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Research Article

On the Rational Recursive Sequence $x_{n+1} = (A + \sum_{i=0}^{k}\alpha_i x_{n-i})/(B + \sum_{i=0}^{k}\beta_i x_{n-i})$

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Recommended by Martin J. Bohner

The main objective of this paper is to study the boundedness character, the periodic character, the convergence, and the global stability of the positive solutions of the difference equation $x_{n+1} = (A + \sum_{i=0}^{k}\alpha_i x_{n-i})/(B + \sum_{i=0}^{k}\beta_i x_{n-i})$, $n = 0,1,2,\ldots$, where A, B, α_i, β_i and the initial conditions $x_{-k},\ldots,x_{-1}$, x_0 are arbitrary positive real numbers, while k is a positive integer number.

1. Introduction

Our goal in this paper is to investigate the boundedness character, the periodic character, the convergence and the global stability of the positive solutions of the difference equation

$$x_{n+1} = \frac{A + \sum_{i=0}^{k}\alpha_i x_{n-i}}{B + \sum_{i=0}^{k}\beta_i x_{n-i}}, \quad n = 0,1,2,\ldots, \tag{1.1}$$

where A, B, α_i, β_i and the initial conditions $x_{-k},\ldots,x_{-1}$, x_0 are arbitrary positive real numbers, while k is a positive integer number. The case where any of A, B, α_i, β_i is allowed to be zero gives different special cases of (1.1) which are studied by many authors (see, e.g., [1–14]). For the related work, see [15–26]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are paramount importance in their own right. Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that the difference equation (1.1) has been extensively studied in the special case $k = 1$ in the monograph [14]. So, the results presented in our paper are new.

Definition 1.1. The equilibrium point $\widetilde{x}$ of the difference equation (1.1) is the point that satisfies the condition $\widetilde{x} = F(\widetilde{x},\widetilde{x},\dots,\widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of the difference equation (1.1).

Definition 1.2. Let $\widetilde{x} \in (0,\infty)$ be an equilibrium point of the difference equation (1.1). Then, the following hold

(i) The equilibrium point $\widetilde{x}$ of the difference equation (1.1) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x_{-k},\dots,x_{-1},x_0 \in (0,\infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\widetilde{x}$ of the difference equation (1.1) is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that $x_{-k},\dots,x_{-1}$, $x_0 \in (0,\infty)$ with $|x_{-k} - \widetilde{x}| + \cdots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then $x_n \to \widetilde{x}$ as $n \to \infty$.

(iii) The equilibrium point $\widetilde{x}$ of the difference equation (1.1) is called global attractor if for every $x_{-k},\dots,x_{-1},x_0 \in (0,\infty)$ one has $x_n \to \widetilde{x}$ as $n \to \infty$.

(iv) The equilibrium point $\widetilde{x}$ of the equation (1.1) is called globally asymptotically stable if it is locally stable and global attractor.

(v) The equilibrium point $\widetilde{x}$ of the difference equation (1.1) is called unstable if it is not locally stable.

Definition 1.3. Say that the sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded and persists if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \forall n \geq -k. \tag{1.2}$$

Definition 1.4. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Assume that $\widetilde{a} = \sum_{i=0}^{k} \alpha_i$, $\overline{a} = \sum_{i=0}^{k} (-1)^i \alpha_i$, $\widetilde{b} = \sum_{i=0}^{k} \beta_i$, and $\overline{b} = \sum_{i=0}^{k} (-1)^i \beta_i$. Then the equilibrium point $\widetilde{x}$ of the difference equation (1.1) is the solution of the equation

$$\widetilde{x} = \frac{A + \widetilde{a}\widetilde{x}}{B + \widetilde{b}\widetilde{x}}. \tag{1.3}$$

Consequently, the positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is given by

$$\widetilde{x} = \frac{(\widetilde{a} - B) + \sqrt{(\widetilde{a} - B)^2 + 4A\widetilde{b}}}{2\widetilde{b}}. \tag{1.4}$$

Let $F : (0,\infty)^{k+1} \to (0,\infty)$ be a continuous function defined by

$$F(u_0,u_1,\dots,u_k) = \frac{A + \sum_{i=0}^{k} \alpha_i u_i}{B + \sum_{i=0}^{k} \beta_i u_i}. \tag{1.5}$$

