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Research Article

An Integral Representation of Standard Automorphic L Functions for Unitary Groups

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Let F be a number field, G a quasi-split unitary group of rank n. We show that given an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, its (partial) L function $L^S(s,\pi,\sigma)$ can be represented by a Rankin-Selberg-type integral involving cusp forms of π, Eisenstein series, and theta series.

1. Introduction

Let F be a number field, G the general linear group of degree n defined over F. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. In [1–3], a Rankin-Selberg-type integral is constructed to represent the L function of π. That the integrals of Jacquet, Piatetski-Shapiro, and Shalika are Eulerian follows from the uniqueness of Whittaker models and the fact that cuspidal representations of GL_n are always generic. For other reductive group whose cuspidal representations are not always generic, in [4], Piatetski-Shapiro and Rallis construct a Rankin-Selberg integral for symplectic group $G = \mathrm{Sp}_{2n}$ to represent the partial L function of a cuspidal representation π of $G(\mathbb{A})$. In this paper, we apply similar method to the quasi-split unitary group of rank n.

Let F be a number field, E a quadratic field extension of F. Let V be a $2n$-dimensional vector space over E with an anti-Hermitian form

$$\eta_{2n} = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \tag{1.1}$$

on it. Let $G = U(\eta_{2n})$ be the unitary group of η_{2n}. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, f a cusp form belonging to the isotypic space of π. The

Rankin-Selberg-type integral is defined by

$$\int_{G(F)\backslash G(\mathbb{A})} f(g)E(g,s)\theta(g)dg, \tag{1.2}$$

where $E(g,s)$ is an Eisenstein series associated with a degenerate principle series, θ is a theta series defined by the Weil representation of $\mathrm{Sp}(V \otimes W)$, where W is a nondegenerate Hermitian space of dimension n. We show in Theorem 6.3 that (1.2) represent the standard partial L function $L^S(s,\pi,\sigma)$ of π.

In [4], after showing the Rankin-Selberg integral has a Euler product decomposition, Piatetski-Shapiro and Rallis continued to show that if $n/2+1$ is a pole of partial L function, then theta lifting is nonvanishing [4, Proposition on page 120]. There should be a parallel application of our paper, that is, relate the largest possible pole with nonvanishing of period integral.

2. Notations and conventions

Let F be a field of characteristic 0, E a commutative F-algebra with rank two. Let ρ be an F-linear automorphism of E. We are interested in (E,ρ) of the following two types:

(1) E is a quadratic field extension of F, ρ is the nontrivial element of $\mathrm{Gal}(E/F)$;

(2) $E = F \oplus F$, $(x,y)^\rho = (y,x)$.

Let tr be the trace of E over F, that is, it is defined by

$$\mathrm{tr}(z) = z + z^\rho, \quad z \in E. \tag{2.1}$$

Let V be a left E-module, $\varphi : V \times V \to E$ a nonsingular ε-Hermitian form on V, here $\varepsilon = \pm 1$. The unitary group of φ is

$$U(\varphi) = \{\alpha \in \mathrm{GL}(V,E) \mid \varphi(x\alpha, y\alpha) = \varphi(x,y),\ \forall x,y \in V\}. \tag{2.2}$$

Let $\varepsilon' = -\epsilon$ so that $\varepsilon\varepsilon' = -1$. Let (W,φ') be a nonsingular ε'-Hermitian space. Put

$$\mathbb{W} = V \otimes W. \tag{2.3}$$

Then $\mathbb{W}$ is a nonsingular symplectic space over F with symplectic form

$$\phi = \mathrm{tr} \circ (\varphi \otimes \varphi'). \tag{2.4}$$

Let $G = U(\varphi)$, $G' = U(\varphi')$ be the unitary groups corresponding to φ and φ', respectively. It is well known that $G \times G'$ embeds as a dual pair in $\mathrm{Sp}(\phi)$.

We often express various objects by matrices. For a matrix x with entries in E, put

$$x^* = {}^t x^\rho, \qquad x^{-\rho} = (x^\rho)^{-1}, \qquad \hat{x} = {}^t x^{-\rho}, \tag{2.5}$$

assuming x to be square and invertible if necessary. Assume that $V \cong E^\ell$ for some nonzero positive integer ℓ. Let φ_0 be an $\ell \times \ell$ matrix satisfying $\varphi_0^* = \varepsilon\varphi_0$. We can define an ε-Hermitian form φ on V by requiring

$$\varphi(x,y) = x\varphi_0 y^*. \tag{2.6}$$

Then the unitary group $U(\varphi)$ is isomorphic to the subgroup of $\mathrm{GL}_\ell(E)$ consisting elements g satisfying

$$g\varphi_0 g^* = \varphi_0. \tag{2.7}$$

In the following we let $\varepsilon = -1$. Then φ is a nonsingular skew-Hermitian form, hence $\ell = 2n$ for some positive integer n. Let $e_1, \ldots, e_{2n}$ be a basis of V such that φ is represented by

$$\eta_{2n} = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}. \tag{2.8}$$

Put

$$X = \oplus_{i=1}^{n} Ee_i, \qquad Y = \oplus_{n+1}^{2n} Ee_i. \tag{2.9}$$

Then X, Y are maximal isotropic spaces of V. Let P be the maximal parabolic subgroup of G preserving Y. Then

$$P(F) = \left\{ \begin{pmatrix} g & gu \\ & \hat{g} \end{pmatrix} \mid g \in \mathrm{GL}_n(E),\ u \in S(F) \right\}. \tag{2.10}$$

Here

$$S(F) = \{ b \in M_{n\times n}(E) \mid b^* = b \} \tag{2.11}$$

is the set of Hermitian matrices of degree n. Let N be the unipotent radical of P. Then $N(F)$ consists of elements of the following type:

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad \text{with } b \in S(F). \tag{2.12}$$

Let

$$M = \{ g \in P \mid Xg \subset X,\ Yg \subset Y \}. \tag{2.13}$$

Then M is a Levi subgroup of P. The F-rational points $M(F)$ of M consists of elements of the following form:

$$m(a) = \begin{pmatrix} a & \\ & \hat{a} \end{pmatrix}, \quad \text{with } a \in \mathrm{GL}_n(E). \tag{2.14}$$

Define an action of $\mathrm{GL}_n(E)$ on $S(F)$ by

$$(a, b) \longrightarrow aba^*, \quad \text{with } a \in \mathrm{GL}_n(E),\ b \in S(F). \tag{2.15}$$

It is equivalent to the adjoint action of M on N, since

$$m(a)n(b)m(a)^{-1} = n(aba^*). \tag{2.16}$$

We will say “the action of $M(F)$ on $S(F)$” if no confusion is caused.

Let O be the unique open orbit of $M(F)\backslash S(F)$, then

$$O = \{b \in S(F) \mid \det b \neq 0\}. \tag{2.17}$$

For $\beta \in O$, let M_β be the stabilizer of β. Since β is a nonsingular Hermitian matrix,

$$M_\beta \cong U(\beta) \tag{2.18}$$

is the unitary group of β.

Let $\mathbb{Y} = Y \otimes W$. For $w \in \mathbb{Y}$, let us write

$$w = \sum_{i=1}^{n} e_{n+i} \otimes w_i, \quad \text{with } w_i \in W,\ i = 1,\dots,n. \tag{2.19}$$

Define the moment map $\mu : \mathbb{Y} \to S(F)$ by

$$\mu(w) = \left(\varphi'\left(w_i, w_j\right)\right)_{1 \le i,j \le n}. \tag{2.20}$$

It is clear that if $m = m(a) \in M(F)$, then

$$\mu(wm) = {}^{t}a\mu(w)a^{\rho}. \tag{2.21}$$

Denote the image of μ by $\mathscr{C}$, then it is invariant under $M(F)$. Let T be a Hermitian matrix representing φ'. If $\dim W = n$, then $T \in \mathscr{C} = O$. In particular, from (2.18),

$$M_T = G'. \tag{2.22}$$

3. Localization of various objects

Let F be a number field, E a quadratic field extension of F. Let $\mathbf{v}$ be the set of all places of F, $\mathbf{a}$, $\mathbf{f}$ be the sets of Archimedean and non-Archimedean places, respectively. Then $\mathbf{v} = \mathbf{a} \cup \mathbf{f}$. For $v \in \mathbf{v}$, let F_v be the v-completion of F, $\mathbb{O}_v$ the valuation ring of F_v if v is finite. Let $\mathbb{A}$, $\mathbb{A}_E$ be the rings of adeles of F and E, respectively.

Let ρ be the generator of Gal(E/F). For $v \in \mathbf{v}$, let $E_v = E \otimes F_v$. We may extend ρ to E_v, denote it by ρ_v. Then E_v is a quadratic extension of F_v, ρ_v is an F_v-automorphism of E_v of order 2. Corresponding to v is split in E or not, the couple (E_v,ρ_v) belongs to one of the following two cases.

