

SOLITARY-WAVE PROPAGATION AND INTERACTIONS FOR A SIXTH-ORDER GENERALIZED BOUSSINESQ EQUATION

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We study the solitary waves and their interaction for a six-order generalized Boussinesq equation (SGBE) both numerically and analytically. A shooting method with appropriate initial conditions, based on the phase plane analysis around the equilibrium point, is used to construct the solitary-wave solutions for this nonintegrable equation. A symmetric three-level implicit finite difference scheme with a free parameter θ is proposed to study the propagation and interactions of solitary waves. Numerical simulations show the propagation of a single solitary wave of SGBE, and two solitary waves pass by each other without changing their shapes in the head-on collisions.

1. Introduction

There has been considerable interest in nonlinear lattice dynamics since the second half of the last century. Such lattice models, in spite of their relative simplicity, are associated with rather important problems in physics [13]. Usually, a quasicontinuum approximation for a nonlinear lattice gives rise to the Boussinesq-type equation (see (2.7) in the subsequent section). In fact, (2.7) is a “bad” Boussinesq equation (BaBE) as the sign in front of the fourth-order derivative yields unpleasant numerical features [2, 9]. According to Rosenau [12], this is simply due to a bad expansion in the discrete-continuum transition. Ostrovskii et al. introduced an improved Boussinesq equation (IBE) with a mixed fourth-order space-time derivative while incorporating the effect of lateral inertia in the one-dimensional dynamical description of elastic rods [10]. We note that the works in [3, 7] have numerically studied the dynamical behavior of IBEs.

In the numerical study of the Fermi-Pasta-Ulam problem [5], Kruskal and Zabusky avoided the difficulty of BaBE problem by considering its unidirectional variant, that is, the KdV equation [14]. The intriguing findings regarding the KdV equation led to the development of soliton and nonlinear wave theory [4]. On the other hand, Maugin et al. proposed a sixth-order Boussinesq equation (SGBE) by considering a sixth-order term in the expansion of discrete-continuum transition and corrected the bad numerical features [2, 9]. In the present paper, we study the solitary-wave propagation and interactions of the SGBE. In Section 2, we will derive the Boussinesq equation and the

sixth-order Boussinesq equation (SGBE) arising in the one-dimensional nonlinear lattice. In Section 3, we seek traveling wave solutions of the SGBE numerically. In contrast to the Boussinesq equation, the SGBE admits the solitary-wave solutions in a very narrow range, whose reason is still unknown theoretically. In Section 4, we propose a conservative implicit finite difference method for the SGBE and analyze the method with respect to the accuracy and the linear stability. The conservative quantities of the SGBE are also checked for this method. In Section 5, by virtue of the solitary-wave solutions obtained in Section 3, we perform the numerical simulations of propagations and head-on collisions of solitary waves. The results indicate that these solitary solutions are stable from the fact that head-on collisions of two solitary waves are elastic. In Section 6, we give a brief summary and further aspects for our future study.

2. The continuum approximation of the one-dimensional nonlinear lattice

We consider a one-dimensional lattice of N particles with equal mass m with nearest-neighbor interactions. In the equilibrium position, the particles are uniformly spaced each at distance a from its nearest neighbor. We denote the longitudinal displacement of n th particle from its equilibrium position by x_n , the relative displacement between n th and $(n + 1)$ th particles by $r_n = x_n - x_{n+1}$. The interaction potential between two adjacent particles is assumed to be $\Phi(r_n)$. The equations of motion of the n th particle is

$$m\ddot{x}_n = \Phi'(r_{n-1}) - \Phi'(r_n), \quad n = 1, \dots, N. \tag{2.1}$$

In the general case, the interaction potential Φ may be chosen to have the form of a standard atomic potential model like the Morse or Lennard-Jones potential or the Toda potential, here we only consider the potential has the simple form

$$\Phi(r) = \frac{1}{2}\kappa r^2 + \frac{1}{3}\kappa\alpha r^3. \tag{2.2}$$

Thus, (2.1) becomes

$$m\ddot{x}_n = \kappa(x_{n+1} - 2x_n + x_{n-1})[1 + \alpha(x_{n+1} - x_{n-1})]. \tag{2.3}$$

