S_{i+1}^*,\dots,S_n^*)$ has derivative $DF(S_i^*)$, $i\in\underline{n}$; in that case, the ith partial derivative of G at S^* is defined to be $D_iG(S^*)=DF(S_i^*)$, $i\in\underline{n}$.

Definition 2.3. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to be *differentiable* at S^* if all the partial derivatives $D_iG(S^*)$, $i\in\underline{n}$, exist and

$$G(S)=G(S^*)+\sum_{i=1}^{n}\langle D_iG(S^*),\chi_{S_i}-\chi_{S_i^*}\rangle+W_G(S,S^*), \tag{2.3}$$

where $W_G(S,S^*)$ is $o(d(S,S^*))$ for all $S\in\mathbb{A}^n$.

It was shown by Morris [33] that for any triple $(S,T,\lambda)\in\mathbb{A}\times\mathbb{A}\times[0,1]$, there exist sequences $\{S_k\}$ and $\{T_k\}$ in $\mathbb{A}$ such that

$$\chi_{S_k}\xrightarrow{w^*}\lambda\chi_{S\setminus T},\qquad \chi_{T_k}\xrightarrow{w^*}(1-\lambda)\chi_{T\setminus S} \tag{2.4}$$

imply

$$\chi_{S_k\cup T_k\cup(S\cap T)}\xrightarrow{w^*}\lambda\chi_S+(1-\lambda)\chi_T, \tag{2.5}$$

where $\xrightarrow{w^*}$ denotes weak* convergence of elements in $L_\infty(X,\mathbb{A},\mu)$, and $S\backslash T$ is the complement of T relative to S. The sequence $\{V_k(\lambda)\} = \{S_k \cup T_k \cup (S\cap T)\}$ satisfying (2.4) and (2.5) is called the *Morris sequence* associated with (S,T,λ).

Definition 2.4. A function $F:\mathbb{A}^n \to \mathbb{R}$ is said to be (strictly) convex if for every $(S,T,\lambda) \in \mathbb{A}^n \times \mathbb{A}^n \times [0,1]$, there exists a Morris sequence $\{V_k(\lambda)\}$ in $\mathbb{A}^n$ such that

$$\limsup_{k\to\infty} F\big(V_k(\lambda)\big)(<) \leqq \lambda F(S) + (1-\lambda)F(T). \tag{2.6}$$

It was shown in [8, 33] that if a differentiable function $F:\mathbb{S}\to\mathbb{R}$ is (strictly) convex, then

$$F(S)(>) \geqq F(T) + \sum_{i=1}^{n} \langle D_iF(T), \chi_{S_i} - \chi_{T_i}\rangle \tag{2.7}$$

for all $S,T\in\mathbb{A}^n$.

Following the introduction of the notion of convexity for set functions by Morris [33] and its extension for n-set functions by Corley [8], various generalizations of convexity for set and n-set functions were proposed in [4, 24, 25, 28, 35, 36, 40, 44, 45]. More specifically, quasiconvexity and pseudoconvexity for set functions were defined in [25], and for n-set functions in [28]; generalized ρ-convexity for n-set functions was defined in [40], $(\mathscr{F},\rho)$-convexity in [35], b-vexity in [4], (ρ,b)-vexity in [36], $(\mathscr{F},\rho,\theta)$-convexity for nondifferentiable set functions in [24], and $(\mathscr{F},\alpha,\rho,\theta)$-$V$-convexity in [44, 45]. For predecessors and point-function counterparts of these convexity concepts, the reader is referred to the original papers where the extensions to set and n-set functions are discussed. A survey of recent advances in the area of generalized convex functions and their role in developing optimality conditions and duality relations for optimization problems is given in [34].

For the purpose of formulating and proving various collections of sufficiency criteria for (P), in this study, we will use a new class of generalized convex n-set functions, called $(\mathscr{F},b,\phi,\rho,\theta)$-*univex functions*, which will be defined later in this section. This class of functions may be viewed as a combination of several previously defined types of generalized convex functions. Its main ingredients are $\mathscr{F}$-convex functions and univex functions, which were introduced in [3, 13], respectively. These functions were proposed as generalizations of the class of invex functions.

Prior to giving the definitions of the new classes of n-set functions, it will be useful for purposes of reference and comparison to recall the definitions of the point function analogues of the principal components of these functions mentioned above. We will keep this review to a bare minimum because our primary objective is only to put a number of interrelated generalized convexity concepts into proper perspective. For this reason, we will only reproduce the essential forms of the definitions without elaborating on their refinements, variants, special cases, and other manifestations. For full discussions of the consequences and applications of the underlying ideas, the reader may consult the original sources. We begin by defining an invex function, which occupies a pivotal position in a vast array of generalized convex functions, some of which are specified in the following definitions.

Definition 2.5 [12]. Let f be a real-valued differentiable function defined on an open subset $\mathbb{S}$ of $\mathbb{R}^n$. Then f is said to be *η-invex* (*invex with respect to* η) at x^* if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for each $x \in \mathbb{S}$,

$$f(x) - f(x^*) \geqq \nabla f(x^*)^T \eta(x, x^*), \tag{2.8}$$

where $\nabla f(x^*)$ is the gradient of f at x^*, and T denotes transposition; f is said to be *η-invex* (*invex with respect to* η) on $\mathbb{S}$ if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for all $x, y \in \mathbb{S}$,

$$f(x) - f(y) \geqq \nabla f(y)^T \eta(x, y). \tag{2.9}$$

From the above definition, it is clear that every real-valued differentiable function is invex with respect to $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [12] who showed that for a nonlinear programming problem of the form

(P_0) Minimize $f(x)$ subject to $g_i(x) \leqq 0$, $i \in \underline{m}$, $x \in \mathbb{R}^n$,

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are invex with respect to the same function η, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [9] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasi-invex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been extended in several directions. Some recent surveys and syntheses of results pertaining to various generalizations of invex functions and their applications along with extensive lists of relevant references are available in [10, 11, 20, 31, 34]. Two of the earliest generalizations of invex functions are $\mathcal{F}$-convex and (ρ, η)-invex functions. An $\mathcal{F}$-convex function is defined in terms of a sublinear function, that is, a function that is subadditive and positively homogeneous.

Definition 2.6. A function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is said to be *sublinear (superlinear)* if $\mathcal{F}(x+y) \leqq (\geqq)$ $\mathcal{F}(x) + \mathcal{F}(y)$ for all $x, y \in \mathbb{R}^n$, and $\mathcal{F}(ax) = a\mathcal{F}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$.

Now combining the definitions of $\mathcal{F}$-convex and (ρ, η)-invex functions given in [13, 17], respectively, we can define $(\mathcal{F}, \rho)$-convex, $(\mathcal{F}, \rho)$-pseudoconvex, and $(\mathcal{F}, \rho)$-quasiconvex functions.

Let g be a real-valued differentiable function defined on the open subset $\mathbb{S}$ of $\mathbb{R}^n$, and assume that for each $x, y \in \mathbb{S}$, the function $\mathcal{F}(x, y; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear.