Now, we have

$$y_{n+1} = \sum_{j=0}^{k} \frac{\partial F(\widetilde{x},\dots,\widetilde{x})}{\partial u_j} y_{n-j}, \tag{1.6}$$

and then the linearized equation is

$$y_{n+1} + \sum_{j=0}^{k} b_j y_{n-j} = 0, \tag{1.7}$$

where

$$b_j = \frac{\beta_j \widetilde{x} - \alpha_j}{B + \widetilde{b}\widetilde{x}}. \tag{1.8}$$

2. The main results

In this section, we establish some results which show that the positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is globally asymptotically stable and every positive solution of the difference equation (1.1) is bounded and has prime period two.

Theorem 2.1 (see [4, 10, 13, 17]). *Assume that $a,b \in R$ and $k \in \{0,1,2,\dots\}$. Then*

$$|a| + |b| < 1 \tag{2.1}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + ax_n + bx_{n-k} = 0, \quad n = 0,1,\dots. \tag{2.2}$$

Remark 2.2 (see [13]). Theorem 2.1 can be easily extended to a general linear difference equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, \quad n = 0,1,2,\dots, \tag{2.3}$$

where $p_1, p_2, \dots, p_k \in R$ and $k \in \{1,2,\dots\}$. Then equation 2.3 is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{2.4}$$

Theorem 2.3. *Assume that $B > \widetilde{a}$ holds. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of the difference equation (1.1) such that for some $n_0 \geq 0$,*

$$\text{either } x_n \geq \widetilde{x} \quad \text{for } n \geq n_0 \tag{2.5}$$

$$\text{or } x_n \leq \widetilde{x} \quad \text{for } n \geq n_0. \tag{2.6}$$

Then $\{x_n\}$ converges to $\widetilde{x}$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} x_n = \widetilde{x}. \tag{2.7}$$

Proof. Assume that (2.5) holds. The case where (2.6) holds is similar and will be omitted. Then for $n \geq n_0 + k$, we deduce that

$$x_{n+1} = \frac{A + \sum_{i=0}^{k} \alpha_i x_{n-i}}{B + \sum_{i=0}^{k} \beta_i x_{n-i}} = \left[\sum_{i=0}^{k} \alpha_i x_{n-i}\right]\left[\frac{1 + (A/\sum_{i=0}^{k} \alpha_i x_{n-i})}{B + \sum_{i=0}^{k} \beta_i x_{n-i}}\right] \leq \left[\sum_{i=0}^{k} \alpha_i x_{n-i}\right]\frac{[1 + (A/\tilde{a}\tilde{x})]}{(B + \tilde{b}\tilde{x})} = \left[\sum_{i=0}^{k} \alpha_i x_{n-i}\right]\frac{(A + \tilde{a}\tilde{x})}{\tilde{a}\tilde{x}(B + \tilde{b}\tilde{x})}. \tag{2.8}$$

With the aid of (1.3), the last inequality becomes

$$x_{n+1} \leq \sum_{i=0}^{k} \alpha_i x_{n-i}/\tilde{a}, \tag{2.9}$$

and so

$$x_{n+1} \leq \max_{0 \leq i \leq k} \{x_{n-i}\} \quad \text{for } n \geq n_0 + k. \tag{2.10}$$

Set

$$y_n = \max_{0 \leq i \leq k} \{x_{n-i}\} \quad \text{for } n \geq n_0 + k. \tag{2.11}$$

Then clearly

$$y_n \geq x_{n+1} \geq \tilde{x} \quad \text{for } n \geq n_0 + k. \tag{2.12}$$

Next, we claim that

$$y_{n+1} \leq y_n \quad \text{for } n \geq n_0 + k. \tag{2.13}$$

Now, we have

$$y_{n+1} = \max_{0 \leq i \leq k} \{x_{n+1-i}\} = \max\left\{x_{n+1}, \max_{0 \leq i \leq k} \{x_{n-i}\}\right\} \leq \max\{x_{n+1}, y_n\} = y_n. \tag{2.14}$$