(1) Case NS: v remains prime in E. Hence E_v is a quadratic field extension of $F_v, \rho_v \in$ Gal(E/F) is the nontrivial element.

(2) Case S: v splits in E. Then $E_v = F_v \oplus F_v$ and $(x,y)^{\rho_v} = (y,x)$ for $(x,y) \in E_v$.

Let γ be a nontrivial Hecke character of E, that is, it is a continuous homomorphism

$$\gamma : \mathbb{A}_E^{\times} \longrightarrow \mathbf{S}^1 \tag{3.1}$$

such that $\gamma(E^{\times}) = 1$. For $v \in \mathbf{v}$, Let γ_v be the restriction of γ to $E_v^{\times}$, then $\gamma = \otimes_v \gamma_v$.

For an algebraic group H defined over F, we let $H(F_v)$ be the set of F_v-points of H. Put

$$H_{\mathbf{a}} = \prod_{v \in \mathbf{a}} H(F_v), \qquad H_{\mathbf{f}} = \prod_{v \in \mathbf{f}}{}' H(F_v), \tag{3.2}$$

where the prime indicates restricted product with respect to $H(\mathbb{O}_v)$. Then

$$H(\mathbb{A}) = H_{\mathbf{a}}H_{\mathbf{f}}. \tag{3.3}$$

Let $G = U(\eta_n)$ be the quasi-split even unitary group of rank n defined over F. We have defined the standard Siegel parabolic subgroup $P = MN$ of G in Section 2. Keep notations of last section. For $v \in \mathbf{f}$, the localization of these algebraic groups are as follows.

(1) Case NS: v remains prime in E. In this case,

$$\begin{aligned} G(F_v) &= U(\eta_n)(F_v), \\ M(F_v) &= \{m(a) \mid a \in \mathrm{GL}_n(E_v)\}, \\ N(F_v) &= \{n(X) \mid X \in S(F_v)\}. \end{aligned} \tag{3.4}$$

(2) Case S: v splits in E. In this case,

$$\begin{gathered} G(F_v) = \mathrm{GL}_{2n}(F_v), \\ M(F_v) = \left\{ m(A,B) \mid m(A,B) = \begin{pmatrix} A & \\ & B^{-1} \end{pmatrix}, A,B \in \mathrm{GL}_n(F_v) \right\}, \\ N(F_v) = \left\{ n(X) \mid n(X) = \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}, X \in M_{n\times n}(F_v) \right\}. \end{gathered} \tag{3.5}$$

If $v \in \mathbf{f}$ is a finite place, let $K_{0,v} = G(\mathbb{O}_v)$ be a maximal open compact subgroup of $G(F_v)$. For $g \in G(F_v)$, we have Iwasawa decomposition

$$\begin{aligned} (\text{Case NS}) \quad g &= n(X)m(a)k, \\ (\text{Case S}) \quad g &= n(X)m(A,B)k \end{aligned} \tag{3.6}$$

for some $k \in K_{0,v}$, $n(X)m(a)$ or $n(X)m(A,B)$ belong to $P(F_v)$.

4. Local computation

Our result relies heavily on the L function of unitary group in [5] derived by Li. So in this section, we review the doubling method of Gelbart et al. [6] briefly and the main theorem of [5].

Let F be non-Archimedean local field with characteristic 0, $\mathbb{O}$ the valuation ring of F with uniformizer ϖ. Let $|\cdot|$ be the normalized absolute value of F. Let (E,ρ) be a couple as in Section 1. If E is a field extension of F, let $\mathbb{O}_E$ be the ring of integer of E with uniformizer ϖ_E, $|\cdot|_E$ the normalized absolute value of E.

Let V be $2n$-dimensional space over E with skew-Hermitian form $\varphi = \eta_{2n}$, $G = U(V)$. Then

$$\begin{aligned} G(F) &= U(\eta_{2n}), \quad \text{Case NS}; \\ G(F) &= \mathrm{GL}_{2n}, \quad \text{Case S}. \end{aligned} \tag{4.1}$$

Let $-V$ be the space V with Hermitian form $-\varphi$. Define

$$\mathbb{V} = V \oplus -V. \tag{4.2}$$

Then $\varphi \oplus (-\varphi)$ is a nonsingular skew-Hermitian form on $\mathbb{V}$. Let $H = U(\mathbb{V})$ be the unitary group of $\mathbb{V}$. Then $K = H(\mathbb{O})$ is a maximal open compact subgroup of $H(F)$. We embed $G \times G$ into H as a closed subgroup.

Define two maximal isotropic subspaces of $\mathbb{V}$ as follows:

$$\underline{X} = \{(v, -v) \mid v \in V\}, \qquad \underline{Y} = \{(v, v) \mid v \in V\}. \tag{4.3}$$

Then $\mathbb{V} = \underline{X} \oplus \underline{Y}$. Let Q be the maximal parabolic subgroup of H preserving $\underline{Y}$. Following [5], we define a rational character x of Q by

$$x(p) = \det\left(p|_{\underline{Y}}\right)^{-1}, \quad p \in \mathrm{Q}. \tag{4.4}$$

Choose a basis of $\mathbb{V}$ compatible with the decomposition (4.3), we can write p as a matrix:

$$p = \begin{pmatrix} a & * \\ & \hat{a} \end{pmatrix}, \quad \text{with } a \in \mathrm{GL}_{2n}. \tag{4.5}$$

Then $x(p) = \det(a)^{\rho}$.

Let γ be an unramified character of $F^{\times}$. Then $p \mapsto \gamma(x(p))$ is a character of $\mathrm{Q}(F)$. For $s \in \mathbb{C}$, let $I(s, \gamma)$ be the space of smooth functions $f : H(F) \to \mathbb{C}$ satisfying

$$f(pg) = \gamma(x(p)) \left| x(p) \right|^{s+(4n+1)/2} f(g), \quad p \in \mathrm{Q}(F),\ g \in G(F). \tag{4.6}$$

$H(F)$ acts on $I(s, \gamma)$ by right multiplication. Let $I(s, \gamma)^K$ be the subspace of K-invariant elements of $I(s, \gamma)$. Since γ is unramified, by Frobenius reciprocity,

$$\dim_{\mathbb{C}} I(s, \gamma)^K = 1. \tag{4.7}$$

Let $\Phi_{K,s}$ be the unique K-invariant function in $I(s, \gamma)$ such that

$$\Phi_{K,s}(1) = 1. \tag{4.8}$$

One important property of $\Phi_{K,s}$ is the following.

Lemma 4.1 (see [5, Lemma 3.2]). *Let $K_0 = G(\mathbb{O})$ be a maximal open compact subgroup of $G(F)$. Then for $k_1, k_2 \in K_0$, $g \in G(F)$,*

$$\Phi_{K,s}(k_1 g k_2, 1) = \Phi_{K,s}(g, 1), \tag{4.9}$$

here $(g, 1) \in G \times G \hookrightarrow H$.

4.1. L functions. Let (π, V) be an unramified irreducible representation of $G(F)$, $(\check{\pi}, \check{V})$ the contragredient of π. Let $\langle \cdot, \cdot \rangle_{\pi}$ be the canonical pairing between V and $\check{V}$. For $v \in V$, $\check{v} \in \check{V}$, define a matrix coefficient of π by

$$\omega_{\pi}(g; v, \check{v}) = \langle gv, \check{v} \rangle_{\pi}, \quad g \in G(F). \tag{4.10}$$

If v and $\check{v}$ are K_0-fixed elements of π and $\check{\pi}$, respectively, then $\omega_\pi(g;v,\check{v})$ is a spherical function of π. In addition, if $\langle v,\check{v}\rangle_\pi = 1$, then $\omega_\pi(1;v,\check{v}) = 1$, we get the zonal spherical function ω_π of π.

Let ${}^LG$ be the dual group of G. Then

$$\begin{aligned} {}^LG &= \mathrm{GL}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}(E/F), \quad \text{Case NS} \\ {}^LG &= \mathrm{GL}_{2n}(\mathbb{C}), \quad \text{Case S.} \end{aligned} \tag{4.11}$$

For Case NS, the action of $\mathrm{Gal}(E/F)$ on GL_{2n} is given by

$$g^\rho = \Phi_{2n}\, {}^tg^{-1}\Phi_{2n}^{-1}, \quad g \in \mathrm{GL}_{2n}(\mathbb{C}). \tag{4.12}$$

Here

$$\Phi_{2n} = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \vdots & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}. \tag{4.13}$$

Since π is an unramified irreducible representation of $G(F)$, it determines a unique semisimple conjugacy class (a_π,ρ)(Case NS) or a_π(Case S) in ${}^LG$ [7]. We can take a representative of a_π as follows:

$$\begin{aligned} a_\pi &= \mathrm{diag}\left(a_1,\dots,a_n,1,\dots,1\right), \quad \text{Case NS,} \\ a_\pi &= \mathrm{diag}\left(a_1,\dots,a_{2n}\right), \quad \text{Case S,} \end{aligned} \tag{4.14}$$

with $a_i \in \mathbb{C}^\times$, $i = 1,\dots,2n$ [7, Section 6.9].