Definition 2.7. The function g is said to be *$(\mathcal{F}, \rho)$-convex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$g(x) - g(y) \geqq \mathcal{F}(x, y; \nabla g(y)) + \rho \|x - y\|^2. \tag{2.10}$$

Definition 2.8. The function g is said to be $(\mathcal{F},\rho)$-*pseudoconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$\mathcal{F}(x,y;\nabla g(y)) \geqq -\rho\|x-y\|^2 \Longrightarrow g(x) \geqq g(y). \tag{2.11}$$

Definition 2.9. The function g is said to be $(\mathcal{F},\rho)$-*quasiconvex* at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,

$$g(x) \leqq g(y) \Longrightarrow \mathcal{F}(x,y;\nabla g(y)) \leqq -\rho\|x-y\|^2. \tag{2.12}$$

Evidently, if in Definitions 2.7–2.9 we choose $\mathcal{F}(x,y;\nabla g(y)) = \nabla g(y)^T\eta(x,y)$, where $\eta : \mathbb{S}\times\mathbb{S} \to \mathbb{R}^n$ is a given function, and set $\rho = 0$, then we see that they reduce to the definitions of η-invexity, η-pseudoinvexity, and η-quasi-invexity for the function g.

The foregoing classes of generalized convex functions have been utilized for establishing numerous sets of sufficient optimality conditions and a variety of duality results for several categories of static and dynamic optimization problems. For a wealth of information as well as long lists of references concerning these results, the reader is referred to [20, 34].

Another significant generalization of the notion of invexity, called *univexity*, which subsumes a number of previously proposed classes of generalized convex functions, was recently given in [3]. We recall the definitions of univex, pseudounivex, and quasiunivex functions.

Let h be a real-valued differentiable function defined on an open subset $\mathbb{S}$ of $\mathbb{R}^n$, let η be a function from $\mathbb{S}\times\mathbb{S}$ to $\mathbb{R}^n$, let Φ be a real-valued function defined on $\mathbb{R}$, and let b be a function from $\mathbb{S}\times\mathbb{S}$ to $\mathbb{R}_+ \setminus \{0\} \equiv (0,\infty)$.

Definition 2.10 [3]. The function h is said to be *univex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$b(x,y)\Phi(h(x)-h(y)) \geqq \nabla h(y)^T\eta(x,y). \tag{2.13}$$

Definition 2.11 [3]. The function h is said to be *pseudounivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$\nabla h(y)^T\eta(x,y) \geqq 0 \Longrightarrow b(x,y)\Phi(h(x)-h(y)) \geqq 0. \tag{2.14}$$

Definition 2.12 [3]. The function h is said to be *quasiunivex* at y with respect to η, Φ, and b if for each $x \in \mathbb{S}$,

$$\Phi(h(x)-h(y)) \leqq 0 \Longrightarrow b(x,y)\nabla h(y)^T\eta(x,y) \leqq 0. \tag{2.15}$$

Finally, we are in a position to give our definitions of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univex n-set functions. They are formulated by combining the n-set versions of Definitions 2.5–2.12.

Let S, $S^* \in \mathbb{A}^n$, and assume that the function $F : \mathbb{A}^n \to \mathbb{R}$ is differentiable at S^*.

Definition 2.13. The function F is said to be *(strictly)* $(\mathcal{F},b,\phi,\rho,\theta)$*-univex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$,

$$\phi(F(S)-F(S^*))(>)\geqq\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))+\rho d^2(\theta(S,S^*)). \tag{2.16}$$

Definition 2.14. The function F is said to be *(strictly)* $(\mathcal{F},b,\phi,\rho,\theta)$*-pseudounivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$ $(S\neq S^*)$,

$$\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))\geqq-\rho d^2(\theta(S,S^*))\Longrightarrow\phi(F(S)-F(S^*))(>)\geqq 0. \tag{2.17}$$

Definition 2.15. The function F is said to be *(prestrictly)* $(\mathcal{F},b,\phi,\rho,\theta)$*-quasiunivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$,

$$\phi(F(S)-F(S^*))(<)\leqq 0\Longrightarrow\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))\leqq-\rho d^2(\theta(S,S^*)). \tag{2.18}$$

From the above definitions it is clear that if F is $(\mathcal{F},b,\phi,\rho,\theta)$-univex at S^*, then it is both $(\mathcal{F},b,\phi,\rho,\theta)$-pseudounivex and $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, if F is $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, then it is prestrictly $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, and if F is strictly $(\mathcal{F},b,\phi,\rho,\theta)$-pseudounivex at S^*, then it is $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^*.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivexity can be defined in the following equivalent way:

F is said to be $(\mathcal{F},b,\phi,\rho,\theta)$-quasiunivex at S^* if for each $S\in\mathbb{A}^n$,

$$\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))>-\rho d^2(\theta(S,S^*))\Longrightarrow\phi(F(S)-F(S^*))>0. \tag{2.19}$$

Needless to say that the new classes of generalized convex n-set functions specified in Definitions 2.13–2.15 contain a variety of special cases; in particular, they subsume all the previously defined types of generalized n-set functions. This can easily be seen by appropriate choices of $\mathcal{F}$, b, ϕ, ρ, and θ.

In the sequel, we will also need a consistent notation for vector inequalities. For all $a,b\in\mathbb{R}^m$, the following order notation will be used: $a\geqq b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$; $a\geqslant b$ if and only if $a_i\geqq b_i$ for all $i\in\underline{m}$, but $a\neq b$; $a>b$ if and only if $a_i>b_i$ for all $i\in\underline{m}$; and $a\ngeqslant b$ is the negation of $a\geqslant b$.

Throughout the sequel, we will deal exclusively with the efficient solutions of (P). An $x^*\in\mathcal{X}$ is said to be an *efficient solution* of (P) if there is no other $x\in\mathcal{X}$ such that $\varphi(x)\leqslant\varphi(x^*)$, where φ is the objective function of (P).

Next, we recall a set of parametric necessary efficiency conditions for (P).

Theorem 2.16 [44]. *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exist $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0, \quad j \in \underline{q}, \tag{2.20}$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle D_k F_\ell(S^*) - \lambda_\ell^* D_k G_\ell(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0. \tag{2.21}$$

If S^* is an efficient solution of (P) and $\lambda_i^* = \varphi(S^*)$, $i \in \underline{p}$, then there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0,\ \sum_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathbb{R}_+^q$ such that

$$\begin{gathered} \sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^* \left[D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*) \right] + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0 \quad \forall S \in \mathbb{A}^n, \\ v_j^* H_j(S^*) = 0, \quad j \in \underline{q}. \end{gathered} \tag{2.22}$$

The above theorem contains two sets of parameters u_i^* and λ_i^*, $i \in \underline{p}$, which were introduced as a consequence of an indirect approach in [44] requiring two intermediate auxiliary problems. It is possible to eliminate one of these two sets of parameters and thus obtain a semiparametric version of Theorem 2.16. Indeed, this can be accomplished by simply replacing λ_i^* by $F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$, and redefining u^* and v^*. This result is stated in the next theorem.