In the continuum limit, we approximate $x_n(t)$ by a continuous function $w(x, t)$ and by substituting the Taylor series expansion

$$w_{n\pm 1} = w \pm aw_x + \frac{a^2}{2}w_{xx} \pm \frac{a^3}{3!}w_{xxx} + \frac{a^4}{4!}w_{xxxx} \pm \frac{a^5}{5!}w_{xxxxx} + \frac{a^6}{6!}w_{xxxxxx} + \dots \tag{2.4}$$

into (2.3). When we consider the Taylor series expansion including forth-order terms, and setting $c_0 = \kappa a^2/m$, we obtain

$$w_{tt} - c_0^2 w_{xx} (1 + 2\alpha a w_x) - \left(\frac{c_0^2 a^2}{12}\right) w_{xxxx} = 0. \tag{2.5}$$

On the other hand, by truncating the Taylor series expansion up to the sixth order, we arrive at the following equation:

$$w_{tt} - c_0^2 w_{xx} (1 + 2\alpha a w_x) - \left(\frac{c_0^2 a^2}{12}\right) w_{xxxx} - \left(\frac{c_0^2 a^4}{360}\right) w_{xxxxxx} = 0. \tag{2.6}$$

Applying a transformation $x \rightarrow (a/\sqrt{12})x$, $t \rightarrow (a/\sqrt{12}c_0)t$, $w \rightarrow (6/\alpha a)w$, and setting $w_x = u$, we obtain

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0, \tag{2.7}$$

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} - 0.4u_{xxxxxx} = 0, \tag{2.8}$$

from (2.5) and (2.6), respectively. Equation (2.7) is the classical Boussinesq equation (BE) [1]. In fact, the initial-value problem of this equation is ill-posed. In order to correct the bad numerical feature of the Boussinesq equation, Maugin introduced (2.8) and referred to it as the sixth-order Boussinesq equation (SGBE). This can be understood this way. Assuming (2.7) and (2.8) have the small-amplitude solution

$$u \sim \exp(i\kappa x + i\omega t), \tag{2.9}$$

after substituting, one can get the linearized dispersion relations

$$\omega^2 = \kappa^2 - \kappa^4, \tag{2.10}$$

$$\omega^2 = \kappa^2 - \kappa^4 + 0.4\kappa^6, \tag{2.11}$$

respectively, from (2.7) and (2.8). It is obvious that $\omega^2 > 0$ for any wave number κ in (2.11), whereas, in (2.10), ω^2 becomes negative if $\kappa > 1.0$. This fact explains why (2.7) is ill-posed, but (2.8) is well-posed.

3. Traveling-wave solutions for the generalized Boussinesq equation

In this communication, we seek the numerical solitary-wave solutions of the SGBE and consider the corresponding fundamental properties of soliton solutions. As the similar procedure in [6], we introduce a new function $v(x, t)$ and rewrite (2.8) into the following simultaneous equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial x}, \\ \frac{\partial v}{\partial t} &= -\frac{\partial}{\partial x} (u + 6u^2 + u_{xx} + 0.4u_{xxxx}). \end{aligned} \tag{3.1}$$

For a solitary-wave solution defined by the conditions that $u(x, t)$, $v(x, t)$, and their derivatives vanish at $t \rightarrow \infty$ and $|x| \rightarrow \infty$, we obtain the following conservation laws by integrating (3.3):

$$\begin{aligned} \int_{-\infty}^{\infty} u(x, t) dx &= \text{const}, \\ \int_{-\infty}^{\infty} v(x, t) dx &= \text{const}, \\ \int_{-\infty}^{\infty} u(x, t)v(x, t) dx &= \text{const}. \end{aligned} \tag{3.2}$$

Assuming $u(x, t) = u(x - ct)$, $v(x, t) = v(x - ct)$, we obtain the following relations:

$$\begin{aligned} cu(x, t) &= v(x, t), \\ cv(x, t) &= u + 6u^2 + u_{xx} + 0.4u_{xxxx}. \end{aligned} \tag{3.3}$$

From above equations, we have

$$c^2 = \frac{\int_{-\infty}^{\infty} u dx + 6 \int_{-\infty}^{\infty} u^2 dx}{\int_{-\infty}^{\infty} u dx}, \tag{3.4}$$

which shows that the velocity is greater than unity provided that $u(x, t) > 0$, that is, the solitary wave existing in one-dimensional lattice is supersonic.

Next, we look for solitary-wave solutions of SGBE numerically. By introducing the coordinate $z = x - ct$ moving with a velocity c , one can reduce (2.8) to an ordinary differential equation of the sixth order by setting $u \equiv U(z)$. Integrating twice with respect to z and considering the boundary conditions $U = U' = U'' = U''' = U'''' = 0$ at infinity, we obtain

$$0.4U'''' + U'' + 6U^2 + (1 - c^2)U = 0, \tag{3.5}$$

where the prime denotes a differentiation with respect to z .