Let r be the natural action of $\mathrm{GL}_{2n}(\mathbb{C})$ on $\mathbb{C}^{2n}$, σ the induced representation

$$\begin{aligned} \sigma &= \mathrm{Ind}_{\mathrm{GL}_{2n}(\mathbb{C})}^{{}^LG}(r), \quad \text{Case NS,} \\ \sigma &= \mathrm{Ind}_{\mathrm{GL}_{2n}(\mathbb{C})}^{\mathrm{GL}_{2n}(\mathbb{C})\times\mathbb{Z}/2\mathbb{Z}}\, r, \quad \text{Case S,} \end{aligned} \tag{4.15}$$

respectively. Associate a local L function $L(s,\pi,\sigma)$ to π by

$$\begin{aligned} \text{Case NS} : L(s,\pi,\sigma) &= \det\left(1 - \sigma\left(a_\pi,\rho\right)q^{-s}\right)^{-1} \\ &= \prod_{i\le n}\left[\left(1 - a_iq^{-2s}\right)\left(1 - a_i^{-1}q^{-2s}\right)\right]^{-1}, \\ \text{Case S} : L(s,\pi,\sigma) &= \det\left(1 - \sigma\left(a_\pi\right)q^{-s}\right)^{-1} \\ &= \prod_{i\le 2n}\left[\left(1 - a_iq^{-s}\right)\left(1 - a_i^{-1}q^{-s}\right)\right]^{-1}, \end{aligned} \tag{4.16}$$

where q is the cardinality of residue field of F.

The relation between the functions $\Phi_{K,s}$, ω_π, and $L(s,\pi,\sigma)$ is as follows.

Theorem 4.2 (see [5, Theorem 3.1]). *Notations as above. For $s \in \mathbb{C}$,*

$$\int_{G(F)} \Phi_{K,s}(g,1)\omega_\pi(g) = \frac{L(s+1/2,\pi,\sigma)}{d_H(s)}. \tag{4.17}$$

Here

$$\begin{gathered}\text{(Case NS)}\quad d_H(s) = \frac{L(2s+1,\epsilon_{E/F})}{L(2s+2n+1,\epsilon_{E/F})}\prod_{0\le j<n}\xi(2s+2n-2j)L(2s+2n-2j+1,\epsilon_{E/F}),\\ \text{(Case S)}\quad d_H(s) = \prod_{j=1}^{2n}(2s+j).\end{gathered} \tag{4.18}$$

$\xi(s)$ is the zeta function of F, $\epsilon_{E/F}$ is the character of order 2 associated to the extension E/F by local class field theory, $L(s,\chi)$ is the local Hecke L function for a character χ of $F^\times$.

We will derive a formula from (4.17) which is applicable for our computation later. For this purpose, for $g \in G(F)$, let

$$\begin{aligned}\text{(Case NS)}\quad \delta(g) &= \operatorname{diag}(\varpi_E^{l_1},\dots,\varpi_E^{l_n}), \quad l_1 \ge \cdots \ge l_n \ge 0,\\ \text{(Case S)}\quad \delta(g) &= \operatorname{diag}(\varpi^{l_1},\dots,\varpi^{l_{2n}}), \quad l_1 \ge \cdots \ge l_{2n},\end{aligned} \tag{4.19}$$

such that $g \in K_0 m(\delta(g))K_0$(Case NS) or $g \in K_0\delta(g)K_0$(Case S). Define a function $\Delta(g)$ on $G(F)$ by

$$\begin{aligned}\text{(Case NS)}\quad \Delta(g) &= |\det\delta(g)|_E^{-1},\\ \text{(Case S)}\quad \Delta(g) &= |\det\delta(g)|^{-1}.\end{aligned} \tag{4.20}$$

By Lemma 4.1,

$$\begin{aligned}\text{(Case NS)}\quad \Phi_{K,s}(g,1) &= \Phi_{K,s}(m(\delta(g),1)),\\ \text{(Case S)}\quad \Phi_{K,s}(g,1) &= \Phi_{K,s}(\delta(g),1).\end{aligned} \tag{4.21}$$

Furthermore, reasoning as in [5, page 197], one can show that

$$\Phi_{K,s}(g,1) = \Delta(g)^{-(s+n)}. \tag{4.22}$$

Hence Theorem 4.2 is equivalent to the following.

Theorem 4.3. *For $s \in \mathbb{C}$,*

$$\int_{G(F)} \Delta(g)^{-(s+n)}(g)\omega_\pi(g)dg = \frac{L(s+1/2,\pi,\sigma)}{d_H(s)}. \tag{4.23}$$

Here $d_H(s)$ is the meromorphic functions in Theorem 4.2.

Before we end this section, we record a formula for the value on $\Delta(g)$ for some special elements in $G(F)$. For $\beta \in M_{n\times n}(F)$, let $L(\beta)$ be the set of all minors of β.

Lemma 4.4 (see [8, Proposition 3.9]). (1) *(Case NS) Let*

$$g = \begin{pmatrix} \hat{w} & \\ & w \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \begin{pmatrix} v^* & \\ & v^{-1} \end{pmatrix} \in G(F) \tag{4.24}$$

with $v, w \in \mathrm{GL}_n(E) \cap M_{n\times n}(\mathbb{O}_E)$. *Then*

$$\Delta(g) = \left| \det(vw) \right|_E^{-1} \max_{C\in L(\beta)} \left| \det C \right|_E. \tag{4.25}$$

(2) *(Case S) Let*

$$g = \begin{pmatrix} w^{-1} & \\ & v \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix} \begin{pmatrix} v' & \\ & w'^{-1} \end{pmatrix} \in G(F) \tag{4.26}$$

with $v, v', w, w' \in \mathrm{GL}_n(F) \cap M_{n\times n}(\mathbb{O})$. *Then*

$$\Delta(g) = \left| \det(vv'ww') \right|^{-1} \left(\max_{C\in L(\beta)} \left| \det C \right| \right)^2. \tag{4.27}$$

5. Fourier coefficients

In this section, we will compute Fourier coefficients of $\Delta(g)$. Our method is similar to that of [4].

Notations are as in the last section. Let ψ be a nontrivial additive character of F. Let (π, V_0) be an unramified irreducible admissible representation of $G(F)$, T a square matrix such that $T \in S(F)$(Case NS) or $T \in M_{n\times n}(F)$(Case S). Let l_T be a linear functional on V_0 satisfying

$$l_T\left(\pi\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} v\right) = \overline{\psi(\mathrm{tr}(XT))}\, l_T(v) \tag{5.1}$$

for all $v \in V_0$, $X \in S(F)$(Case NS) or $X \in M_{n\times n}(F)$(Case S).

Example 5.1. Let F be a number field, π an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ for a moment [9]. Then $\pi = \otimes_v' \pi_v$ is a restricted product of irreducible admissible representations π_v of $G(F_v)$, for almost all $v \in \mathbf{v}$, π_v is unramified irreducible admissible representation. Let f be a cusp form in $A(G(F) \setminus G(\mathbb{A}))_\pi$, the isotypic space of π. Let $v \in \mathbf{f}$ such that π_v is unramified irreducible admissible representation of $G(F_v)$. Let $T_v \in S(F_v)$(Case NS) or $T_v \in M_{n\times n}(F_v)$. Define a linear functional L_{T_v} on $A(G(F) \setminus G(\mathbb{A}))_\pi$ by

$$l_{T_v}(f) = \int f\left(\begin{pmatrix} 1 & X_v \\ & 1 \end{pmatrix}\right) \psi\left(\mathrm{tr}\,(X_v T_v)\right) dX_v, \tag{5.2}$$

where the integral is taken on $S(F_v)$(Case NS) or $M_{n\times n}(F_v)$(Case S). We see that $l_{T_v}(f)$ is independent of $f|_{G(F_w)}$ for $w \in \mathbf{v}$, $w \neq v$. But $\pi_v = \pi|_{G(F_v)}$, so l_{T_v} is a linear functional on π_v satisfying (5.1).