Theorem 2.17. *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exist $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0, \quad j \in \underline{q}, \tag{2.23}$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle G_i(S^*) D_k F_\ell(S^*) - F_i(S^*) D_k G_\ell(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0. \tag{2.24}$$

If S^* is an efficient solution of (P), then there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that

$$\begin{gathered} \sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^* \left[G_i(S^*) D_k F_i(S^*) - F_i(S^*) D_k G_i(S^*) \right] \right. \\ \left. + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0 \quad \forall S \in \mathbb{A}^n, \\ v_j^* H_j(S^*) = 0, \quad j \in \underline{q}. \end{gathered} \tag{2.25}$$

The form and contents of the necessary efficiency conditions given in Theorem 2.17 provide clear guidelines for devising numerous sets of semiparametric sufficient efficiency criteria as well as for constructing various types of semiparametric duality models for (P).

3. Sufficient efficiency conditions

In this section, we present several sets of sufficient efficiency conditions for (P) under a variety of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univexity hypotheses. We begin by introducing some notation.

Let the functions $f_i(\cdot,S^*)$, $i\in \underline{p}$, $f(\cdot,S^*,u^*)$, and $h(\cdot,v^*):\mathbb{A}^n\to\mathbb{R}$ be defined, for fixed S^*, u^*, and v^*, by

$$\begin{aligned} f_i(T,S^*) &= G_i(S^*)F_i(T)-F_i(S^*)G_i(T), \quad i\in\underline{p},\\ f(T,S^*,u^*) &= \sum_{i=1}^{p}u_i^*\left[G_i(S^*)F_i(T)-F_i(S^*)G_i(T)\right],\\ h(T,v^*) &= \sum_{j=1}^{q}v_j^*H_j(T). \end{aligned} \tag{3.1}$$

For given $u^*\in U_0=\{u\in\mathbb{R}_+^q:\sum_{i=1}^p u_i=1\}$ and $v^*\in\mathbb{R}_+^q$, let $I_+(u^*)=\{i\in\underline{p}:u_i^*>0\}$ and $J_+(v^*)=\{j\in\underline{q}:v_j^*>0\}$.

Theorem 3.1. *Let $S^*\in\mathbb{F}$ with $F_i(S^*)\geqq 0$, $i\in\underline{p}$, and assume that F_i,G_i, $i\in\underline{p}$, and H_j, $j\in\underline{q}$, are differentiable at S^*, and that there exist $u^*\in U$ and $v^*\in\mathbb{R}_+^q$ such that*

$$\mathcal{F}\left(S,S^*;\sum_{i=1}^{p}u_i^*\left[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)\right]+\sum_{j=1}^{q}v_j^*DH_j(S^*)\right)\geqq 0 \quad \forall S\in\mathbb{F}, \tag{3.2}$$

$$v_j^*H_j(S^*)=0,\quad j\in\underline{q}, \tag{3.3}$$

where $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function. Assume, furthermore, that any of the following three sets of hypotheses is satisfied:

(a) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar\phi,\bar\rho_i,\theta)$-univex at S^*, and $-G_i$ is $(\mathcal{F},b,\bar\phi,\hat\rho_i,\theta)$-univex at S^*, $\bar\phi$ is superlinear, and $\bar\phi(a)\geqq 0\Rightarrow a\geqq 0$;*

(ii) *for each $j\in J_+\equiv J_+(v^*)$, H_j is $(\mathcal{F},b,\tilde\phi_j,\tilde\rho_j,\theta)$-quasiunivex at S^*, $\tilde\phi_j$ is increasing, and $\tilde\phi_j(0)=0$;*

(iii) *$\rho^*+\sum_{j\in J_+}v_j^*\tilde\rho_j\geqq 0$, where $\rho^*=\sum_{i=1}^p u_i^*[G_i(S^*)\bar\rho_i+F_i(S^*)\hat\rho_i]$;*

(b) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar\phi,\bar\rho_i,\theta)$-univex at S^*, and $-G_i$ is $(\mathcal{F},b,\bar\phi,\hat\rho_i,\theta)$-univex at S^*, $\bar\phi$ is superlinear, and $\bar\phi(a)\geqq 0\Rightarrow a\geqq 0$;*

(ii) *$h(\cdot,v^*)$ is $(\mathcal{F},b,\tilde\phi,\tilde\rho,\theta)$-quasiunivex at S^*, $\tilde\phi$ is increasing, and $\tilde\phi(0)=0$;*

(iii) *$\rho^*+\tilde\rho\geqq 0$;*

(c) (i) *the Lagrangian-type function*

$$T \longrightarrow L(T,S^*,u^*,v^*) = \sum_{i=1}^{p} u_i^* [G_i(S^*)F_i(T) - F_i(S^*)G_i(T)] + \sum_{j=1}^{q} v_j^* H_j(T) \tag{3.4}$$

is $(\mathcal{F},b,\phi,0,\theta)$*-pseudounivex at* S^* *and* $\phi(a) \geqq 0 \Rightarrow a \geqq 0$.
Then S^* *is an efficient solution of (P).*

Proof. (a) Let S be an arbitrary feasible solution of (P). Using the hypotheses specified in (i), we see that for each $i \in \underline{p}$,

$$\begin{aligned} \bar{\phi}(F_i(S) - F_i(S^*)) &\geqq \mathcal{F}(S,S^*;b(S,S^*)DF_i(S^*)) + \bar{\rho}_i d^2(\theta(S,S^*)), \\ \bar{\phi}(-G_i(S) + G_i(S^*)) &\geqq \mathcal{F}(S,S^*;-b(S,S^*)DG_i(S^*)) + \hat{\rho}_i d^2(\theta(S,S^*)). \end{aligned} \tag{3.5}$$

Since $u^* > 0$, $\sum_{i=1}^{p} u_i^* = 1$, $F_i(S^*) \geqq 0$, $G_i(S^*) > 0$, $\bar{\phi}$ is superlinear, and $\mathcal{F}(S,S^*;\cdot)$ is sublinear, we deduce from these inequalities that

$$\begin{aligned} &\bar{\phi}\left(\sum_{i=1}^{p} u_i^*[G_i(S^*)F_i(S) - F_i(S^*)G_i(S)] - \sum_{i=1}^{p} u_i^*[G_i(S^*)F_i(S^*) - F_i(S^*)G_i(S^*)]\right) \\ &\quad \geqq \mathcal{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p} u_i^*[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)]\right) \\ &\qquad + \sum_{i=1}^{p} u_i^*[G_i(S^*)\bar{\rho}_i + F_i(S^*)\hat{\rho}_i]d^2(\theta(S,S^*)). \end{aligned} \tag{3.6}$$

As $S \in \mathbb{F}$, it follows from (3.3) that for each $j \in J_+$, $H_j(S) \leqq 0 = H_j(S^*)$, and so using the properties of $\tilde{\phi}_j$, we get for each $j \in J_+$, $\tilde{\phi}_j(H_j(S) - H_j(S^*)) \leqq 0$, which in view of (ii) implies that $\mathcal{F}(S,S^*;b(S,S^*)DH_j(S^*)) \leqq -\tilde{\rho}_j d^2(\theta(S,S^*))$. Because $v^* \geqq 0$, $v_j^* = 0$ for each $j \in \underline{q} \setminus J_+$, and $\mathcal{F}(S,S^*;\cdot)$ is sublinear, these inequalities yield

$$\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \leqq -\sum_{j\in J_+} v_j^* \tilde{\rho}_j d^2(\theta(S,S^*)). \tag{3.7}$$

