THE COMPLETE PRODUCT OF ANNIHILATINGLY UNIQUE DIGRAPHS

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Let G be a digraph with n vertices and let $A(G)$ be its adjacency matrix. A monic polynomial $f(x)$ of degree at most n is called an annihilating polynomial of G if $f(A(G)) = 0$. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial. Difans and diwheels are two classes of annihilatingly unique digraphs. In this paper, it is shown that the complete product of difan and diwheel is annihilatingly unique.

1. Introduction

All graphs under consideration in this paper are directed, connected, finite, loopless, and without multiple arcs. By a *digraph* $G = (V,E)$, we mean a finite set V (the elements of which are called vertices) together with a set E of ordered pairs of elements of V (these pairs are called arcs). Two vertices are said to be adjacent if they are connected by an arc. Undefined terms and notations can be found in [1, 2, 4].

A *diwalk* in a digraph is an alternating sequence of vertices and arcs, $v_0, x_1, v_1, \dots, x_k, v_k$ in which each arc x_i is (v_{i-1}, v_i). The length of such diwalk is k, the number of occurrences of arcs in it.

A *dipath* P_n is a diwalk of order n in which no vertex is repeated.

A *directed cycle* C_n of order n is a digraph with vertex set $\{v_1, \dots, v_n\}$ having arcs (v_i, v_{i+1}), $i = 1, 2, \dots, n-1$, and (v_n, v_1).

A *difan* F_n is a digraph consisting essentially of a dipath P_{n-1} of $n-1$ vertices labelled $1, 2, \dots, n-1$ and an additional vertex n, where there is an arc from n to each of the vertices of P_{n-1}. Vertex n is called the *hub* of the difan whereas the arc (n,k) from the hub n to the vertex k of P_{n-1} is called a *spoke*. Vertices with labelling $i = 1, 2, \dots, n-1$ are called the *rim vertices* of the difan.

The *diwheel* W_n of order n consists of a directed cycle C_{n-1} with an additional vertex n joined from it to all the others. Vertex n is called the hub of the diwheel, whereas the arc (n,k) from the hub n to the vertex k of C_{n-1} is called a spoke. Vertices with labelling i where $i = 1, 2, \dots, n-1$ are called the rim vertices of the diwheel.

Let G be a digraph with n vertices. The *adjacency matrix* $A(G) = (a_{ij})$ of G is a square matrix of order n where the (i, j)-entry, a_{ij}, is equal to the number of arcs starting at

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the vertex i and terminating at the vertex j. Let $A^k(G) = (a_{ij}^k)$ where k is a positive integer and the (i,j)-entry a_{ij}^k of $A^k(G)$ is the number of different diwalks at length k from the vertex i to vertex j.

The determinant of a square matrix A is denoted by $|A|$ or $(\det)A$. The characteristic polynomial $|xI - A(G)|$ of the adjacency matrix $A(G)$ of G is called the *characteristic polynomial* of G and is denoted by $\psi(x)$. A monic polynomial $f(x)$ of degree at most n with $f(A(G)) = 0$ is called an *annihilating polynomial* of G. The existence of annihilating polynomial of G is guaranteed by its characteristic polynomial. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial.

The following result is well known in linear algebra (see [1, 5, 6]).

Theorem 1.1. *Let A be an $n\times n$ matrix, $m(x)$ its minimum polynomial, and $\psi(x)$ its characteristic polynomial.*

(1) $\psi(A) = 0$.

(2) *If $f(x)$ is any polynomial with $f(A) = 0$, then $m(x)$ divides $f(x)$; in particular $m(x)$ divides $\psi(x)$.*

(3) *Let $\{x_1,x_2,\dots,x_k\}$ be the set of distinct eigenvalues of A, x_i having algebraic multiplicity c_i. Then*

$$\begin{aligned}\psi(x) &= (x-x_1)^{c_1}(x-x_2)^{c_2}\cdots(x-x_k)^{c_k},\\ m(x) &= (x-x_1)^{q_1}(x-x_2)^{q_2}\cdots(x-x_k)^{q_k},\end{aligned}\tag{1.1}$$

where q_i satisfies $0 < q_i \le c_i$ $(1,2,\dots,k)$. Furthermore, if $k = n$, then

$$m(x) = \psi(x) = (x-x_1)(x-x_2)\cdots(x-x_n).\tag{1.2}$$

A matrix for which the minimum polynomial is equal to the characteristic polynomial is called *nonderogatory*; otherwise *derogatory*. The following result follows from Theorem 1.1.