Back to the assumption that F is non-Archimedean local field, (π, V_0) is an unramified irreducible representation of $G(F)$. Define a subset $M(\mathbb{O})$ of $M_{2n}(E)$(Case NS) or of $M_{2n}(F)$(Case S) as follows:

$$\begin{gathered}
\text{(Case NS)} \quad M(\mathbb{O}) = \left\{ m(a) = \begin{pmatrix} a & \\ & \hat{a} \end{pmatrix} \mid a \in M_{n\times n}(\mathbb{O}_E) \cap \mathrm{GL}_n(E) \right\}; \\
\text{(Case S)} \quad M(\mathbb{O}) = \left\{ m(A,B) = \begin{pmatrix} A & \\ & B^{-1} \end{pmatrix} \mid A, B \in M_{n\times n}(\mathbb{O}) \cap \mathrm{GL}_n(F) \right\}.
\end{gathered} \tag{5.3}$$

Let γ_0 be a function on $M(\mathbb{O})$ defined by

$$\begin{gathered}
\text{(Case NS)} \quad \gamma_0(m(a)) = |\det a|_E, \\
\text{(Case S)} \quad \gamma_0(m(A,B)) = |\det A \det B|.
\end{gathered} \tag{5.4}$$

Lemma 5.2. *Let ψ be an unramified additive character of F. Let T be a square matrix such that $T \in S(F)$(Case NS) or $T \in M_{n\times n}(F)$(Case S). Let (π, V_0) be an unramified irreducible admissible representation of $G(F)$. Take $0 \neq f_0 \in V_0^{K_0}$, where $K_0 = G(\mathbb{O})$ is a maximal compact subgroup of $G(F)$. Let l_T be a linear functional on V_0 satisfying (5.1). Then for $s \in \mathbb{C}$,*

$$\int_{G(F)} \Delta^{-(s+n)}(g) l_T(\pi(g) f_0)\, dg = l_T(f_0) \frac{L(s+1/2, \pi, \sigma)}{d_H(s)}. \tag{5.5}$$

Proof. As in [3], the convergence of left-hand side of the equation when $\mathrm{Re}\, s$ is sufficiently large comes from the vanishing of $l_T(\pi(a) f_0)$ when a is sufficiently large, here a belongs to the maximal F-torus consisting of diagonal elements in $G(F)$.

Since both sides are meromorphic functions of s, we only need to show the equation for $\mathrm{Re}\, s$ sufficiently large. We first claim that

$$\int_{K_0} l_T(\pi(kg) f_0)\, dk = l_T(f_0) \omega_\pi(g), \quad g \in G(F). \tag{5.6}$$

In fact, the left-hand side is a bi-K_0-invariant matrix coefficient of π, so there is some $\lambda \in \mathbb{C}$ such that

$$\int_{K_0} l_T(\pi(kg) f_0)\, dk = \lambda \omega_\pi(g), \quad g \in G(F). \tag{5.7}$$

Let $g = 1$, then $\lambda = l_T(f_0)$.

Back to the proof of the lemma. If $\mathrm{Re}\, s$ is sufficiently large, the left-hand side of (5.5) converges absolutely. Hence

$$\begin{aligned}
\text{L.H.S of (5.5)} &= \int_{G(F)} \int_{K_0} \Delta^{-(s+n)}(kg) l_T(\pi(g) f_0)\, dk\, dg \\
&= \int_{G(F)} \int_{K_0} \Delta^{-(s+n)}(g) l_T(\pi(kg) f_0)\, dk\, dg
\end{aligned} \tag{5.8}$$

we have computed the inside integral in (5.6), so

$$\begin{aligned}(5.8) &= l_T(f_0)\int_{G(F)}\Delta^{-(s+n)}(g)\omega_\pi(g)dg \\ &= l_T(f_0)\frac{L(s+1/2,\pi,\sigma)}{d_H(s)}, \quad \text{by Theorem 4.3.}\end{aligned} \tag{5.9}$$

□

Apply Iwasawa decomposition (3.6) $g = n(X)m(a)k$ in the integrand of (5.5). When $\operatorname{Re} s$ is sufficiently large,

$$\begin{aligned}\int_{G(F)}\Delta^{-(s+n)}(g)l_T(\pi(g)f_0)df = \int_{K_0\times M(F)\times N(F)} &\Delta^{-(s+n)}(n(X)m(a)k)l_T(\pi(n(X)m(a)k)f_0) \\ &\times\delta_P(m(a))^{-1}dn(X)dm(a)dk.\end{aligned} \tag{5.10}$$

Here $\delta_P(m(a))$ is the modular function of $P(F)$, hence $\delta_P(m(a)) = |\det a|_E^n$(Case NS) or $\delta_P(m(A,B)) = |\det A\det B|^n$(Case S). Note that f_0 is K_0 invariant, Δ is bi-K_0 invariant,

$$\begin{aligned}(5.10) = \int_{M(F)\times N(F)} &\Delta^{-(s+n)}(n(X)m(a))\overline{\psi(\operatorname{tr}(XT))} \\ &\times l_T(\pi(m(a))f_0)\delta_P(m(a))^{-1}dn(X)dm(a).\end{aligned} \tag{5.11}$$

If we let

$$J_T(s,a) = \int_{N(F)}\Delta^{-(s+n)}(n(X)m(a))\overline{\psi(\operatorname{tr}(XT))}dn(X), \tag{5.12}$$

for $m(a)\in M(F)$, then

$$(5.11) = \int_{M(F)} J_T(s,a)l_T(\pi(m(a))f_0)\delta_P^{-1}(m(a))dm(a). \tag{5.13}$$

Properties of $J_T(s,a)$, such as convergent when s sufficiently large, having meromorphic continuation to $\mathbb{C}$, is discussed by Shimura [10], for example, Proposition 3.3 there.

Lemma 5.3. *Let ψ be an unramified character of F. Let T be a square matrix such that $T\in \mathrm{GL}_{n\times n}(\mathbb{O}_E)\cap S(F)$ or $T\in\mathrm{GL}_n(\mathbb{O})$(Case S). Then*

$$J_T(s,a) = \begin{cases}\gamma_0(m(a))^{s+n}j_T(s), & a\in M(\mathbb{O}),\\ 0, & \textit{if else}.\end{cases} \tag{5.14}$$

Here

$$\begin{aligned}
&\text{(Case NS)}\quad j_T(s)=\int_{S(F)}\Delta^{-(s+n)}(n(X))\overline{\psi(\operatorname{tr}(TX))}\,dX\\
&\qquad\qquad\qquad=\prod_{r=0}^{n-1}L\left(2s+2n-r,\epsilon_{E/F}^{r}\right),\\
&\text{(Case S)}\quad j_T(s)=\int_{M_{n\times n}(F)}\Delta^{-(s+n)}(n(X))\overline{\psi(\operatorname{tr}(TX))}\,dX\\
&\qquad\qquad\qquad=\prod_{r=0}^{n-1}\zeta(2s+2n-r).
\end{aligned}\tag{5.15}$$

Proof. Both sides of (5.14) are meromorphic functions for a given $m(a)\in M(F)$. We only need to prove this lemma for $\operatorname{Re}s$ sufficiently large.

(Case NS). Let $a\in \mathrm{GL}_n(E)$. By the principle of elementary divisors, $a={}^tw^{-1}\,{}^tv$ with $v,w\in M_{n\times n}(\mathbb{O}_E)$, $v=k\delta_1, w=k'\delta_2$ with $k,k'\in\mathrm{GL}_n(\mathbb{O}_E)$ and

$$\begin{aligned}
\delta_1&=\operatorname{diag}\left(\varpi_E^{m_1},\dots,\varpi_E^{m_i},1,\dots,1\right),\\
\delta_2&=\operatorname{diag}\left(1,\dots,1,\varpi_E^{m_{i+1}},\dots,\varphi_E^{m_n}\right)
\end{aligned}\tag{5.16}$$

with $m_1\geq\cdots\geq m_i\geq 0$, $m_{i+1}\geq\cdots\geq m_n\geq 0$ for some $0\leq i\leq n$. Then

$$\begin{aligned}
J_T(s,a)&=J_T\left(s,{}^tw^{-1t}v\right)\\
&=\int_{S(F)}\Delta^{-(s+n)}\left(n(X)m\left({}^tw^{-1t}v\right)\right)\overline{\psi(\operatorname{tr}(XT))}\,dX\\
&=\int_{S(F)}\Delta^{-(s+n)}\left(m\left({}^tw^{-1}\right)m\left({}^tw^{-1}\right)^{-1}n(X)m\left({}^tw^{-1t}v\right)\right)\\
&\quad\times\overline{\psi(\operatorname{tr}(XT))}\,dX\\
&=|\det(w)|_E^{-n}\int_{S(F)}\Delta^{-(s+n)}\left(m\left({}^tw^{-1}\right)n(X)m\left({}^tv\right)\right)\\
&\qquad\times\overline{\psi\left(\operatorname{tr}\left(Xw^{-\rho}T^tw^{-1}\right)\right)}\,dX.
\end{aligned}\tag{5.17}$$