From the sublinearity of $\mathcal{F}(S,S^*;\cdot)$, (3.2), and the fact that $b(S,S^*) > 0$, it is clear that

$$\begin{aligned} &\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p} u_i^*[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)]\right) \\ &\quad + \mathcal{F}\left(S,S^*;b(S,S^*)\sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \geqq 0. \end{aligned} \tag{3.8}$$

Combining (3.6)–(3.8) and using (iii), we obtain

$$\bar{\phi}\left(\sum_{i=1}^{p} u_i^*\left[G_i(S^*)F_i(S) - F_i(S^*)G_i(S)\right]\right) \geqq \left(\rho^* + \sum_{j\in J_+} v_j^*\tilde{\rho}_j\right)d^2\left(\theta(S,S^*)\right) \geqq 0. \tag{3.9}$$

Since $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i^*\left[G_i(S^*)F_i(S) - F_i(S^*)G_i(S)\right] \geqq 0. \tag{3.10}$$

Since $u^* > 0$, (3.10) implies that

$$\left(G_1(S^*)F_1(S) - F_1(S^*)G_1(S),\dots,G_p(S^*)F_p(S) - F_p(S^*)G_p(S)\right) \nleq (0,\dots,0), \tag{3.11}$$

which in turn implies that

$$\begin{aligned}\varphi(S) &= \left(\frac{F_1(S)}{G_1(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)\\ &\nleq \left(\frac{F_1(S^*)}{G_1(S^*)},\dots,\frac{F_p(S^*)}{G_p(S^*)}\right) = \varphi(S^*).\end{aligned} \tag{3.12}$$

Because $S \in \mathbb{F}$ was arbitrary, we conclude that S^* is an efficient solution of (P).

(b) Since $v^* \geqq 0$, it follows that for any $S \in \mathbb{F}$, $v_j^* H_j(S) \leqq 0$ for each $j \in \underline{q}$, and so by (3.3), we have that $h(S,v^*) \leqq h(S^*,v^*)$, which in view of the properties of $\tilde{\phi}$ can be expressed as $\tilde{\phi}(h(S,v^*) - h(S^*,v^*)) \leqq 0$. Because of (ii) the last inequality implies that

$$\mathcal{F}\left(S,S^*;b(S,S^*)\sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \leqq -\tilde{\rho}d^2\left(\theta(S,S^*)\right). \tag{3.13}$$

Now proceeding as in the proof of part (a) and using this inequality instead of (3.7), we will obtain (3.10), which leads to the desired conclusion that S^* is an efficient solution of (P).

(c) Since (3.2) holds, $\mathcal{F}(S,S^*;\cdot)$ is sublinear, and $b(S,S^*) > 0$, we have

$$\mathcal{F}\left(S,S^*;b(S,S^*)\left\{\sum_{i=1}^{p} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \sum_{j=1}^{q} v_j^* DH_j(S^*)\right\}\right) \geqq 0, \tag{3.14}$$

which because of our $(\mathcal{F},b,\phi,0,\theta)$-pseudounivexity assumption implies that

$$\phi\left(L(S,u^*,v^*) - L(S^*,u^*,v^*)\right) \geqq 0. \tag{3.15}$$

But $\phi(a) \geqq 0 \Rightarrow a \geqq 0$, and hence $L(S,u^*,v^*) \geqq L(S^*,u^*,v^*) = 0$, where the equality follows from (3.3). Since $v_j^* H_j(S) \leqq 0$ for each $j \in \underline{q}$, the inequality reduces to (3.10), which leads, as seen in the proof of part (a), to the conclusion that S^* is an efficient solution of (P). □

In Theorem 3.1, separate $(\mathscr{F},b,\phi,\rho,\theta)$-univexity conditions were imposed on the functions F_i and $-G_i$, $i \in \underline{p}$. In the remainder of this section, we will present a number of sufficiency results in which various generalized $(\mathscr{F},b,\phi,\rho,\theta)$-univexity requirements will be placed on certain combinations of these functions.

Theorem 3.2. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *$f(\cdot,S^*,u^*)$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *for each $j \in J_+ \equiv J_+(v^*)$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
  (iii) *$\bar{\rho} + \sum_{j\in J_+} v_j^* \tilde{\rho}_j \geqq 0$;*

(b) (i) *$f(\cdot,S^*,u^*)$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *$h(\cdot,v^*)$ is $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
  (iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$;*

(c) (i) *$f(\cdot,S^*,u^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *for each $j \in J_+$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
  (iii) *$\bar{\rho} + \sum_{j\in J_+} v_j^* \tilde{\rho}_j > 0$;*

(d) (i) *$f(\cdot,S^*,u^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *$h(\cdot,v^*)$ is $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
  (iii) *$\bar{\rho} + \tilde{\rho} > 0$;*

(e) (i) *$f(\cdot,S^*,u^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *for each $j \in J_+$, H_j is strictly $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$;*
  (iii) *$\bar{\rho} + \sum_{j\in J_+} v_j^* \tilde{\rho}_j \geqq 0$;*

(f) (i) *$f(\cdot,S^*,u^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
  (ii) *$h(\cdot,v^*)$ is strictly $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$-pseudounivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$;*
  (iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$.*

Then S^* is an efficient solution of (P).

Proof. (a) Let S be an arbitrary feasible solution of (P). Then, as seen in the proof of Theorem 3.1, our hypotheses in (ii) lead to (3.7), which when combined with (3.8) yields

$$\begin{aligned} &\mathscr{F}\left(S,S^*;b(S,S^*)\sum_{i=1}^{p} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right]\right) \\ &\qquad \geqq \left(\bar{\rho} + \sum_{j\in J_+} v_j^* \tilde{\rho}_j\right) d^2(\theta(S,S^*)) \geqq -\bar{\rho} d^2(\theta(S,S^*)), \end{aligned} \tag{3.16}$$

where the second inequality follows from (iii). By virtue of (i), this inequality implies that $\bar{\phi}(f(S,S^*,u^*) - f(S^*,S^*,u^*)) \geqq 0$, which because of the properties of the function $\bar{\phi}$, reduces to $f(S,S^*,u^*) \geqq f(S^*,S^*,u^*)$. But $f(S^*,S^*,u^*) = 0$, and hence we have that $f(S,S^*,u^*) \geqq 0$, which is precisely (3.10). Therefore, we conclude, as in the proof of Theorem 3.1, that S^* is an efficient solution of (P).

(b) The proof is similar to that of part (a).

(c) As seen in the proof of Theorem 3.1, our hypotheses in (ii) lead to (3.7), which when combined with (3.8) yields

$$\begin{aligned}\mathcal{F}\bigg(S,S^*;b(S,S^*)\sum_{i=1}^{p}u_i^*\big[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)\big]\bigg)\\ \geqq\bigg(\rho^*+\sum_{j\in J_+}v_j^*\tilde{\rho}_j\bigg)d^2\big(\theta(S,S^*)\big)>-\bar{\rho}d^2\big(\theta(S,S^*)\big),\end{aligned}\tag{3.17}$$

where the second inequality follows from (iii). By (i), this inequality implies that $\bar{\phi}(f(S,S^*,u^*)-f(S^*,S^*,u^*))\geqq 0$. But $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$, and hence we get $f(S,S^*,u^*)\geqq f(S^*,S^*,u^*)=0$, which is (3.10), and therefore we conclude, as in Theorem 3.1, that S^* is an efficient solution of (P).