A new type of solitary waves with oscillatory tails was firstly found by one of the authors [8]. Here, a similar shooting method is used to find solitary-wave solutions numerically. To this end, (3.5) is rewritten into a set of first-order equations:

$$\begin{aligned} U' &= V, \\ V' &= W, \\ W' &= S, \\ S' &= -2.5[6U^2 + (1 - c^2)U + W]. \end{aligned} \tag{3.6}$$

A possible solution describes a curve in the four-dimensional phase space (U, V, W, S) . The origin $O(0, 0, 0, 0)$ of this phase space is an equilibrium of (3.6) and the dynamical

property around O is determined by the roots of the characteristic equation corresponding to the linearized version of (3.5) around O .

$$0.4\mu^4 + \mu^2 + (1 - c^2) = 0. \tag{3.7}$$

We start iteration from $z \approx -\infty$ and assume the starting values as

$$\begin{aligned} U &= \epsilon, \\ V &= \mu_0\epsilon, \\ W &= \mu_0^2\epsilon, \\ S &= \mu_0^3\epsilon, \end{aligned} \tag{3.8}$$

where μ_0 is the maximum positive root of (3.7) and the classical fourth-order Runge-Kutta method is used.

Figure 3.1 shows the numerical results, for solitary waves corresponding to $c \approx 1.01, 1.03,$ and $1.05,$ respectively. The monotone solitary-wave solutions only exist in a very narrow range, that is, they exist only when the velocity c is very close to unity within 5%, but larger than unity, whose reason we still do not understand.

It is well known that the single soliton solution for the Boussinesq equation (2.7) can be expressed as

$$u(x, t) = \frac{c^2 - 1}{4} \operatorname{sech}^2 \left[\sqrt{c^2 - 1} (x \mp ct - x_0) \right]. \tag{3.9}$$

We give a comparison for the solitary-wave solutions for the Boussinesq equation and the SGBE with the same velocity $c = 1.03$ in Figure 3.2. We also plot a curve governed by a $\operatorname{sech}^{2/3}$ -type function

$$u(x, t) = \frac{c^2 - 1}{4} \operatorname{sech}^{2/3} \left[\sqrt{c^2 - 1} (x \mp ct - x_0) \right]. \tag{3.10}$$

It is obvious that for the same velocity, the solitary waves for the Boussinesq equation and the SGBE are the same amplitude, but the solitary wave for the Boussinesq equation is narrower. It is very interesting to note that solitary-wave solution for the SGBE is very similar to the $\operatorname{sech}^{2/3}$ type function (with the same velocity, amplitude, and width) except for some subtle differences.

4. A conservative finite difference scheme for SGBE

We consider an implicit finite difference scheme for the SGBE (2.8). For convenience, an equidistant mesh in space and time will be used with Δt and $\Delta x,$ respectively, in the (t, x) plane. Let L be the spatial period and T the computation time. Then the grid will

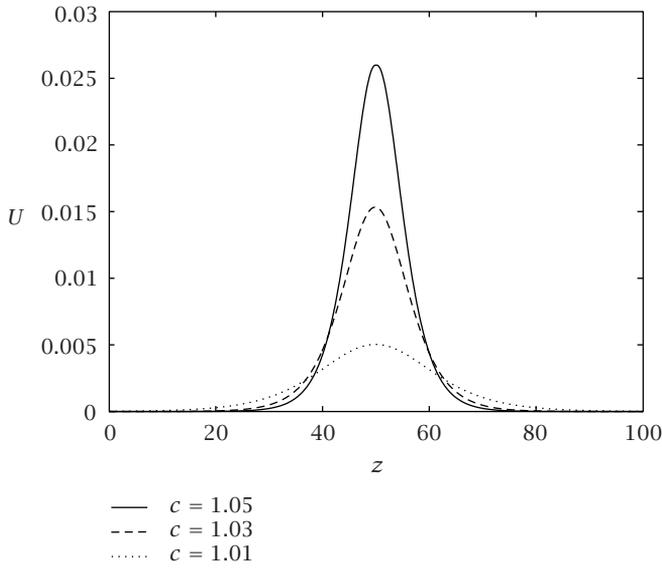


Figure 3.1. Solitary waves of the SGBE by numerical integration.