Let $S(\mathbb{O})$ be the set of elements in $S(F)$ with entries in $\mathbb{O}_E$. Let $\mathcal{I}$ be a set of representative of $S(F)/S(\mathbb{O})$. Decompose the integral in (5.17) as a sum of integrals indexed by $\mathcal{I}$:

$$(5.17)=|\det w|_E^{-n}\sum_{\xi\in\mathcal{I}}\int_{\xi+S(\mathbb{O})}\Delta^{-(s+n)}\left(m\left({}^tw^{-1}\right)n(X)m\left({}^tv\right)\right)\times\overline{\psi\left(\operatorname{tr}\left(Xw^{-\rho}T^tw^{-1}\right)\right)}\,dX.\tag{5.18}$$

Let $\xi\in S(F)$. If $\xi\notin S(\mathbb{O})$, by Lemma 4.4,

$$\Delta^{-(s+n)}\left(m\left({}^tw^{-1}\right)n(\xi+X)m\left({}^tv\right)\right)=\left|\det v^\rho w^\rho\right|_E^{s+n}\Delta^{-(s+n)}(n(\xi))\tag{5.19}$$

for all $X\in S(\mathbb{O})$, since

$$\max_{C\in L(\xi+X)}|\det C|_E=\max_{C\in L(\xi)}|\det C|_E\tag{5.20}$$

for $\xi \notin S(\mathbb{O})$. If $\xi \in S(\mathbb{O})$, then $\Delta(n(\xi)) = 1$,

$$\Delta^{-(s+n)}(m({}^t w^{-1})n(\xi+X)m({}^t v)) = |\det(vw)^\rho|_E^{s+n}\Delta^{-(s+n)}(n(\xi)) = |\det(vw)^\rho|_E^{s+n}. \tag{5.21}$$

Hence for all $\xi \in S(F)$, $X \in S(\mathbb{O})$,

$$\Delta^{-(s+n)}(m({}^t w^{-1})n(\xi+X)m({}^t v)) = |\det(vw)^\rho|_E^{s+n}\Delta^{-(s+n)}(n(\xi)). \tag{5.22}$$

Apply (5.22) to (5.18), we then get

$$\begin{aligned}(5.18) = |\det w|_E^{-n}|\det(vw)^\rho|_E^{s+n}&\sum_{\xi\in\mathcal{J}}\Delta^{-(s+n)}(n(\xi))\\ \times\overline{\psi(\operatorname{tr}(\xi w^{-\rho}T^t w^{-1}))}&\int_{S(\mathbb{O})}\overline{\psi(\operatorname{tr}(Xw^{-\rho}T^t w^{-1}))}dX.\end{aligned} \tag{5.23}$$

If $a \notin M_{n\times n}(\mathbb{O}_E)$, then $|\det w|_E < 1$ and $w^{-\rho}T\,{}^t w^{-1} \in S(\mathbb{O})$. Hence

$$\int_{S(\mathbb{O})}\overline{\psi(\operatorname{tr}(Xw^{-\rho}T\,{}^t w^{-1}))}dX = 0, \tag{5.24}$$

and $J_T(s,a) = 0$. If $a \in \mathrm{GL}_n(E)\cap M_{n\times n}(\mathbb{O}_E)$, we compute $J_T(s,a)$ directly:

$$\begin{aligned}J_T(s,a) &= \int_{S(F)}\Delta^{-(s+n)}(n(X)m(a))\overline{\psi(\operatorname{tr}(XT))}dX\\ &= |\det a|_E^{s+n}\int_{S(F)}\Delta^{-(s+n)}(n(X))\overline{\psi(\operatorname{tr}(XT))}dX, \quad \text{by Lemma 4.4}\\ &= |\det a|_E^{s+n}j_T(s),\end{aligned} \tag{5.25}$$

here

$$\begin{aligned}j_T(s) &= \int_{S(F)}\Delta^{-(s+n)}(n(X))\overline{\psi(\operatorname{tr}(TX))}dX\\ &= \prod_{r=0}^{n-1}L(2s+2n-r,\epsilon_{E/F}^r),\end{aligned} \tag{5.26}$$

where the second equality comes from [10, Proposition 6.2] by Shimura.

The proof for Case S is similar, and we omit it here. □

Theorem 5.4. *Let ψ be an unramified character of F, (π,V_0) an unramified irreducible admissible representation of $G(F)$. Let T be a square matrix such that $T\in\mathrm{GL}_n(\mathbb{O}_E)\cap S(F)$(Case NS) or $T\in\mathrm{GL}_n(\mathbb{O})$(Case S). Let l_T be a linear functional on V_0 satisfying (5.1). Then for $0\neq f_0\in V_0^{K_0}$,*

$$\int_{M(\mathbb{O})}y_0^s(m(a))l_T(\pi(m(a))f_0)dm(a) = l_T(f_0)\frac{L(s+1/2,\pi,\sigma)}{j_T(s)d_H(s)}, \tag{5.27}$$

where $d_H(s)$ and $j_T(s)$ are given in Theorem 4.2 and Lemma 5.3.

Proof. Lemma 5.2 and the paragraph after Lemma 5.2 have shown that

$$\begin{aligned} l_T(f_0)\frac{L(s+1/2,\pi,\sigma)}{d_H(s)} &= \int_{G(F)} \Delta^{-(s+n)}(g) l_T(\pi(g)f_0)\,dg \\ &= \int_{M(F)} J_T(s,a) l_T(\pi(m(a))f_0)\delta_P^{-1}(m(a))\,dm(a). \end{aligned} \tag{5.28}$$

By Lemma 5.3, $J_T(s,a)$ vanishes when $a \notin M(\mathbb{O})$. Substitute the formula of $J_T(s,a)$ for $a \in M(\mathbb{O})$ and δ_P^{-1}, the conclusion follows. □

6. Global computation

Let F be a number field, E a quadratic field extension of F. As usual, let $\mathbf{v}$ be the set of all places of F, $\mathbf{a},\mathbf{f}$ the set of archimedean and non-archimedean places of F respectively. Let F_v be the localization of F at the place v of $\mathbf{v}$, $E_v = E \otimes F_v$. If $v \in \mathbf{f}$, let $\mathbb{O}_v$ be the ring of integers of F_v. If v remains prime in E, then E_v is a quadratic field extension of F_v, let $\mathbb{O}_{E_v}$ be the ring of integer of E_v. The ring of adeles of F (resp., E) is denoted by $\mathbb{A}$ (resp., $\mathbb{A}_E$). Denote by $|\cdot|$ (resp., $|\cdot|_E$) the normalized absolute value of $\mathbb{A}^\times$ (resp., $\mathbb{A}_E^\times$). Let ψ be a nontrivial continuous character of $\mathbb{A}$ trivial on F.

Let V be a $2n$-dimensional vector space over E with an anti-Hermitian form η_{2n} on it. Let W be an n-dimensional vector space over E with a nonsingular Hermitian form T. Let $G = U(\eta_{2n})$, $G' = U(T)$ be the corresponding unitary groups. Then $G \times G'$ is a dual pair in $\mathrm{Sp}(\mathbb{W})$, where $\mathbb{W} = V \otimes W$ is symplectic space with symplectic form $\mathrm{tr}_{E/F}(\eta_{2n} \otimes T)$.

Let $P = MN$ be the maximal parabolic subgroup of G defined in Section 2. For $v \in \mathbf{v}$, let K_v be a maximal compact subgroup of $G(F_v)$ such that for almost all $v \in \mathbf{v}$, $K_v = G(\mathbb{O}_v)$. Let $K_{\mathbb{A}} = \prod_{v\in\mathbf{v}} K_v$. Then $G(\mathbb{A}) = P(\mathbb{A})K_{\mathbb{A}}$. For $v \in \mathbf{v}$, let dk_v be the Haar measure on K_v such that $\int_{K_v} dk_v = 1$. Then $dk = \prod_v dk_v$ is an Haar measure on $K_{\mathbb{A}}$ such that $\int_{K_{\mathbb{A}}} dk = 1$. Let $d_l(p_v)$ be a left Haar measure on $P(F_v)$ for $v \in \mathbf{v}$. Then $d_l p = \prod_v d_l(p_v)$ is a left Haar measure on $P(\mathbb{A})$. Since $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})$, $d_l p = |\det a|_E^{-n} d^\times a\, dX$ if $p = m(a)n(X)$ for $a \in \mathrm{GL}_n(\mathbb{A}_E)$, $X \in S(\mathbb{A})$, where $d^\times a$, dX are Haar measure on $\mathrm{GL}_n(\mathbb{A}_E)$, $S(\mathbb{A})$, respectively. We then let $dg = d_l p\, dk$ be an Haar measure on $G(\mathbb{A})$.