Theorem 1.2. *The annihilating polynomial $f(x)$ of any digraph G with adjacency matrix $A(G)$ is unique if and only if $A(G)$ is nonderogatory.*

Corollary 1.3. *Let G be a digraph with n vertices and $A(G)$ its adjacency matrix. Then G is annihilatingly unique if $A(G)$ has n distinct eigenvalues.*

2. The complete product of digraphs

In this section, we will consider a binary operation on digraphs: the complete product of digraphs.

The *direct sum* $G_1 + G_2$ of digraphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ $(V_1 \cap V_2 = \emptyset)$ is the digraph $G(V,E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

The *complete product* $G_1 \otimes G_2$ of digraphs G_1 and G_2 is the digraph obtained from $G_1 + G_2$ by joining every vertex of G_1 to every vertex of G_2.

Let G_1 and G_2 be digraphs with n_1 and n_2 vertices, respectively. The vertices of $G_1 \otimes G_2$ are labelled as follows.

The vertices of G_2 are labelled $1,2,\ldots,n_2$ and vertices of G_1 are labelled $n_2+1, n_2+2,\ldots,n_2+n_1$. Suppose that $A(G_i)$ is the adjacency matrix of G_i, $i=1,2$. Then the adjacency matrix of $G_1 \otimes G_2$ is of the form

$$\begin{bmatrix} A(G_2) & 0_{n_1}^{(n_2)} \\ J_{n_2}^{(n_1)} & A(G_1) \end{bmatrix}, \tag{2.1}$$

where $J_{n_2}^{(n_1)}$ is an $n_1 \times n_2$ matrix with all entries equal to 1 and $0_{n_1}^{(n_2)}$ is an $n_2 \times n_1$ matrix with all the entries equal to zero. It follows immediately that the characteristic polynomial of $G_1 \otimes G_2$ is the product of the characteristic polynomials of G_1 and G_2.

A question arises: is it true that the complete product of two annihilatingly unique digraphs possesses a unique annihilating polynomial?

The complete product of annihilatingly unique digraphs was first studied by Lam [3]. In particular, Lam proved that $P_q \otimes P_r$, $P_q \otimes C_r$, $C_r \otimes P_q$, $P_q \otimes W_r$, $W_r \otimes P_q$ are annihilatingly unique. Furthermore, if q and r are relatively prime, then $C_q \otimes C_r$, $C_q \otimes W_{r+1}$, $W_{r+1} \otimes C_q$, and $W_{q+1} \otimes W_{r+1}$ are annihilatingly unique.

In general, it is not true that the complete product of two annihilatingly unique digraphs possesses a unique annihilating polynomial. For example, $C_3 \otimes W_4$ of order 7 is not annihilatingly unique.

Motivated by the above results, we will prove that the complete product of difans and diwheels are annihilatingly unique in the next section.

3. The complete product of difans and diwheels

In this section, we show that the complete products $F_q \otimes W_r$ and $W_q \otimes F_r$ of order $q+r$ are annihilatingly unique. The characteristic polynomials for difan of order q and diwheel of order r are x^q and $x^r - x$, respectively.

Theorem 3.1. *$F_q \otimes W_r$ and $W_q \otimes F_r$ are annihilatingly unique.*

Proof. We divide the proof of Theorem 3.1 into two parts and in each part, we will prove that the given complete product of digraphs is annihilatingly unique by showing the equality of its characteristic and minimum polynomials.

Part 1 ($F_q \otimes W_r$ is annihilatingly unique). The characteristic polynomial of $F_q \otimes W_r$ is given by $\psi(x) = x^q(x^r - x) = x^{q+1}(x^{r-1} - 1)$. From the digraph of $F_q \otimes W_r$, we observe that $a_{r,r-1}^{(r-1)} = 1$ and $a_{r+q,1}^{(q+1)} > 1$ (this is because there exist at least two diwalks of length $q+1$ from vertex $r+q$ to vertex 1. For example, $r+q \to r+1 \to r+2 \to r+3 \to \cdots \to r+q-2 \to r+q-1 \to r \to 1$ and $r+q \to r+1 \to r+2 \to r+3 \to \cdots \to r+q-2 \to r+q-1 \to r-1 \to 1$ are two such diwalks of length $q+1$ from vertex $r+q$ to vertex 1). This observation implies that A^{q+1} and $A^{r-1} - I$ are not equal to 0 and x^{q+1} and $x^{r-1} - 1$ are therefore not the minimum polynomials of $\psi(x)$.