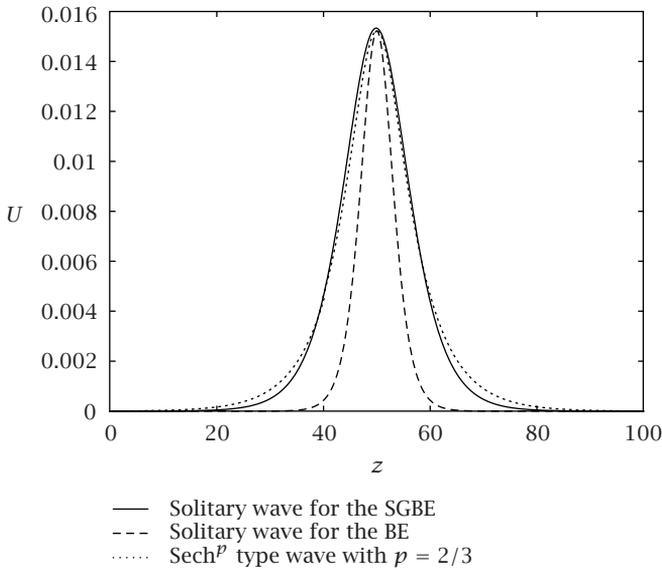


Figure 3.2. Comparison for the solitary waves of the Boussinesq equation and the SGBE.

be the points $(t_n, x_l) = (n\Delta t, l\Delta x)$. The solution value $u(x, t)$ will be approximated as $u_l^n \approx u(l\Delta x, n\Delta t)$, where $l = 0, \dots, N = [L/\Delta x]$ and $n = 1, \dots, [T/\Delta t]$.

Conventionally, the following difference operators are used:

$$\begin{aligned}
 \Delta_t u_l^n &= \frac{u_l^{n+1} - u_l^n}{\Delta t}, \\
 \nabla_t u_l^n &= \frac{u_l^n - u_l^{n-1}}{\Delta t}, \\
 \Delta_x u_l^n &= \frac{u_{l+1}^n - u_l^n}{\Delta x}, \\
 \nabla_x u_l^n &= \frac{u_l^n - u_{l-1}^n}{\Delta x}, \\
 \delta_t^2 u_l^n &= \frac{u_l^{n+1} - 2u_l^n + u_l^{n-1}}{(\Delta t)^2}, \\
 \delta_x^2 u_l^n &= \frac{u_{l+1}^n - 2u_l^n + u_{l-1}^n}{(\Delta x)^2}, \\
 \delta_x^4 u_l^n &= \frac{u_{l+2}^n - 4u_{l+1}^n + 6u_l^n - 4u_{l-1}^n + u_{l-2}^n}{(\Delta x)^4}, \\
 \delta_x^6 u_l^n &= \frac{u_{l+3}^n - 6u_{l+2}^n + 15u_{l+1}^n - 20u_l^n + 15u_{l-1}^n - 6u_{l-2}^n + u_{l-3}^n}{(\Delta x)^6}.
 \end{aligned} \tag{4.1}$$

For (2.8) subject to smooth initial conditions

$$u(x, 0) = u^0(x), \quad u(x, \Delta t) = u^1(x) \quad (0 \leq x \leq L) \tag{4.2}$$

and periodic boundary conditions

$$u_l^n = u_m^n \quad \text{if } l \equiv m \pmod{N}, \tag{4.3}$$

we propose the implicit finite difference scheme for (2.8):

$$\begin{aligned}
 \delta_t^2 u_l^n - \delta_x^2 [(1 - 2\theta)u_l^n + \theta(u_l^{n-1} + u_l^{n+1})] - 6\delta_x^2 [(1 - 2\theta)u_l^{n2} + \theta((u_l^{n-1})^2 + (u_l^{n+1})^2)] \\
 - \delta_x^4 [(1 - 2\theta)u_l^n + \theta(u_l^{n-1} + u_l^{n+1})] - 0.4\delta_x^6 [(1 - 2\theta)u_l^n + \theta(u_l^{n-1} + u_l^{n+1})] = 0.
 \end{aligned} \tag{4.4}$$

Here, θ is a parameter with the value $0 \leq \theta \leq 1/2$.