Let $s \in \mathbb{C}$, let γ be a Hecke character of E. Denote by $I(s,\gamma)$ the set of smooth functions $f : G(\mathbb{A}) \to \mathbb{C}$ satisfying

(i) $f(pg) = \gamma(x(p))|x(p)|_E^{s+n/2} f(g)$, for $p \in P(\mathbb{A})$, $g \in G(\mathbb{A})$,

(ii) f is K_v-finite for all $v \in \mathbf{a}$.

$G(\mathbb{A})$ acts on $I(s,\gamma)$ by right multiplication. Let $\Phi(g,s)$ be a smooth function in $I(s,\gamma)$ holomorphic at s. The Eisenstein series associated to $\Phi(g,s)$ is given by

$$E(g,s;\gamma,\Phi) = \sum_{\xi \in P(F)\backslash G(F)} \Phi(\xi g, s). \tag{6.1}$$

In [9], it has been shown that (6.1) is convergent when $\mathrm{Re}\, s > n/2$ and has a meromorphic continuation to the whole complex plane.

Let π be a cusp automorphic representation of $G(\mathbb{A})$ (cf. [9]). Let f be cusp form in the isotypic space of π. Let $\beta \in S(F)$. The βth Fourier coefficient of f is

$$f_\beta(g) = \int_{S(F)\backslash S(\mathbb{A})} f(n(X)g)\psi(\operatorname{tr}(X\beta))\,dX, \quad g \in G(\mathbb{A}). \tag{6.2}$$

If $\beta_1, \beta_2 \in S(F)$, $\beta_1 = {}^t a^\rho \beta_2 a$ for some $a \in \mathrm{GL}_n(E)$, then

$$f_{\beta_1}(g) = f_{\beta_2}(m(a)g), \quad g \in G(\mathbb{A}). \tag{6.3}$$

Let χ be a Hecke character of E satisfying $\chi|_{\mathbb{A}^\times/F^\times} = \epsilon_{E/F}^n$, where $\epsilon_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times$ by global class field theory. Associate with ψ a Weil representation ω_ψ of $G(\mathbb{A})$ acting on $\mathcal{S}(\mathbb{Y}(\mathbb{A}))$, the set of Schwartz-Bruhat functions on $\mathbb{Y}(\mathbb{A})$. In fact, ω_ψ is the restriction of Weil representation (associated with ψ) of $\widetilde{\mathrm{Sp}(\mathbb{W})}(\mathbb{A})$ to $G(\mathbb{A})$ (see Section 2 for the definition of $\mathbb{Y}, \mathbb{W}$). We will omit the subscript ψ when ψ is clear from the context. The explicit formula of ω is given in [11], we cite here the formula on $P(\mathbb{A})$. Let $\phi \in \mathcal{S}(\mathbb{Y}(\mathbb{A}))$, $a \in \mathrm{GL}_n(\mathbb{A}_E)$, $n(X) \in N(\mathbb{A})$, then

$$\begin{aligned} \omega(m(a))\phi(y) &= \chi(\det a)|\det a|_E^{n/2}\phi(ya), \\ \omega(n(X))\phi(y) &= \psi(\operatorname{tr}(b\mu(y)))\phi(y), \quad y \in \mathbb{Y}(\mathbb{A}). \end{aligned} \tag{6.4}$$

Here $\mu = \prod_v \mu_v : \mathbb{Y}(\mathbb{A}) \to \mathcal{S}(\mathbb{A})$, μ_v is the moment map defined at Section 2 for local field F_v.

The theta series θ_ϕ for $\phi \in \mathcal{S}(\mathbb{Y}(\mathbb{A}))$ is a smooth function on $G(\mathbb{A})$ of moderate growth

$$\theta_\phi(g) = \sum_{\xi \in S(F)} \omega(g)\phi(\xi), \quad g \in G(\mathbb{A}). \tag{6.5}$$

6.1. Vanishing lemma. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. We make the following assumption: There is some cusp form f in the isotypic space of π such that

$$\int_{N(F)\backslash N(\mathbb{A})} f(n(X)g)\psi(\operatorname{tr}(XT)) \neq 0. \tag{6.6}$$

In [4], Piatetski-Shapiro and Rallis do not propose this assumption, because Li has shown in [12] that every cusp forms supports some nonsingular symmetric matrix.

For $\phi \in \mathcal{S}(\mathbb{Y}(\mathbb{A}))$, $\Phi(g,s) \in I(s,\gamma)$, $f \in A(G(F)\backslash G(\mathbb{A}))_\pi$ the isotypic space of π in the space of automorphic forms on $G(\mathbb{A})$, define

$$I(s,\phi,\Phi,f) = \int_{G(F)\backslash G(\mathbb{A})} f(g)E(g,s,\Phi)\theta_\phi(g)\,dg. \tag{6.7}$$

Although θ_ϕ is slowly increasing function on $G(\mathbb{A})$, $E(g,s,\Phi)$ is of moderate growth, but f is rapidly decreasing on $G(\mathbb{A})$, (6.7) is convergent at s where the Eisenstein series is holomorphic. We will show that when we choose appropriate ϕ, Φ, f, $I(s,\phi,\Phi,f)$ is product of meromorphic function with partial L function of π.

Substitute Eisenstein series (6.1), theta series (6.5) into (6.7), then

$$\begin{aligned}(6.7) &= \int_{P(F)\backslash G(\mathbb{A})} f(g)\Phi(g,s) \sum_{\xi\in\mathbb{Y}(F)} \omega(g)\phi(\xi)dg \\ &= \int_{K_{\mathbb{A}}}\int_{P(F)\backslash P(\mathbb{A})} f(pk)\Phi(pk,s) \sum_{\xi\in\mathbb{Y}(F)} \omega(pk)\phi(\xi)d_l p\, dk.\end{aligned} \tag{6.8}$$

By the assumption that $\Phi(g,s) \in I(s,\gamma)$, $\Phi(pk,s) = \gamma(x(p))|x(p)|_E^{s+n/2}\Phi(k,s)$. Apply the formula of Weil representation (6.4) to (6.8), then

$$\begin{aligned}(6.8) = &\int_{K_{\mathbb{A}}}\int_{M(F)\backslash M(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})} f(n(X)m(a)k)\Phi(k,s) \\ &\times (\gamma\chi|\cdot|_E^s)(\det a) \sum_{\xi\in\mathbb{Y}(F)} \psi(\operatorname{tr}(b\mu(\xi)))\omega(k)\phi(\xi a)dX\, d^\times a\, dk.\end{aligned} \tag{6.9}$$

Recall that in Section 2, we let $\mathscr{C} \subset S(F)$ be the image of moment map, which is invariant under the action of $M(F)$. Let $\mathscr{J}$ be a set of representatives of orbits $\mathscr{C}/M(F)$ such that $T \in \mathscr{J}$. We then write (6.9) as a sum of integrals indexed by $\mathscr{J}$:

$$\begin{aligned}(6.9) &= \int_{K_{\mathbb{A}}}\int_{M(F)\backslash M(\mathbb{A})} \sum_{\beta\in\mathscr{C}}\sum_{\xi\in\mu^{-1}(\beta)} f_\beta(m(a)k)\Phi(k,s) \\ &\quad\times (\gamma\chi|\cdot|_E^s)(\det a)\omega(k)\phi(\xi a)d^\times a\, dk \\ &= \sum_{\beta\in\mathscr{J}}\int_{K_{\mathbb{A}}}\int_{M(F)\backslash M(\mathbb{A})} \sum_{a'\in M_\beta(F)\backslash M(F)}\sum_{\xi\in\mu^{-1}(\beta)} f_\beta(m(a')m(a)k)\Phi(k,s) \\ &\quad\times (\gamma\chi|\cdot|_E^s)(\det a)\omega(k)\phi(\xi a' a)d^\times a\, dk.\end{aligned} \tag{6.10}$$

Here f_β is βth Fourier coefficient of f, M_β is the stabilizer of β under the action of M (cf. Section 2). For $\beta \in \mathscr{J}$, let

$$\begin{aligned}I_\beta(s) = &\int_{K_{\mathbb{A}}}\int_{M(F)\backslash M(\mathbb{A})} \sum_{a'\in M_\beta(F)\backslash M(F)}\sum_{\xi\in\mu^{-1}(\beta)} f_\beta(m(a')m(a)k)\Phi(k,s) \\ &\times (\gamma\chi|\cdot|_E^s)(\det a)\omega(k)\phi(\xi a' a)d^\times a\, dk.\end{aligned} \tag{6.11}$$

Then

$$I(s,\phi,\Phi,f) = \sum_{\beta\in\mathscr{J}} I_\beta(s). \tag{6.12}$$

Lemma 6.1. *$I_\beta(s) = 0$ for all $\beta \in \mathscr{J}$ with* $\det\beta = 0$.