(d)–(f) The proofs are similar to those of parts (a)–(c). □

Theorem 3.3. *Let $S^*\in\mathbb{F}$ and assume that F_i,G_i, $i\in\underline{p}$, and H_j, $j\in\underline{q}$, are differentiable at S^*, and that there exist $u^*\in U_0$ and $v^*\in\mathbb{R}_+^q$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $i\in I_+\equiv I_+(u^*)$, $f_i(\cdot,S^*)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j\in J_+\equiv J_+(v^*)$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j^*\tilde{\rho}_j\geqq 0$, where $\rho^\circ=\sum_{i\in I_+}u_i^*\bar{\rho}_i$;*

(b) (i) *for each $i\in I_+$, $f_i(\cdot,S^*)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *$h(\cdot,v^*)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^\circ+\tilde{\rho}\geqq 0$;*

(c) (i) *for each $i\in I_+$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j\in J_+$, H_j is strictly $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j^*\tilde{\rho}_j\geqq 0$;*

(d) (i) *for each $i\in I_+$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *$h(\cdot,v^*)$ is strictly $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-pseudounivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^\circ+\tilde{\rho}\geqq 0$;*

(e) (i) *for each $i\in I_+$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j\in J_+$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j^*\tilde{\rho}_j>0$;*

(f) (i) *for each $i\in I_+$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) $h(\cdot,v^*)$ *is* $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$*-quasiunivex at* S^*, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0)=0$;
(iii) $\rho^\circ+\tilde{\rho}>0$.
Then S^* *is an efficient solution of (P).*

Proof. (a) Suppose to the contrary that S^* is not an efficient solution of (P). Then there exists $\bar{S}\in\mathbb{F}$ such that $\varphi_i(\bar{S})\leqq\varphi_i(S^*)$ for each $i\in\underline{p}$, and $\varphi_\ell(\bar{S})<\varphi_\ell(S^*)$ for some $\ell\in\underline{p}$. From these inequalities it can easily be seen that for each $i\in I_+$,

$$G_i(S^*)F_i(\bar{S})-F_i(S^*)G_i(\bar{S})\leqq 0=G_i(S^*)F_i(S^*)-F_i(S^*)G_i(S^*), \tag{3.18}$$

which in view of the properties of $\bar{\phi}_i$ can be expressed as

$$\bar{\phi}_i\big(G_i(S^*)F_i(\bar{S})-F_i(S^*)G_i(\bar{S})-[G_i(S^*)F_i(S^*)-F_i(S^*)G_i(S^*)]\big)\leqq 0. \tag{3.19}$$

By (i), this implies that for each $i\in I_+$,

$$\mathscr{F}(\bar{S},S^*;b(\bar{S},S^*)[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)])<-\bar{\rho}_i d^2(\theta(\bar{S},S^*)). \tag{3.20}$$

Since $u^*\geqq 0$, $u_i^*=0$ for each $i\in\underline{p}\setminus I_+$, $\sum_{i\in I_+}u_i^*=1$, and $\mathscr{F}(\bar{S},S^*;\cdot)$ is sublinear, the above inequalities yield

$$\mathscr{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p}u_i^*[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)]\right)<-\sum_{i\in I_+}u_i^*\bar{\rho}_i d^2(\theta(\bar{S},S^*)). \tag{3.21}$$

Now combining (3.7) (which is valid for the present case because of (ii)), (3.8), and (3.21), and using the sublinearity of $\mathscr{F}(\bar{S},S^*;\cdot)$, we obtain

$$\begin{aligned}\mathscr{F}\Bigg(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p}u_i^*[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)]+\sum_{j=1}^{q}v_j^*DH_j(S^*)\Bigg)&\\<-\Bigg(\rho^\circ+\sum_{i\in J_+}v_j^*\tilde{\rho}_j\Bigg)d^2(\theta(\bar{S},S^*)),&\end{aligned} \tag{3.22}$$

which in view of (iii) contradicts (3.2). Hence, S^* is an efficient solution of (P).

(b)–(d) The proofs are similar to that of part (a).

(e) Proceeding as in the proof of part (a) and using the conditions set forth in (i), we arrive at the inequality

$$\begin{aligned}\mathscr{F}\Bigg(\bar{S},S^*;b(\bar{S},S^*)\sum_{i=1}^{p}u_i^*[G_i(S^*)DF_i(S^*)-F_i(S^*)DG_i(S^*)]\Bigg)&\\\leqq-\sum_{i\in I_+}u_i^*\bar{\rho}_i d^2(\theta(\bar{S},S^*)).&\end{aligned} \tag{3.23}$$

Combining this inequality with (3.8) and using (iii), we obtain

$$\begin{aligned}
&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{j=1}^{q}v_j^*DH_j(S^*)\right)\\
&\qquad \geqq \sum_{i\in I_+}u_i^*\bar{\rho}_i d^2(\theta(\bar{S},S^*)) > -\sum_{j\in J_+}v_j^*\tilde{\rho}_j d^2(\theta(\bar{S},S^*)),
\end{aligned} \tag{3.24}$$

which contradicts (3.7). Therefore, we conclude that S^* is an efficient solution of (P).

(f) The proof is similar to that of part (e). □

We close this section by stating a variant of Theorem 4.4; its proof is similar to that of Theorem 4.4 and hence omitted.

Theorem 3.4. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U_0$ and $v^* \in \mathbb{R}_+^q$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following four sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_{1+} \neq \emptyset$, $f_i(\cdot,S^*)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, and for each $i \in I_+ \equiv I_+(u^*)$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $j \in J_+ \equiv J_+(v^*)$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^\circ + \sum_{j\in J_+} v_j^*\tilde{\rho}_j \geqq 0$, where $\rho^\circ = \sum_{i\in I_+} u_i^*\bar{\rho}_i$;*

(b) (i) *for each $i \in I_{1+} \neq \emptyset$, $f_i(\cdot,S^*)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, and for each $i \in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *$h(\cdot,v^*)$ is $(\mathcal{F},\tilde{b},\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^\circ + \tilde{\rho} \geqq 0$;*

(c) (i) *for each $i \in I_+$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}(0)=0$;*

(ii) *for each $j \in J_{1+} \neq \emptyset$, H_j is strictly $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at S^*, for each $j \in J_{2+}$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, and for each $j \in J_+$, $\tilde{\phi}_j$ is increasing and $\tilde{\phi}_j(0)=0$, where $\{J_{1+},J_{2+}\}$ is a partition of J_+;*

(iii) *$\rho^\circ + \sum_{j\in J_+} v_j^*\tilde{\rho}_j \geqq 0$;*

(d) (i) *for each $i \in I_{1+}$, $f_i(\cdot,S^*)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $f_i(\cdot,S^*)$ is $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at S^*, and for each $i \in I_+ \equiv I_+(u^*)$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $j \in J_{1+}$, H_j is strictly $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at S^*, for each $j \in J_{2+}$, H_j is $(\mathcal{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at S^*, and for each $j \in J_+$, $\tilde{\phi}_j$ is increasing and $\tilde{\phi}_j(0)=0$, where $\{J_{1+},J_{2+}\}$ is a partition of J_+;*

(iii) *$\rho^\circ + \sum_{j\in J_+} v_j^*\tilde{\rho}_j \geqq 0$;*

(iv) *$I_{1+} \neq \emptyset$, $J_{1+} \neq \emptyset$, or $\rho^\circ + \sum_{j\in J_+} v_j^*\tilde{\rho}_j > 0$.*

Then S^* is an efficient solution of (P).