The nonlinear part in (4.4) can be linearized by virtue of Taylor's expansion of $(u^2)_l^{n-1}$ and $(u^2)_l^{n+1}$ about the n th time level. After some manipulation of algebra, the scheme (4.4) can be rewritten in the following system of linear equations:

$$au_{j+3}^{n+1} + bu_{j+2}^{n+1} + cu_{j+1}^{n+1} + du_j^{n+1} + cu_{j-1}^{n+1} + bu_{j-2}^{n+1} + au_{j-3}^{n+1} = e_j^n, \tag{4.5}$$

with the coefficients given by

$$\begin{aligned} a &= -0.4\theta s, & b &= 6\theta s - \theta r, \\ c &= -15\theta s + 4\theta r - \theta p, & d &= 20\theta s - 6\theta r + 2\theta p + 1, \end{aligned} \tag{4.6}$$

$$\begin{aligned} e_i^n &= 0.4s[(1 - 2\theta)(u_{i+3}^n - 6u_{i+2}^n + 15u_{i+1}^n - 20u_i^n + 15u_{i-1}^n - 6u_{i-2}^n + u_{i-3}^n) \\ &\quad + \theta(u_{i+3}^{n-1} - 6u_{i+2}^{n-1} + 15u_{i+1}^{n-1} - 20u_i^{n-1} + 15u_{i-1}^{n-1} - 6u_{i-2}^{n-1} + u_{i-3}^{n-1})] \\ &\quad + r[(1 - 2\theta)(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n) \\ &\quad + \theta(u_{i+2}^{n-1} - 4u_{i+1}^{n-1} + 6u_i^{n-1} - 4u_{i-1}^{n-1} + u_{i-2}^{n-1})] + 2u_i^n - u_i^{n-1} \\ &\quad + p[(1 - 2\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1})] \\ &\quad + 6p[(u_{i+1}^n)^2 - 2(u_i^n)^2 + (u_{i-1}^n)^2]. \end{aligned} \tag{4.7}$$

The parameters p , r , and s in above equation (4.7) are defined as

$$p = \frac{(\Delta t)^2}{(\Delta x)^2}, \quad r = \frac{(\Delta t)^2}{(\Delta x)^4}, \quad s = \frac{(\Delta t)^2}{(\Delta x)^6}, \quad \text{respectively.} \tag{4.8}$$

Scheme (4.4) is a *general three-level implicit* scheme with a free parameter θ . The parameter $\theta = 1/4$ gives a method discussed by Richtmyer and Morton [11] for linear hyperbolic equation.

Next, we consider the truncation error and the linearized stability of the scheme. If the exact solution of the differential equations is substituted into the difference scheme and the Taylor expansion is used, we obtain the following truncation error:

$$\begin{aligned} u_{tt} &= u_{xx} + 6(u^2)_{xx} + u_{xxxx} + 0.4u_{xxxxxx} \\ &\quad + \left[-\frac{(\Delta t)^2}{12}u_{tttt} + \frac{(\Delta x)^2}{12}u_{xxxx} + \theta(\Delta t)^2u_{xxtt} + \frac{(\Delta x)^2}{2}(u^2)_{xxxx} \right. \\ &\quad \left. + \frac{(\Delta x)^2}{12}u_{xxxxxx} + \theta(\Delta t)^2u_{xxxxtt} + \frac{(\Delta x)^2}{30}u_{xxxxxxx} + \frac{2\theta(\Delta t)^2}{5}u_{xxxxxxtt} \right] \\ &\quad + O((\Delta x)^4 + (\Delta t)^4 + (\Delta x)^2(\Delta t)^2). \end{aligned} \tag{4.9}$$

Therefore, the local truncation error for the scheme (4.4) is $O((\Delta x)^2 + (\Delta t)^2)$.

The following theorem gives the result of the linearized stability analysis of the scheme (4.4).

THEOREM 4.1. *If $\theta \geq 1/4$, then the scheme (4.5) is unconditionally linear stable.*

Proof. Firstly, we freeze one variable in the nonlinear part, namely, $(u^2)_{xx} = \bar{U}u_{xx}$ with $\bar{U} = \max |u|$. Substituting $u_i^n = \xi^n e^{i\beta l}$, $\beta \in [-\pi, \pi]$, into (4.5), after some algebraic manipulation, one obtains

$$[1 + \theta f(\beta)]\xi^2 + [f(\beta)(1 - 2\theta) - 2]\xi + [1 + \theta f(\beta)] = 0. \tag{4.10}$$

Here $f(\beta)$ is a real function of β and it is defined as

$$f(\beta) = 2(1 - \cos\beta) \left\{ 0.4[2(1 - \cos\beta)]^2 s - 2(1 - \cos\beta)r + (1 + 6\bar{U})p \right\}. \tag{4.11}$$