From (2.12) and (2.13), it follows that the sequence $\{y_n\}$ is convergent and that

$$y = \lim_{n \to \infty} y_n \geq \tilde{x}. \tag{2.15}$$

Furthermore, we get

$$x_{n+1} \leq \frac{A + \sum_{i=0}^{k} \alpha_i x_{n-i}}{B + \tilde{b}\tilde{x}} \leq \frac{A + \tilde{a} y_n}{B + \tilde{b}\tilde{x}}. \tag{2.16}$$

From this and by using (2.13) we obtain,

$$x_{n+i} \leq \frac{A + \tilde{a} y_{n+i-1}}{B + \tilde{b}\tilde{x}} \leq \frac{A + \tilde{a} y_n}{B + \tilde{b}\tilde{x}} \quad \text{for } i = 1, \ldots, k+1. \tag{2.17}$$

Then

$$y_{n+k+1} = \max_{1\le i\le k+1}\{x_{n+i}\} \le \frac{A+\widetilde{a}y_n}{B+\widetilde{b}\widetilde{x}}, \tag{2.18}$$

and by letting $n \to \infty$, we obtain

$$y \le \frac{A+\widetilde{a}\,y}{B+\widetilde{b}\widetilde{x}}. \tag{2.19}$$

Consequently, we obtain

$$y\left(1-\frac{\widetilde{a}}{B+\widetilde{b}\widetilde{x}}\right) \le \frac{A}{B+\widetilde{b}\widetilde{x}}. \tag{2.20}$$

From (1.3) and (2.20), we deduce that $y \le \widetilde{x}$, and in view of (2.15), we obtain $y = \widetilde{x}$. Thus, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of the difference equation (1.1) and $B > 1$. Then there exist positive constants m and M such that*

$$m \le x_n \le M, \quad n = 0,1,\ldots. \tag{2.21}$$

Proof. From the difference equation (1.1), we have when $B > 1$

$$x_{n+1} \le \frac{A}{B}+\frac{1}{B}\left(\sum_{i=0}^{k}\alpha_i x_{n-i}\right), \quad n = 0,1,\ldots. \tag{2.22}$$

Consider the linear difference equation

$$y_{n+1} = \frac{A}{B}+\frac{1}{B}\left(\sum_{i=0}^{k}\alpha_i y_{n-i}\right), \quad n = 0,1,\ldots \tag{2.23}$$

with the initial conditions $y_i = x_i > 0$, $i = -k,\ldots,-1,0$. It follows by induction that

$$x_n \le y_n. \tag{2.24}$$

First of all, assume that $B > \widetilde{a}$. Then we have $A/(B-\widetilde{a})$ is a particular solution of (2.23) and every solution of the homogeneous equation which is associated with (2.23) tends to zero as $n \to \infty$. Hence

$$\lim_{n\to\infty} y_n = \frac{A}{B-\widetilde{a}}. \tag{2.25}$$

From this and (2.24), it follows that the sequence $\{x_n\}$ is bounded from above by a positive constant M say. That is,

$$x_n \le M, \quad n = 0,1,\ldots. \tag{2.26}$$

Set

$$m = \frac{A}{B+\widetilde{b}M}. \tag{2.27}$$

Then we have

$$x_{n+1} = \frac{A+\sum_{i=0}^{k}\alpha_i x_{n-i}}{B+\sum_{i=0}^{k}\beta_i x_{n-i}} \geq \frac{A}{B+\widetilde{b}M} = m \tag{2.28}$$

and consequently, we get

$$m \leq x_n \leq M, \quad n = 0,1,\ldots, \tag{2.29}$$

which completes the proof of Theorem 2.4 when $B > \widetilde{a}$. Second, consider the case when $B \leq \widetilde{a}$. It suffices to show that $\{x_n\}$ is bounded from above by some positive constant. For the sake of contradiction, assume that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_j}\}$ such that

$$\begin{gathered} \lim_{j\to\infty} n_j = \infty, \qquad \lim_{j\to\infty} x_{1+n_j} = \infty, \\ x_{1+n_j} = \max\{x_n : -k \leq n \leq 1+n_j\}, \quad (j = 0,1,2,\ldots). \end{gathered} \tag{2.30}$$