4. Generalized sufficient efficiency criteria

In this section, we formulate and discuss several families of generalized sufficiency results for (P) with the help of a partitioning scheme that was originally proposed in [32] for constructing generalized dual problems for nonlinear programs with point functions.

Let $\{J_0, J_1, \ldots, J_m\}$ be a partition of the index set $\underline{q}$; thus $J_r \subset \underline{q}$ for each $r \in \{0,1,\ldots,m\}$, $J_r \cap J_s = \emptyset$ for each $r,s \in \{0,1,\ldots,m\}$ with $r \neq s$, and $\cup_{r=0}^{m} J_r = \underline{q}$. In addition, we will make use of the functions $\Psi_i(\cdot, S^*, v^*)$, $\Psi(\cdot, S^*, u^*, v^*)$, and $\Lambda_t(\cdot, v^*) : \mathbb{A}^n \to \mathbb{R}$ defined, for fixed u^*, v^*, and S^*, by

$$\begin{gathered} \Psi_i(T,S^*,v^*) = G_i(S^*)F_i(T) - F_i(S^*)G_i(T) + \sum_{j\in J_0} v_j^* H_j(T), \quad i \in \underline{p}, \\ \Psi(T,S^*,u^*,v^*) = \sum_{i=1}^{p} u_i^* \left[G_i(S^*)F_i(T) - F_i(S^*)G_i(T)\right] + \sum_{j\in J_0} v_j^* H_j(T), \\ \Lambda_t(T,v^*) = \sum_{j\in J_t} v_j^* H_j(T), \quad t \in \underline{m} \cup \{0\}. \end{gathered} \tag{4.1}$$

Using these sets and functions, we next state and prove a number of generalized sufficiency results for (P).

Theorem 4.1. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following four sets of hypotheses is satisfied:*

(a) (i) *$\Psi(\cdot,S^*,u^*,v^*)$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
 (ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot,v^*)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*
 (iii) *$\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;*

(b) (i) *$\Psi(\cdot,S^*,u^*,v^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
 (ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot,v^*)$ is strictly $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*
 (iii) *$\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;*

(c) (i) *$\Psi(\cdot,S^*,u^*,v^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
 (ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot,v^*)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*
 (iii) *$\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t > 0$;*

(d) (i) *$\Psi(\cdot,S^*,u^*,v^*)$ is prestrictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at S^*, and $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$;*
 (ii) *for each $t \in \underline{m_1}$, $\Lambda_t(\cdot,v^*)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$, for each $t \in \underline{m_2} \neq \emptyset$, $\Lambda_t(\cdot,v^*)$ is strictly $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$, where $\{\underline{m_1},\underline{m_2}\}$ is a partition of $\underline{m}$;*
 (iii) *$\bar{\rho} + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$.*

Then S^* is an efficient solution of (P).

Proof. (a) Let S be an arbitrary feasible solution of (P). As $v^* \geqq 0$, it is clear from (3.3) that for each $t \in \underline{m}$,

$$\Lambda(S, v^*) = \sum_{j \in J_t} v_j^* H_j(S) \leqq 0 = \sum_{j \in J_t} v_j^* H_j(S^*) = \Lambda(S^*, v^*), \tag{4.2}$$

and hence using the properties of $\tilde{\phi}_t$, we get $\tilde{\phi}_t(\Lambda_t(S, v^*) - \Lambda_t(S^*, v^*)) \leqq 0$, which by (ii) implies that for each $t \in \underline{m}$,

$$\mathcal{F}\left(S, S^*; b(S, S^*) \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \leqq -\tilde{\rho}_t d^2(\theta(S, S^*)). \tag{4.3}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(S, S^*; \cdot)$, we obtain

$$\mathcal{F}\left(S, S^*; b(S, S^*) \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \leqq -\sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(S, S^*)). \tag{4.4}$$

From the sublinearity of $\mathcal{F}(S, S^*; \cdot)$ and (3.2), it follows that

$$\begin{aligned} &\mathcal{F}\left(S, S^*; b(S, S^*)\left\{\sum_{i=1}^{p} u_i^* [G_i(S^*) DF_i(S^*) - F_i(S^*) DG_i(S^*)] + \sum_{j \in J_0} v_j^* DH_j(S^*)\right\}\right) \\ &\quad + \mathcal{F}\left(S, S^*; b(S, S^*) \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* DH_j(S^*)\right) \geqq 0. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) and using (iii), we obtain the inequality

$$\begin{aligned} &\mathcal{F}\left(S, S^*; b(S, S^*)\left\{\sum_{i=1}^{p} u_i^* [G_i(S^*) DF_i(S^*) - F_i(S^*) DG_i(S^*)] + \sum_{j \in J_0} v_j^* DH_j(S^*)\right\}\right) \\ &\quad \geqq \sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(S, S^*)) \geqq -\bar{\rho} d^2(\theta(S, S^*)), \end{aligned} \tag{4.6}$$

which in view of (i) implies that $\bar{\phi}(\Psi(S, S^*, u^*, v^*) - \Psi(S^*, S^*, u^*, v^*)) \geqq 0$. Due to the properties of the function $\bar{\phi}$, this inequality yields $\Psi(S, S^*, u^*, v^*) - \Psi(S^*, S^*, u^*, v^*) \geqq 0$. But in view of (3.2) and (3.3), $\Psi(S^*, S^*, u^*, v^*) = 0$, and so we have that $\Psi(S, S^*, u^*, v^*) \geqq 0$. Since $v_j^* H_j(S) \leqq 0$ for each $j \in \underline{q}$, the last inequality reduces to

$$\sum_{i=1}^{p} u_i^* [G_i(S^*) F_i(S) - F_i(S^*) G_i(S)] \geqq 0, \tag{4.7}$$

which is precisely (3.10). Hence, we conclude, as in the proof of Theorem 3.1, that S^* is an efficient solution of (P).

(b)–(d) The proofs are similar to that of part (a). □

Theorem 4.2. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U_0$ and $v^* \in \mathbb{R}^q_+$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\Psi_i(\cdot, S^*, v^*)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(b) (i) *for each $i \in I_+$, $\Psi_i(\cdot, S^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(c) (i) *for each $i \in I_+$, $\Psi_i(\cdot, S^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t > 0$;

(d) (i) *for each $i \in I_{1+} \neq \emptyset$, $\Psi_i(\cdot, S^*, v^*)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $\Psi_i(\cdot, S^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, and for each $i \in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(e) (i) *for each $i \in I_+$, $\Psi_i(\cdot, S^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m_1} \neq \emptyset$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in \underline{m_2}$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in \underline{m}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{\underline{m_1}, \underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(f) (i) *for each $i \in I_{1+}$, $\Psi_i(\cdot, S^*, v^*)$ is strictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, for each $i \in I_{2+}$, $\Psi_i(\cdot, S^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at S^*, and for each $i \in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0) = 0$, where $\{I_{1+}, I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t \in \underline{m_1}$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in \underline{m_2}$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in \underline{m}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{\underline{m_1}, \underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;

(iv) $I_{1+} \neq \emptyset$, $\underline{m_1} \neq \emptyset$, *or* $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t > 0$.