Define

$$g(\xi) = [1 + \theta f(\beta)]\xi^2 + [f(\beta)(1 - 2\theta) - 2]\xi + [1 + \theta f(\beta)] \tag{4.12}$$

and $\xi^* = 1/\bar{\xi}$, then we have

$$\begin{aligned} g(\xi^*) &= [1 + \theta f(\beta)]\bar{\xi}^{-2} + [f(\beta)(1 - 2\theta) - 2]\bar{\xi}^{-1} + [1 + \theta f(\beta)], \\ \overline{g(\xi^*)} &= [1 + \theta f(\beta)]\xi^{-2} + [f(\beta)(1 - 2\theta) - 2]\xi^{-1} + [1 + \theta f(\beta)]. \end{aligned} \tag{4.13}$$

Therefore, with $g^*(\xi) = \xi^2 \overline{g(\xi^*)}$, we can show

$$g^*(\xi) = [1 + \theta f(\beta)]\xi^2 + [f(\beta)(1 - 2\theta) - 2]\xi + [1 + \theta f(\beta)] = g(\xi). \tag{4.14}$$

Then the Bezont resultant

$$\check{g}(\xi) = \frac{g^*(0)g(\xi) - g(0)g^*(\xi)}{\xi} \tag{4.15}$$

turns out to be zero equivalently.

For g to be Von Neumann, it is sufficient to show that

- (1) $\check{g} \equiv 0$, and
- (2) g' is Von Neumann.

Now $\check{g} \equiv 0$, then for g Von Neumann, we are required to show that $g'(\xi)$ is Von Neumann. From (4.12),

$$g'(\xi) = 2[1 + \theta f(\beta)]\xi + [f(\beta)(1 - 2\theta) - 2]. \tag{4.16}$$

Hence, we require the zero of g' to be

$$|\xi| = \left| \frac{(1 - 2\theta)f(\beta) - 2}{2[1 + \theta f(\beta)]} \right| \leq 1, \quad \forall \beta. \tag{4.17}$$

Noting $f(\beta) \geq 0$ for all β , one can easily verify that if $\theta \geq 1/4$; the above inequality (4.17) is satisfied, thus the zeros of $g(\xi)$ lie on the unit circle. In summary, Von Neumann analysis gives the conclusion that the scheme (4.5) is linear stable. \square

THEOREM 4.2. Assume u_l^n and v_l^n are N -periodic mesh functions, that is, $u_l^n = u_{l+N}^n$, $v_l^n = v_{l+N}^n$, then for $\theta = 1/4$, the difference scheme (4.4) admits the following three invariants:

$$\sum_{l=1}^N u_l^n, \quad \sum_{l=1}^N v_l^n, \quad \sum_{l=1}^N u_l^n v_l^n. \tag{4.18}$$

Proof. Firstly, under the assumptions $\sigma(u) = u + 6u^2$ and $\sigma_l^n = \sigma(u_l^n)$, the scheme (4.4) is equivalent to the following implicit scheme in the case of $\theta = 1/4$:

$$\begin{aligned} \Delta_t u_l^n &= \frac{1}{2} \Delta_x (v_l^n + v_l^{n+1}), \\ \nabla_t v_l^n &= \frac{1}{2} \nabla_x [(\sigma_l^n + \sigma_l^{n+1}) + \delta_x^2 (u_l^n + u_l^{n+1}) + 0.4\delta_x^4 (u_l^n + u_l^{n+1})]. \end{aligned} \tag{4.19}$$

Multiplying both sides of (4.19) by Δt and summing over l , one obtains the invariants $\sum_{l=1}^N u_l^n$ and $\sum_{l=1}^N v_l^n$ directly. Also,

$$\begin{aligned} \sum_{l=1}^N u_l^{n+1} v_l^{n+1} - \sum_{l=1}^N u_l^n v_l^n &= \sum_{l=1}^N (u_l^{n+1} v_l^{n+1} - u_l^n v_l^{n+1} + u_l^n v_l^{n+1} - u_l^n v_l^n) \\ &= v_l^{n+1} \sum_{l=1}^N (u_l^{n+1} - u_l^n) - u_l^n \sum_{l=1}^N (v_l^{n+1} - v_l^n). \end{aligned} \tag{4.20}$$

From the first two invariants, we conclude that the third invariant $\sum_{l=1}^N u_l^n v_l^n = \text{const.}$ \square

Although we have just shown that the two conserved quantities $\int u dx$ and $\int v dx$ and the total energy $\int uv dx$ over a period are conserved in the special case $\theta = 1/4$, the practical computation shows that the scheme (4.5) seems to be a conservative linear stable difference scheme for $\theta \geq 1/4$.