Notice that the only repeated factor of the characteristic polynomial of $F_q \otimes W_r$ is x of multiplicity $q+1$. We claim that $\psi(x) = m(x) = x^{q+1}(x^{r-1} - 1)$. Suppose that $m(x) = x^q(x^{r-1} - 1) = x^{q+r-1} - x^q$. We will show that $m(A) \neq 0$ to obtain a contradiction. From the digraph of $F_q \otimes W_r$, we observe that $a_{r+q,r}^{(r+q-1)} = 0$ and $a_{r+q,r}^{(q)} = 1$. This implies that

$m(A) = A^{q+r-1} - A^q \neq 0$, a contradiction. Therefore, $\psi(x)$ is equal to the minimum polynomial of $F_q \otimes W_r$. Hence, $F_q \otimes W_r$ is annihilatingly unique.

Part 2 ($W_q \otimes F_r$ is annihilatingly unique). The characteristic polynomial of $W_q \otimes F_r$ is given by $\psi(x) = x^r(x^q - x) = x^{r+1}(x^{q-1} - 1)$. Like the case of $F_q \otimes W_r$, from the digraph of $W_q \otimes F_r$, we have $a^{(q-1)}_{r+q,r+q-1} = 1$ and $a^{(r+1)}_{r+1,r-1} > 1$. This implies that A^{r+1} and $A^{q-1} - I$ are not equal to 0 and x^{r+1} and $x^{q-1} - 1$ are therefore not the minimum polynomials of $\psi(x)$.

Notice that the only repeated factor of $\psi(x)$ is x of multiplicity $r + 1$. We claim that $\psi(x) = m(x) = x^{r+1}(x^{q-1} - 1)$. Suppose on the contrary that $m(x) = x^r(x^{q-1} - 1)$. Again, we will show that $m(A) \neq 0$ and obtain a contradiction. To this end, we claim that

$$a^{(k)}_{r+q,r-1} = \begin{cases} (q-1)(2r-3)+1 & \text{if } k = r, \\ 2(q-1)(r-1) & \text{if } k > r. \end{cases} \tag{3.1}$$

When $k = r$, the number of diwalks from the vertex $r + q$ to vertex $r - 1$ can be enumerated in the following three ways.

(i) If the rim vertices of W_q are avoided.

In this case, there is only one such diwalk, namely $r + q \to r \to 1 \to 2 \to \cdots \to r - 2 \to r - 1$.

(ii) Move along some rim vertices of W_q with the hub vertex r of F_r being included in the diwalk.

For each rim vertex of W_q, since the hub vertex r of F_r is included in the diwalk, the possible number of rim vertices of F_r covered in the diwalk is at most $r - 2$. From the multiplication principle, since W_q has $q - 1$ rim vertices, the number of diwalks from vertex $r + q$ to vertex $r - 1$ in case (ii) is $(q - 1)(r - 2)$.

(iii) Move along some rim vertices of W_q with the hub vertex r of F_r being excluded in the diwalk.

In this case, since the hub vertex r of F_r is excluded in the diwalk, the possible number of rim vertices of F_r contained in the diwalk is $r - 1$. This implies that the number of diwalks from vertex $r + q$ to vertex $r - 1$ in case (iii) is $(q - 1)(r - 1)$.

Thus, for the case $k = r$, $a^{(k)}_{r+q,r-1} = 1 + (q - 1)(r - 2) + (q - 1)(r - 1) = (q - 1)(2r - 3) + 1$.

Likewise, the similar enumeration method for the case when $k = r$ can be used to show that $a^{(k)}_{r+q,r-1} = 0 + (q - 1)(r - 1) + (q - 1)(r - 1) = 2(q - 1)(r - 1)$ when $k > r$.

By using the results in the claim, we have

$$\begin{aligned} a^{(r+q-1)}_{r+q,r-1} - a^{(r)}_{r+q,r-1} &= 2(q-1)(r-1) - \{(q-1)(2r-3)+1\} \\ &= q - 2 \neq 0. \end{aligned} \tag{3.2}$$

This implies that $m(A) = A^{r+q-1} - A^r \neq 0$, a contradiction. Therefore $\psi(x)$ is equal to the minimum polynomial of $W_q \otimes F_r$, and $W_q \otimes F_r$ is annihilatingly unique and the proof of Part 2 is complete. □

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