Proof. If $\beta = 0$, then for all $g \in G(\mathbb{A})$,

$$f_\beta(g) = \int_{N(F)\backslash N(\mathbb{A})} f(ng)dn = 0 \tag{6.13}$$

since f is a cusp form. Hence

$$I_\beta(s) = \int_{K_{\mathbb{A}}} \int_{M(F)\backslash M(\mathbb{A})} \sum_{a' \in M_\beta(F)\backslash M(F)} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a')m(a)k)\Phi(k,s) \\ \times (\gamma\chi|\cdot|_E^s)(\det a)\omega(k)\phi(\xi a' a)d^\times a\, dk = 0. \tag{6.14}$$

Let $0 \neq \beta \in \mathcal{J}$ with $\det\beta = 0$. Then

$$\begin{aligned} I_\beta(s) &= \int_{K_{\mathbb{A}}} \int_{M(F)\backslash M(\mathbb{A})} \sum_{a' \in M_\beta(F)\backslash M(F)} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a')m(a)k)\Phi(k,s) \\ &\quad \times (\gamma\chi|\cdot|_E^s)(\det a)\omega(k)\phi(\xi a' a)d^\times a\, dk \\ &= \int_{K_{\mathbb{A}}} \int_{M_\beta(\mathbb{A})\backslash M(\mathbb{A})} \int_{M_\beta(F)\backslash M_\beta(\mathbb{A})} f_\beta(m_1 m k)\Phi(k,s) \\ &\quad \times (\gamma\chi|\cdot|_E^s)(x(m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k)\phi(\xi m_1 m)dm_1\, dm\, dk. \end{aligned} \tag{6.15}$$

Let $x \in \mathbb{Y}$ such that $\beta = \mu(x) = {}^t x^\rho T x$, $r = \operatorname{rank}(\beta)$. Then $r < n$. Let $a \in \mathrm{GL}_n(F)$ such that

$${}^t A^\rho \beta A = \begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix}, \tag{6.16}$$

where T' is a nondegenerate $r \times r$ Hermitian matrix. So without loss of generality, we assume that $\beta = \operatorname{diag}(0_{n-r}, T')$. Then

$$M_\beta = \left\{ m\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in M \mid D \in U(T'),\ {}^t C^\rho T' C = 0,\ {}^t C^\rho T' D = 0 \right\}. \tag{6.17}$$

Define two subgroups M_1, L of M_β:

$$\begin{aligned} M_1 &= \left\{ m\left(\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \right) \in M \mid D \in U(T'),\ {}^t C^\rho T' C = 0,\ {}^t C^\rho T' D = 0 \right\}, \\ \mathrm{L} &= \left\{ m\left(\begin{pmatrix} 1_{n-r} & B \\ 0 & 1_r \end{pmatrix} \right) \in M \mid B \in M_{n-r\times n-r}(E) \right\}. \end{aligned} \tag{6.18}$$

Then $M_\beta = M_1 \cdot \mathrm{L}$. We use this decomposition to compute the inner integral over $M_\beta(F)\backslash M_\beta(\mathbb{A})$ of (6.15),

$$\int_{M_\beta(F)\backslash M(\mathbb{A})} f_\beta(m_1 m k)(\gamma\chi|\cdot|_E^s)(x(m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k)\phi(\xi m_1 m)dm_1. \tag{6.19}$$

(Here because $\Phi(k,s)$ is independent of m_1 so we remove it from the integral over $M_\beta(F)\backslash M(\mathbb{A})$.) The above integral equals to

$$\begin{aligned}&\int_{M_1(F)\backslash M_1(\mathbb{A})}\int_{\mathrm{L}(F)\backslash \mathrm{L}(\mathbb{A})}\int_{S(F)\backslash S(\mathbb{A})} f(n(X)\ell m_1 mk)\psi(\mathrm{tr}(X\beta))\\ &\quad\times(\gamma\chi|\cdot|_E^s)(x(\ell m_1 m))\sum_{\xi\in\mu^{-1}(\beta)}\omega(k)\phi(\xi\ell m_1 m)\,dX\,d\ell\,dm_1.\end{aligned}\tag{6.20}$$

Let U be the subgroup of N consisting of elements of the following form:

$$n\left(\begin{pmatrix} c & d\\ {}^t d^\rho & 0\end{pmatrix}\right)\quad \text{with } c\in M_{(n-r)\times(n-r)}.\tag{6.21}$$

Then $\mathrm{L}U$ is the unipotent radical of the maximal parabolic group P' preserving the flag $0\subset\otimes_{i=1}^{n-r}Ee_{n+i}\subset Y$ (see Section 2 for the choice of basis of V). On the other hand, let Δ_+ be the set of positive roots of G with respect to the Borel subgroup of G consisting of element of following form:

$$\begin{pmatrix} A & B\\ & \hat{A}\end{pmatrix}\quad \text{with } A \text{ be upper triangular matrix.}\tag{6.22}$$

For $\alpha\in\Delta_+$, let N_α be the 1-parameter unipotent subgroup of G corresponding to α. Set $\Gamma=\{\alpha\in\Delta_+\mid N_\alpha\subset N\}$. Let α_0 be the simple root corresponding to P', $w=s_{\alpha_0}$ be the simple reflection of α_0. Then $U=\prod_{\beta\in\Gamma,w\beta\in\Gamma}N_\beta$. If we put $U_1=\prod_{\beta\in\Gamma,w\beta\in-\Gamma}N_\beta$, then $N=U\cdot U_1$. Hence we have decomposition

$$N(F)\backslash N(\mathbb{A})=U(F)\backslash U(\mathbb{A})\cdot U_1(F)\backslash U_1(\mathbb{A}).\tag{6.23}$$

Corresponding to the decomposition of N, we have a decomposition of $S(F)$:

$$\begin{aligned}S_U(F)&=\left\{\begin{pmatrix} c & d\\ {}^t d^\rho & 0\end{pmatrix}\in S(F)\mid c\in M_{(n-r)\times(n-r)}(F)\right\},\\ S_{U_1}(F)&=\left\{\begin{pmatrix} 0 & 0\\ 0 & d\end{pmatrix}\in S(F)\mid d\in M_{r\times r}(F)\right\}.\end{aligned}\tag{6.24}$$

Then the isomorphism $n: S(F)\to N$ send S_U and S_{U_1} onto U and U_1, respectively.

Substitute the decomposition of $S(F)$ into (6.20), then

$$\begin{aligned}(6.20)=&\int_{M_1(F)\backslash M_1(\mathbb{A})}\int_{\mathrm{L}(F)\backslash \mathrm{L}(\mathbb{A})}\int_{S_{U_1}(F)\backslash S_{U_1}(\mathbb{A})}\int_{S_U(F)\backslash S_U(\mathbb{A})}\\ &\times f(n(X_U+X_{U_1})\ell m_1 mk)\psi(\mathrm{tr}((X_U+X_{U_1})\beta))\\ &\times(\gamma\chi|\cdot|_E^s)(x(\ell m_1 m))\sum_{\xi\in\mu^{-1}(\beta)}\omega(k)\phi(\xi\ell m_1 m)\,dX_U\,dX_{U_1}\,d\ell\,dm_1\,dm.\end{aligned}\tag{6.25}$$

Direct computation shows that L centralizes U_1. We can change the order of the above integration, then

$$\begin{aligned}(6.20) = \int_{M_1(F)\backslash M_1(\mathbb{A})}\int_{S_{U_1}(F)\backslash S_{U_1}(\mathbb{A})}\int_{\mathrm{L}(F)\backslash \mathrm{L}(\mathbb{A})}\int_{S_U(F)\backslash S_U(\mathbb{A})} \\ \times f(n(X_U)\ell n(X_{U_1})m_1mk)\psi(\operatorname{tr}((X_U+X_{U_1})\beta)) \\ \times (\gamma\chi|\cdot|_E^s)(x(\ell m_1 m))\sum_{\xi\in\mu^{-1}(\beta)}\omega(k)\phi(\xi\ell m_1 m)\,dX_U\ell dX_{U_1}d\,dm_1\,dm.\end{aligned} \tag{6.26}$$

Let $X_U = \begin{pmatrix} c & d \\ {}^t d^\rho & 0\end{pmatrix}$ be an element of $S_U(\mathbb{A})$. Then

$$\beta X_U = \begin{pmatrix} 0 & 0 \\ 0 & T'\end{pmatrix}\begin{pmatrix} c & d \\ {}^t d^\rho & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T'\,{}^t d^\rho & 0\end{pmatrix}. \tag{6.27}$$

So

$$\operatorname{tr}(\beta(X_U+X_{U_1})) = \operatorname{tr}(\beta X_{U_1}) \tag{6.28}$$

which is independent of X_U. Since $x(\ell) = 1$ for $\ell \in \mathrm{L}(\mathbb{A})$, we see that

$$(\gamma\chi|\cdot|_E^s)(\ell) = 1, \quad \ell \in \mathrm{L}(\mathbb{A}). \tag{6.29}$$