From (2.22), we deduce that

$$\sum_{i=0}^{k}\alpha_i x_{-i+n_j} \geq Bx_{1+n_j} - A. \tag{2.31}$$

Taking the limit as $j \to \infty$ of both sides of the last inequality, we obtain

$$\lim_{j\to\infty}\sum_{i=0}^{k}\alpha_i x_{-i+n_j} = \infty. \tag{2.32}$$

It is easy enough to show that $x_{-i+n_j} \leq x_{1+n_j}$, $(i = 0,1,2,\ldots,k)$, and then as $\widetilde{a} = \sum_{i=0}^{k}\alpha_i$, we have

$$\sum_{i=0}^{k}\alpha_i x_{-i+n_j} \leq \widetilde{a}x_{1+n_j}. \tag{2.33}$$

From the last inequality and the difference equation (1.1), we obtain

$$0 \leq \widetilde{a}x_{1+n_j} - \sum_{i=0}^{k}\alpha_i x_{-i+n_j} = \frac{\widetilde{a}A + \sum_{i=0}^{k}\alpha_i x_{-i+n_j}\left[\widetilde{a} - B - \sum_{i=0}^{k}\beta_i x_{-i+n_j}\right]}{B+\sum_{i=0}^{k}\beta_i x_{-i+n_j}}. \tag{2.34}$$

Consequently, it follows that

$$\sum_{i=0}^{k}\beta_i x_{-i+n_j} \leq \widetilde{a} - B. \tag{2.35}$$

Then for every $i = 0,1,2,\ldots,k$ for which β_i is positive, the subsequence $\{x_{-i+n_j}\}$ is bounded which implies that the sequence $\{\sum_{i=0}^{k} \alpha_i x_{-i+n_j}\}$ is also bounded. This contradicts (2.32) and the proof of Theorem 2.4 is completed. □

Theorem 2.5. *Assume that $B > \tilde{a}$ holds. Then the positive equilibrium point $\tilde{x}$ of the difference equation (1.1) is globally asymptotically stable.*

Proof. The linearized equation (1.7) with (1.8) can be written in the form

$$y_{n+1} + \sum_{j=0}^{k} \frac{(\beta_j \tilde{x} - \alpha_j)}{(B + \tilde{b}\tilde{x})} y_{n-j} = 0. \tag{2.36}$$

As $B > \tilde{a}$, we get

$$\sum_{j=0}^{k} \left| \frac{\beta_j \tilde{x} - \alpha_j}{B + \tilde{b}\tilde{x}} \right| \le \frac{(\tilde{a} + \tilde{b}\tilde{x})}{(B + \tilde{b}\tilde{x})} < 1. \tag{2.37}$$

Thus, by Remark 2.2, we deduce that the equilibrium point $\tilde{x}$ of the difference equation (1.1) is locally asymptotically stable. It remains to prove that the equilibrium point $\tilde{x}$ is a global attractor. To this end, set $I = \lim_{n\to\infty} \inf x_n$ and $S = \lim_{n\to\infty} \sup x_n$, which by Theorem 2.4 are positive numbers. Then, from the difference equation (1.1), we see that

$$S \le \frac{A + \tilde{a}S}{B + \tilde{b}I}, \qquad I \ge \frac{A + \tilde{a}I}{B + \tilde{b}S}. \tag{2.38}$$

Hence

$$A + (\tilde{a} - B)I \le \tilde{b}IS \le A + (\tilde{a} - B)S. \tag{2.39}$$

From which it follows that $I = S$. Thus, the proof of Theorem 2.5 is completed. □

Theorem 2.6. *The necessary and sufficient condition for the difference equation (1.1) to have positive prime period two solutions is that both inequalities*

$$A(\tilde{b} - \bar{b})^2 - (\tilde{a} + \bar{a})(\tilde{b} - \bar{b})(B + \bar{a}) < \bar{b}(B + \bar{a})^2, \tag{2.40}$$
$$B + \bar{a} < 0 \tag{2.41}$$

are valid.