5. Propagation and interaction of the solitary waves

In this section, in order to check the propagation of the obtained solitary waves and study their interactions, we present the results of some numerical simulations in regard with SGBE. We choose the solitary waves obtained numerically to be initial conditions and take periodic boundary conditions. Because our scheme is an implicit three-level one, in order to have a start of the computation, not only the initial values at $t = 0$ are needed, but also the values at $t = \Delta t$. We solve this problem as follows.

Notice that for small amplitudes, it is possible to reduce SGBE to the KdV equation

$$v_t + 6v v_x + v_{xxx} = 0 \tag{5.1}$$

by expanding along characteristics of the wave equation in one direction. The solutions of the SGBE and the KdV equation (5.1) have the relations

$$\begin{aligned} u(x, t) &= \epsilon v \left(\epsilon^{1/2}(x - t), \frac{1}{2} \epsilon^{3/2} t \right) + O(\epsilon^2) \quad \text{for the positive direction,} \\ u(x, t) &= \epsilon v \left(\epsilon^{1/2}(x + t), \frac{1}{2} \epsilon^{3/2} t \right) + O(\epsilon^2) \quad \text{for the negative direction.} \end{aligned} \tag{5.2}$$

Firstly, by relations (5.2), we transform $u(x, 0)$ to the initial values $v(x, 0)$ of the KdV equation (5.1), then a simple explicit difference scheme for the KdV equation (5.1)

$$v_l^n = v_l^n - \frac{\Delta t}{\Delta x} (v_{l+1}^n + v_l^n + v_{l-1}^n) (v_{l+1} - v_{l-1}) - \frac{\Delta t}{2(\Delta x)^2} (v_{l+2}^n - 2v_{l+1}^n + 2v_{l-1}^n - v_{l-2}^n) \tag{5.3}$$

gives the values at $t = \Delta t$, v_l^1 , $l = 0, \dots, N$. Again, by (5.2), we can give an estimation of values of u_l^1 , $l = 0, \dots, N$, at $t = \Delta t$, permitting an error of $O(\epsilon^2)$.

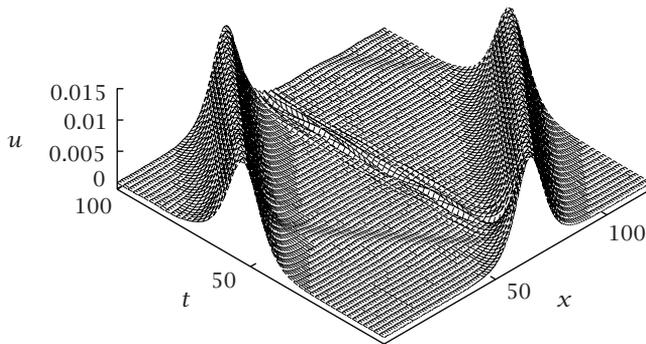


Figure 5.1. Numerical solution for a single solitary wave with $\Delta t = 0.1$ and $\Delta x = 0.1$.

5.1. Propagation of a single solitary wave. A single solitary-wave solution has been obtained in Section 3 with $c = 1.03$. In the numerical calculation that follows, the approximate problem is defined on a finite domain $R = [0, 120]$ with periodic boundary conditions. The numerical solution of the solitary-wave solution for $0 < x < 120$ and $0 < t < 100$, obtained by using (4.5) with $\Delta x = 0.1$ and $\Delta t = 0.1$, is depicted in Figure 5.1. It can be seen from Figure 5.1 that the solitary wave propagates stably without changing its shape for a long time. The ripples are observed, but they don't move with the time. We consider these ripples are due to the error introduced in the starting procedure.

5.2. Collisions of the solitary waves. It is well known that the solitary-wave solutions of the KdV equation move in one direction. Thus in the numerical experiments of solitons collisions, we can only set up a situation, in which one soliton catches up with the other one, and finally passes through it, whereas, for SGBE, it is possible to make numerical experiments of head-on collision of two solitary waves. First, we present the results of numerical simulations of head-on collision between two equal-amplitude solitary waves. Two solitary waves of equal amplitude 1.03 are placed along the x -axis, one on the left is set moving in the right direction, while the other one on the right moving in the left direction. Consequently, these two solitary waves will have a head-on collision later on. Figure 5.2 shows the numerical solution with $\Delta x = 0.1$ and $\Delta t = 0.1$.

We also investigate the head-on collision of two solitary waves with different amplitudes. Figure 5.3 represents the initial conditions with $c = 1.03$ and 1.01, respectively. Figure 5.4 displays the numerical solution for the collision process. From Figures 5.2 and 5.4, we can derive the conclusion that the solitary waves are soliton-like and the collisions are almost elastic.