Then S^* is an efficient solution of (P).

Proof. (a) Suppose to the contrary that S^* is not an efficient solution of (P). As seen in the proof of Theorem 3.3, this supposition leads to the inequalities $G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S}) \leqq 0$ for each $i \in \underline{p}$, and $G_\ell(S^*)F_\ell(\bar{S}) - F_\ell(S^*)G_\ell(\bar{S}) < 0$ for some $\ell \in \underline{p}$, where $\bar{S} \in \mathbb{F}$. Using these inequalities along with the feasibility of $\bar{S}$, nonnegativity of v^*, and

(3.3), we see that for each $i \in \underline{p}$,

$$\begin{aligned}
\Psi_i(\bar{S},S^*,v^*) &= G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S}) + \sum_{j\in J_0} v_j^* H_j(\bar{S}) \\
&\leqq G_i(S^*)F_i(\bar{S}) - \Phi(S^*,u^*)G_i(\bar{S}) \leqq 0 \\
&= G_i(S^*)F_i(S^*) - F_i(S^*)G_i(S^*) + \sum_{j\in J_0} v_j^* H_j(S^*) \\
&= \Psi_i(S^*,S^*,v^*),
\end{aligned} \tag{4.8}$$

with strict inequality holding for at least one index $i \in \underline{p}$. From the properties of $\bar{\phi}_i$, we see that for each $i \in \underline{p}$, $\bar{\phi}_i(\Psi_i(\bar{S},S^*,u^*,v^*) - \Psi_i(S^*,S^*,u^*,v^*)) \leqq 0$, which in view of (i) implies that

$$\begin{aligned}
&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*) + \sum_{j\in J_0} v_j^* DH_j(S^*)\right]\right) \\
&\quad < -\bar{\rho}_i d^2(\theta(\bar{S},S^*)).
\end{aligned} \tag{4.9}$$

Inasmuch as $u^* \geqq 0$, $u_i^* = 0$ for each $i \in \underline{p} \setminus I_+$, $\sum_{i\in I_+} u_i^* = 1$, and $\mathcal{F}(\bar{S},S^*;\cdot)$ is sublinear, these inequalities yield

$$\begin{aligned}
&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\left\{\sum_{i=1}^{p} u_i^*[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)] + \sum_{j\in J_0} v_j^* DH_j(S^*)\right\}\right) \\
&\quad < -\sum_{i\in I_+} u_i^* \bar{\rho}_i d^2(\theta(\bar{S},S^*)).
\end{aligned} \tag{4.10}$$

Now combining this inequality with (4.5) and using (iii), we obtain

$$\begin{aligned}
&\mathcal{F}\left(S,S^*;b(\bar{S},S^*)\sum_{t=1}^{m}\sum_{j\in J_t} v_j^* DH_j(S^*)\right) \\
&\quad > \sum_{i\in I_+} u_i^* \bar{\rho}_i d^2(\theta(\bar{S},S^*)) \geqq -\sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(\bar{S},S^*)),
\end{aligned} \tag{4.11}$$

contradicting (4.4), which is valid for the present case because of our hypotheses in (ii). Hence, S^* is an efficient solution of (P).

(b)–(f) The proofs are similar to that of part (a). □

Each of the ten sets of conditions specified in Theorems 4.1 and 4.2 can be viewed as a collection of sufficiency results for (P). Their special cases can easily be identified by appropriate choices of the partitioning sets J_r, $r = 0,1,\ldots,m$. We illustrate this possibility by stating explicitly some important special cases of Theorem 4.2 (a). They are collected in the following corollary.

Corollary 4.3. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U_0$ and $v^* \in \mathbb{R}_+^q$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following five sets of hypotheses is satisfied:*

(a) *for each $i \in I_+ \equiv I_+(u^*)$, the function $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$; the function $T \to \sum_{j=1}^q v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \tilde{\rho} \geqq 0$;*

(b) *for each $i \in I_+$, the function $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T) + \sum_{j=1}^q v_j^* H_j(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i \geqq 0$;*

(c) *for each $i \in I_+$, $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$; for each $j \in \underline{q}$, $T \to v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{j=1}^q \tilde{\rho}_j \geqq 0$;*

(d) *for each $i \in I_+$, $T \to (S^*)F_i(T) - F_i(S^*)G_i(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$; for each $t \in \underline{m}$, $T \to \sum_{j\in J_t} v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \sum_{t=1}^m \tilde{\rho}_t \geqq 0$;*

(e) *for each $i \in I_+$, $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T) + \sum_{j\in J_0} v_j^* H_j(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$; $T \to \sum_{j\in J_1} v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \tilde{\rho} \geqq 0$.*

Then S^* is an efficient solution of (P).

Proof. In Theorem 4.2 (a), let (a) $J_1 = \underline{q}$, (b) $J_0 = \underline{q}$, (c) $m = q$ and $J_t = \{t\}$, $t \in \underline{q}$, (d) $J_0 = \emptyset$, (e) $J_t = \emptyset$ for $t = 2, 3, \ldots, m$. □

Comparing parts (a) and (c) of the above corollary, we see that they represent two extreme cases with regard to the $(\mathscr{F}, b, \phi, \rho, \theta)$-quasiunivexity assumptions in the sense that in (a) all the functions $T \to v_j^* H_j(T)$ are lumped together, whereas in (c) separate $(\mathscr{F}, b, \phi, \rho, \theta)$-quasiunivexity conditions are imposed on the individual functions. It is also possible to devise sufficiency conditions that lie between these two extremes. For example, one may consider the following variant of (a):

(a) for each $i \in I_+$, $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at S^*, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0) = 0$, $T \to \sum_{j\in J_1}^q v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at S^*, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0) = 0$; for each $j \in J_2$, $T \to v_j^* H_j(T)$ is $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at S^*, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0) = 0$; and $\sum_{i\in I_+} u_i^* \bar{\rho}_i + \tilde{\rho} + \sum_{j\in J_2} \tilde{\rho}_j \geqq 0$, where $\{J_1, J_2\}$ is a partition of $\underline{q}$.

In a similar manner, one can determine numerous special cases and variants of the other nine sets of sufficient efficiency conditions given in Theorems 4.1 and 4.2.

Finally, in the remainder of this section, we present several sets of sufficiency results for (P) that are different from those stated in Theorems 4.1 and 4.2. These results involve generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-univexity assumptions placed on different combinations of the functions $T \to v_j^* H_j(T)$ and $T \to G_i(S^*)F_i(T) - F_i(S^*)G_i(T)$ arising from a partition of the index set $\underline{p}$.