Proof. First, suppose that there exist positive prime period two solutions

$$\ldots, P, Q, P, Q, \ldots \tag{2.42}$$

of the difference equation (1.1). We will prove that the condition (2.40) holds. It follows from the difference equation (1.1) that

$$P = \frac{A+\alpha_0 Q+\alpha_1 P+\alpha_2 Q+\alpha_3 P+\cdots}{B+\beta_0 Q+\beta_1 P+\beta_2 Q+\beta_3 P+\cdots},$$
$$Q = \frac{A+\alpha_0 P+\alpha_1 Q+\alpha_2 P+\alpha_3 Q+\cdots}{B+\beta_0 P+\beta_1 Q+\beta_2 P+\beta_3 Q+\cdots}. \tag{2.43}$$

Consequently, we obtain

$$A+\alpha_0 Q+\alpha_1 P+\alpha_2 Q+\alpha_3 P+\cdots = BP+\beta_0 PQ+\beta_1 P^2+\beta_2 PQ+\beta_3 P^2+\cdots, \tag{2.44}$$

$$A+\alpha_0 P+\alpha_1 Q+\alpha_2 P+\alpha_3 Q+\cdots = BQ+\beta_0 PQ+\beta_1 Q^2+\beta_2 PQ+\beta_3 Q^2+\cdots. \tag{2.45}$$

By subtracting, we deduce after some reduction that

$$P+Q = \frac{-(B+\overline{a})}{\beta_1+\beta_3+\cdots}, \tag{2.46}$$

while by adding we obtain

$$PQ = \frac{A(\beta_1+\beta_3+\cdots)-(\alpha_0+\alpha_2+\cdots)(B+\overline{a})}{\overline{b}(\beta_1+\beta_3+\cdots)}, \tag{2.47}$$

where $B+\overline{a}<0$. Now, it is clear from (2.46) and (2.47) that P and Q are two positive distinct real roots of the quadratic equation

$$t^2-(P+Q)t+PQ=0. \tag{2.48}$$

Thus, we deduce that

$$\left(\frac{-(B+\overline{a})}{\beta_1+\beta_3+\cdots}\right)^2 > 4\left(\frac{A(\beta_1+\beta_3+\cdots)-(\alpha_0+\alpha_2+\cdots)(B+\overline{a})}{\overline{b}(\beta_1+\beta_3+\cdots)}\right). \tag{2.49}$$

From (2.49), we obtain

$$A(\widetilde{b}-\overline{b})^2-(\widetilde{a}+\overline{a})(\widetilde{b}-\overline{b})(B+\overline{a}) < \overline{b}(B+\overline{a})^2, \tag{2.50}$$

and hence the condition (2.40) is valid. Conversely, suppose that the condition (2.40) is valid. Then, we deduce immediately from (2.40) that the inequality (2.49) holds. Consequently, there exist two positive distinct real numbers P and Q such that

$$P = \frac{-(B+\overline{a})}{2(\beta_1+\beta_3+\cdots)} - \frac{1}{2}\sqrt{T_1}, \tag{2.51}$$

$$Q = \frac{-(B+\overline{a})}{2(\beta_1+\beta_3+\cdots)} + \frac{1}{2}\sqrt{T_1}, \tag{2.52}$$

where $T_1 > 0$ which is given by the formula

$$T_1 = \left(\frac{-(B+\overline{a})}{\beta_1+\beta_3+\cdots}\right)^2 - 4\left(\frac{A(\beta_1+\beta_3+\cdots)-(\alpha_0+\alpha_2+\cdots)(B+\overline{a})}{\overline{b}(\beta_1+\beta_3+\cdots)}\right). \tag{2.53}$$

Thus, P and Q represent two positive distinct real roots of the quadratic equation (2.48). Now, we are going to prove that P and Q are positive prime period two solutions of the difference equation (1.1). To this end, we assume that

$$x_{-k} = P, \qquad x_{-k+1} = Q, \ldots, \qquad x_{-1} = Q, \qquad x_0 = P. \tag{2.54}$$

We wish to show that

$$x_1 = Q, \qquad x_2 = P. \tag{2.55}$$

To this end, we deduce from the difference equation (1.1) that

$$\begin{aligned} x_1 &= \frac{A+\alpha_0 x_0+\alpha_1 x_{-1}+\cdots+\alpha_k x_{-k}}{B+\beta_0 x_0+\beta_1 x_{-1}+\cdots+\beta_k x_{-k}} \\ &= \frac{A+P(\alpha_0+\alpha_2+\cdots)+Q(\alpha_1+\alpha_3+\cdots)}{B+P(\beta_0+\beta_2+\cdots)+Q(\beta_1+\beta_3+\cdots)}. \end{aligned} \tag{2.56}$$