6. Summary and conclusions

Numerical solution for the sixth-order Boussinesq equation (SGBE), which corrects the bad numerical features of the Boussinesq equation, has been studied in this paper. A symmetric three-level implicit finite difference scheme with a free parameter θ is proposed. Linearized stability analysis demonstrates that the method for the SGBE is stable

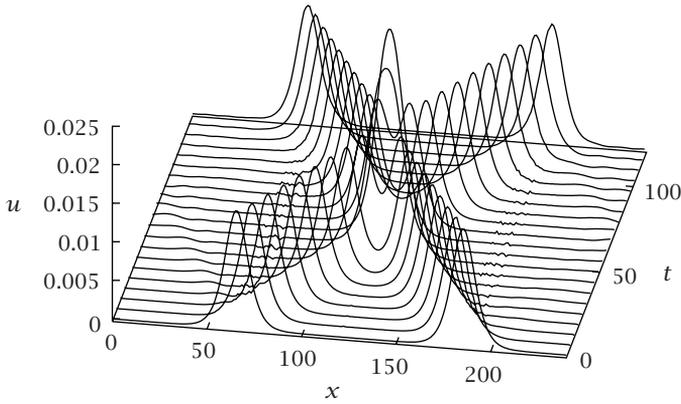


Figure 5.2. Numerical solutions for equal-amplitude solitary waves.

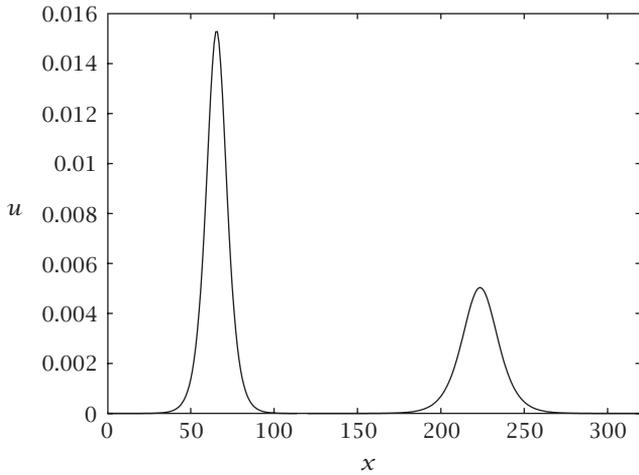


Figure 5.3. Initial conditions for two unequal-amplitude solitary waves.

if $\theta \geq 1/4$. As to the order of accuracy, it has been shown that the method is of order $O((\Delta x)^2 + (\Delta t)^2)$. We also have proved that the conservative quantities $\int u dx$, $\int v dx$, and $\int uv dx$ are invariants in the special case of $\theta = 1/4$.

Solitary-wave solutions for SGBE have been solved numerically, and the results show that they only exist in a very narrow range, whose reason is still unknown. This is an interesting topic for further research. Numerical simulations for the propagation of a single solitary wave of SGBE show that it moves stably without changing its shape. The propagation velocity is consistent with the predicated one. Moreover, we present the numerical results for the head-one collisions of two solitary waves, which are impossible to happen

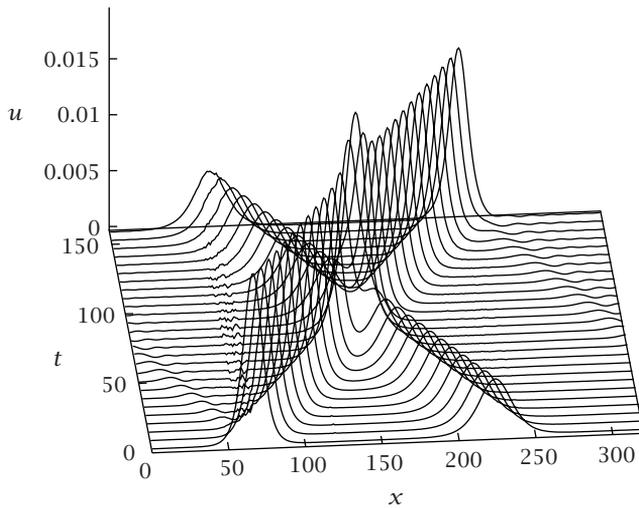


Figure 5.4. Numerical solution with $\Delta x = 0.1$ and $\Delta t = 0.1$.

in the case of solving the KdV equation. Also, the solitary waves turn out to collide almost elastically.

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