Let $\{I_0, I_1, \dots, I_k\}$ be a partition of $\underline{p}$ such that $K \equiv \{0, 1, \dots, k\} \subset M \equiv \{0, 1, \dots, m\}$, and let the function $\Omega_t(\cdot, S^*, u^*, v^*) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed S^*, u^*, and v^*, by

$$\Omega_t(T, S^*, u^*, v^*) = \sum_{i \in I_t} u_i^* [G_i(S^*) F_i(T) - F_i(S^*) G_i(T)] + \sum_{j \in J_t} v_j^* H_j(T), \quad t \in K. \tag{4.12}$$

Theorem 4.4. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.2) and (3.3) hold. Assume, furthermore, that any of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $t \in K$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is strictly $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-pseudounivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (ii) *for each $t \in M \backslash K$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (iii) *$\sum_{t \in M} \rho_t \geqq 0$;*

(b) (i) *for each $t \in K$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (ii) *for each $t \in M \backslash K$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-pseudounivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (iii) *$\sum_{t \in M} \rho_t \geqq 0$;*

(c) (i) *for each $t \in K$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (ii) *for each $t \in M \backslash K$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (iii) *$\sum_{t \in M} \rho_t > 0$;*

(d) (i) *for each $t \in K_1 \neq \emptyset$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is strictly $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in K_2$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in K$, $\bar{\phi}_t$ is increasing and $\bar{\phi}_t(0) = 0$, where $\{K_1, K_2\}$ is a partition of K;*
 (ii) *for each $t \in M \backslash K$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (iii) *$\sum_{t \in M} \rho_t \geqq 0$;*

(e) (i) *for each $t \in K$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$-quasiunivex at S^*, ϕ_t is increasing, and $\phi_t(0) = 0$;*
 (ii) *for each $t \in (M \setminus K)_1 \neq \emptyset$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in (M \setminus K)_2$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in M \setminus K$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{(M \setminus K)_1, (M \setminus K)_2\}$ is a partition of $M \setminus K$;*
 (iii) *$\sum_{t \in M} \rho_t \geqq 0$;*

(f) (i) *for each $t \in K_1$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is strictly $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in K_2$, $\Omega_t(\cdot, S^*, u^*, v^*)$ is $(\mathcal{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in K$, $\bar{\phi}_t$ is increasing and $\bar{\phi}_t(0) = 0$, where $\{K_1, K_2\}$ is a partition of K;*
 (ii) *for each $t \in (M \setminus K)_1$, $\Lambda_t(\cdot, v^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at S^*, for each $t \in (M \setminus K)_2$, $\Lambda_t(\cdot, v^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at S^*, and for each $t \in M \setminus K$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0) = 0$, where $\{(M \setminus K)_1, (M \setminus K)_2\}$ is a partition of $M \setminus K$;*

(iii) $\sum_{t\in M}\rho_t \geqq 0$;
(iv) $K_1 \neq \emptyset$, $(M \setminus K)_1 \neq \emptyset$, *or* $\sum_{t\in M}\rho_t > 0$.

Then S^* is an efficient solution of (P).

Proof. (a) Suppose to the contrary that S^* is not an efficient solution of (P). As seen in the proof of Theorem 3.3, this supposition leads to the inequalities $G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S}) \leqq 0$ for each $i \in \underline{p}$, and $G_\ell(S^*)F_\ell(\bar{S}) - F_\ell(S^*)G_\ell(\bar{S}) < 0$ for some $\ell \in \underline{p}$, for some $\bar{S} \in \mathbb{F}$. Since $u^* > 0$, these inequalities yield

$$\sum_{i\in I_t} u_i^*\left[G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S})\right] \leqq 0, \quad t \in K. \tag{4.13}$$

Inasmuch as $v_j^* H_j(\bar{S}) \leqq 0$ for each $j \in \underline{q}$, and $\bar{S}, S^* \in \mathbb{F}$, it follows from these inequalities and (3.3) that for each $t \in K$,

$$\begin{aligned}\Omega_t(\bar{S},S^*,u^*,v^*) &= \sum_{i\in I_t} u_i^*\left[G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S})\right] + \sum_{j\in J_t} v_j^* H_j(\bar{S}) \leqq 0 \\ &= \sum_{i\in I_t} u_i^*\left[G_i(S^*)F_i(S^*) - F_i(S^*)G_i(S^*)\right] + \sum_{j\in J_t} v_j^* H_j(S^*) \\ &= \Omega_t(S^*,S^*,u^*,v^*),\end{aligned} \tag{4.14}$$

and so using the properties of ϕ_t, we have that for each $t \in K$, $\phi_t(\Omega_t(\bar{S},S^*,u^*,v^*) - \Omega_t(S^*,S^*,u^*,v^*)) \leqq 0$, which in view of (i) implies that for each $t \in K$,

$$\begin{aligned}&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\left\{\sum_{i\in I_t} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \sum_{j\in J_t} v_j^* DH_j(S^*)\right\}\right) \\ &\qquad < -\rho_t d^2(\theta(\bar{S},S^*)).\end{aligned} \tag{4.15}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$, we obtain

$$\begin{aligned}&\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\left\{\sum_{i=1}^{p} u_i^*\left[G_i(S^*)DF_i(S^*) - F_i(S^*)DG_i(S^*)\right] + \sum_{t\in K}\sum_{j\in J_t} v_j^* DH_j(S^*)\right\}\right) \\ &\qquad < -\sum_{t\in K}\rho_t d^2(\theta(\bar{S},S^*)).\end{aligned} \tag{4.16}$$

Since for each $t \in M\backslash K$, $\Lambda_t(\bar{S},v^*) \leqq 0 = \Lambda_t(S^*,v^*)$, it follows from the properties of ϕ_t that $\phi_t(\Lambda_t(\bar{S},v^*) - \Lambda_t(S^*,v^*)) \leqq 0$, which by (ii) implies that

$$\mathcal{F}\left(\bar{S},S^*;b(\bar{S},S^*)\sum_{j\in J_t} v_j^* DH_j(\bar{S})\right) \leqq -\rho_t d^2(\theta(\bar{S},S^*)). \tag{4.17}$$

Adding these inequalities and using the sublinearity of $\mathscr{F}(\bar{S}, S^*; \cdot)$, we get

$$\mathscr{F}\left(\bar{S}, S^*; b(\bar{S}, S^*) \sum_{t \in M \setminus K} \sum_{j \in J_t} v_j^* DH_j(\bar{S})\right) \leqq - \sum_{t \in M \setminus K} \rho_t d^2(\theta(\bar{S}, S^*)). \tag{4.18}$$

Now combining this inequality with (4.16) and using the sublinearity of $\mathscr{F}(\bar{S}, S^*; \cdot)$ and (iii), we see that

$$\begin{aligned} &\mathscr{F}\left(\bar{S}, S^*; b(\bar{S}, S^*)\left\{\sum_{i=1}^{p} u_i^*\left[G_i(S^*) DF_i(S^*) - F_i(S^*) DG_i(S^*)\right] + \sum_{j=1}^{q} v_j^* DH_j(S^*)\right\}\right) \\ &\qquad < - \sum_{t \in M} \rho_t d^2(\theta(\bar{S}, S^*)) \leqq 0, \end{aligned} \tag{4.19}$$

which contradicts (3.2). Therefore, S^* is an efficient solution of (P).

(b)–(f) The proofs are similar to that of part (a). □

Following the approach employed in generating Corollary 4.3, we can easily identify numerous special cases of the six sets of sufficient efficiency conditions given in Theorem 4.4.

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