Dividing the denominator and numerator of (2.56) by $-(B+\overline{a})/(\beta_1+\beta_3+\cdots)$ and using (2.51)–(2.53), we obtain

$$\begin{aligned} x_1 &= \frac{-2A(\beta_1+\beta_3+\cdots)/(B+\overline{a})+[1+\sqrt{K_1}](\alpha_0+\alpha_2+\cdots)+[1-\sqrt{K_1}](\alpha_1+\alpha_3+\cdots)}{-2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})+[1+\sqrt{K_1}](\beta_0+\beta_2+\cdots)+[1-\sqrt{K_1}](\beta_1+\beta_3+\cdots)} \\ &= \frac{\left[\widetilde{a}-2A(\beta_1+\beta_3+\cdots)/(B+\overline{a})\right]+\overline{a}\sqrt{K_1}}{\left[\widetilde{b}-2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})\right]+\overline{b}\sqrt{K_1}}, \end{aligned} \tag{2.57}$$

where

$$K_1 = 1 - \left[\frac{A(\widetilde{b}-\overline{b})^2-(\widetilde{a}+\overline{a})(\widetilde{b}-\overline{b})(B+\overline{a})}{\overline{b}(B+\overline{a})^2}\right], \tag{2.58}$$

and from the condition (2.40), we deduce that $K_1 > 0$. Multiplying the denominator and numerator of (2.57) by

$$\left(\widetilde{b} - \frac{2B(\beta_1+\beta_3+\cdots)}{(B+\overline{a})}\right) - \overline{b}\sqrt{K_1}. \tag{2.59}$$

We have

$$x_1 = \frac{[\widetilde{a} - 2A(\beta_1+\beta_3+\cdots)/(B+\overline{a})][\widetilde{b} - 2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})] - \overline{b}\overline{a}K_1}{[\widetilde{b} - 2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})]^2 - \overline{b}^2 K_1}$$
$$+ \frac{[\widetilde{b}\overline{a} - \widetilde{a}\overline{b} - \overline{a}(2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})) + \overline{b}(2A(\beta_1+\beta_3+\cdots)/(B+\overline{a}))]\sqrt{K_1}}{[\widetilde{b} - 2B(\beta_1+\beta_3+\cdots)/(B+\overline{a})]^2 - \overline{b}^2 K_1}. \tag{2.60}$$

After some reduction, we deduce that

$$x_1 = \frac{-(B+\overline{a})}{2(\beta_1+\beta_3+\cdots)}$$
$$\times \frac{[2(\alpha_1+\cdots)(\beta_0+\cdots) - 2(\alpha_0+\cdots)(\beta_1+\cdots) - (2(\beta_1+\cdots)/(B+\overline{a}))(A\overline{b} - B\overline{a})](1+\sqrt{K_1})}{[2(\alpha_1+\cdots)(\beta_0+\cdots) - 2(\alpha_0+\cdots)(\beta_1+\cdots) - (2(\beta_1+\cdots)/(B+\overline{a}))(A\overline{b} - B\overline{a})]}$$
$$= \frac{-(B+\overline{a})(1+\sqrt{K_1})}{2(\beta_1+\beta_3+\cdots)} = \frac{-(B+\overline{a})}{2(\beta_1+\beta_3+\cdots)} + \frac{1}{2}\sqrt{T_1} = Q. \tag{2.61}$$

Similarly, we can show that

$$x_2 = \frac{A+\alpha_0 x_1 + \alpha_1 x_0 + \cdots + \alpha_k x_{-(k-1)}}{B+\beta_0 x_1 + \beta_1 x_0 + \cdots + \beta_k x_{-(k-1)}} = \frac{A+Q(\alpha_0+\alpha_2+\cdots) + P(\alpha_1+\alpha_3+\cdots)}{B+Q(\beta_0+\beta_2+\cdots)+P(\beta_1+\beta_3+\cdots)} = P. \tag{2.62}$$

By using the mathematical induction, we have

$$x_n = P, \quad x_{n+1} = Q \quad \forall n \geq -k. \tag{2.63}$$

Thus, the difference equation(1.1) has positive prime period two solutions

$$\ldots, P, Q, P, Q, \ldots. \tag{2.64}$$

Hence the proof of Theorem 2.6 is completed. □

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