If $\xi \in \mu^{-1}(\beta)$, then $\operatorname{rank}(\xi) = r$. Let $a_1,\dots,a_n$ be the column vectors of ξ. Recall that the right lower corner of ξ is an $r\times r$ nonsingular matrix T', the space generated by $a_{n-r+1},\dots,a_n$ is of rank r. Hence there is $a \in M_\beta$ (depends on ξ, but it does not affect our computation) such that

$$\xi' = \xi a^{-1} = \begin{pmatrix} 0 & v \\ 0 & u\end{pmatrix} \tag{6.30}$$

for some nonsingular $r\times r$ matrix u. If $\ell = m\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} \in \mathrm{L}$, then

$$\xi'\ell = \begin{pmatrix} 0 & v \\ 0 & u\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} = \xi'. \tag{6.31}$$

The integral for fixed $\xi \in \mu^{-1}(\beta)$ on $\mathrm{L}(F)\backslash\mathrm{L}(\mathbb{A})\times U(F)\backslash U(\mathbb{A})$ in (6.26) is

$$\begin{aligned}\int_{\mathrm{L}(F)\backslash \mathrm{L}(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} f(n(X_U)\ell n(X_{U_1})m_1mk)\psi(\operatorname{tr}((X_U+X_{U_1})\beta)) \\ \times(\gamma\chi|\cdot|_E^s)(\ell m_1 m)\omega(k)\phi(\xi\ell m_1 m)\,dX_U\,d\ell.\end{aligned} \tag{6.32}$$

By (6.28), (6.29), and (6.31),

$$\begin{aligned}(6.32) = \int_{\mathrm{L}(F)\backslash \mathrm{L}(\mathbb{A})}\int_{U(F)\backslash U(\mathbb{A})} f(n(X_U)\ell\, n(X_{U_1})m_1mk)\psi(\operatorname{tr}(X_{U_1}\beta)) \\ \times(\gamma\chi|\cdot|_E^s)(m_1 m)\omega(k)\phi(\xi' m_1 m)\,dX_U\,d\ell,\end{aligned} \tag{6.33}$$

which is 0, since LU is the unipotent radical of P'. This finishes the proof of the lemma. □

By Lemma 6.1, $I_\beta(s) = 0$ if β is singular. Recall that we choose T to be the representative of the open orbit of $\mathscr{C}/M$. The stabilizer M_T is isomorphic to $G' = U(T)$ the unitary group of W. Then (6.12) reduces to

$$\begin{aligned} I(s,\phi,\Phi,f) &= \int_{K_{\mathbb{A}}}\int_{M(F)\backslash M(\mathbb{A})}\sum_{a'\in G'(F)\backslash M(F)} f_T(m(a')m(a)k)\Phi(k,s) \\ &\quad\times (\gamma\chi|\cdot|_E^s)(\det a)\sum_{\xi\in G'(F)}\omega(k)\phi(\xi a' a)d^\times a\,dk \\ &= \int_{K_{\mathbb{A}}}\int_{M(\mathbb{A})} f_T(m(a)k)\Phi(k,s)\omega(k)\phi(\xi a)(\gamma\chi|\cdot|_E^s)d^\times a\,dk. \end{aligned} \tag{6.34}$$

6.2. Main theorem. Let $\gamma_v = \gamma|_{E_v}$, then $\gamma = \prod_v \gamma_v$. Similarly, $\chi = \prod_v \chi_v$. Let Φ_v be a standard section of $I(\gamma,s)$ of $G(F_v)$ for all $v \in \mathbf{v}$. Set $\Phi = \prod_v \Phi_v$. Assume that $\phi = \prod_v \phi_v$ in $\mathscr{S}(\mathbb{Y})$. Let f be a cusp form in the isotypic space of a cuspidal automorphic representation of $G(\mathbb{A})$. Let S be a finite subset of $\mathbf{v}$ containing all archimedean places such that if $v \notin S$, χ_v, γ_v are unramified, $T_v \in \mathrm{GL}_{n\times n}(\mathbb{O}_E) \cap S(F_v)$ and ψ_v is unramified character of F_v. Since $\pi = \otimes_v' \pi_v$ for almost all $v \in \mathbf{v}$, π_v is unramified for almost all places. Assume that π_v is unramified if $v \notin S$ and f is K_v fixed. Moreover, $\phi_v = \mathrm{char}(\mathbb{Y}(\mathbb{O}_v))$ if $v \notin S$.

Let Ω be a finite subset of $\mathbf{v}$ containing S. Put

$$G_\Omega = \prod_{v\in\Omega}, \qquad K_\Omega = \prod_{v\in\Omega} K_v, \qquad M_\Omega = \prod_{v\in\Omega} M_v. \tag{6.35}$$

They embed naturally into $G(\mathbb{A})$, $K_{\mathbb{A}}$, $M(\mathbb{A})$, respectively. If $a \in M(\mathbb{A})$, $a = \prod_v a_v$, put $a_\Omega = \prod_{v'\in\Omega} a_{v'}$. Similarly, if $k \in K_{\Omega\cup\{v\}}$, then $k = k_\Omega \cdot k_v$, for $k_\Omega \in K_\Omega$, $k_v \in K_v$. To compute (6.34), we define

$$I_\Omega(s) = \int_{K_\Omega}\int_{M_\Omega} f_T(m(a)k)\Phi(k,s)\omega(k)\phi(a)(\gamma\chi|\cdot|_E^s)(a)d^\times a\,dk. \tag{6.36}$$

Theorem 6.2. *Notations as above. Then*

$$I_{\Omega\cup\{v\}}(s) = \frac{L(s+1/2,\pi_v,\gamma_v\chi_v,\sigma)}{j_{T_v}(s)d_{H_v}(s)} I_\Omega(s), \tag{6.37}$$

where j_{T_v}, $d_{H_v}(s)$ are $j_T(s)$, $d_H(s)$ in Theorem 5.4 for T_v, H_v, respectively,

$$L\left(s+\frac{1}{2},\pi_v,\gamma_v\chi_v,\sigma\right) = L\left(s+\frac{1}{2}+\lambda_v,\pi_v,\sigma\right), \tag{6.38}$$

where $\lambda_v \in \mathbb{C}$ such that $(\gamma_v\chi_v)(a) = |a|_E^{\lambda_v}$ for all $a \in E_v^\times$ (Case NS), or $(\gamma_v\chi_v)(a) = |a|^{\lambda_v}$ for all $a \in F_v^\times$ (Case S) (See Section 3 for the definition of Case NS and Case S).

Proof. We will apply results in Section 5, F_v will be F there,

$$\begin{aligned} I_{\Omega\cup\{v\}}(s) &= \int_{K_{\Omega\cup\{v\}}}\int_{M_{\Omega\cup\{v\}}} f_T(m(a)k)\Phi(k,s)\omega(k)\phi(a)(\gamma\chi|\cdot|_E^s)(\det a)d^\times a\,dk \\ &= \int_{K_\Omega M_\Omega}\int_{K_v M(F_v)} \Phi(K_\Omega,s)\Phi_v(k_v,s)f_T'(m(a_v)m(a_\Omega)k_v k_\Omega) \\ &\quad\times(\gamma\chi|\cdot|_E^s)(\det a_\Omega a_v)\omega(k_\Omega)\phi_\Omega(a_\Omega)\omega(k_v)\phi_v(a_v)d^\times a_v\, d^\times dk_v a_\Omega\, dk_\Omega. \end{aligned} \tag{6.39}$$

Φ_v is the standard section, then $\Phi_v(k_v,s)=1$ for all $k_v\in K_v$. Moreover, f is K_v-fixed, hence $f_T(m(a_v a_\Omega)k_v k_\Omega)=f_T(m(a_v a_\Omega)k_\Omega)$ for all $k_v\in K_v$. $\phi_v=\operatorname{char}(\mathbb{Y}(\mathbb{O}_v))$ which is K_v fixed element for the Weil representation, hence $\omega(k_v)\phi_v=\phi_v$,

$$\begin{aligned} (6.39) &= \int_{K_\Omega M_\Omega}\int_{K_v M(F_v)} \Phi_\Omega(k_\Omega,s)f_T(m(a_v a_\Omega k_\Omega)) \\ &\quad\times(\gamma\chi|\cdot|_E^s)(\det a_v a_\Omega)\omega(k_\Omega)\phi(a_\Omega)\phi(a_v)d^\times a_v\, dk_v\, da_\Omega\, dk_\Omega \\ &= \int_{K_\Omega M_\Omega}\Phi_\Omega(k_\Omega,s)\omega(k_\Omega)\phi(a_\Omega)(\gamma\chi|\cdot|_E^s)(\det a_\Omega)\int_{M(F_v)} \\ &\quad\times f_T(m(a_v)m(a_\Omega)k_\Omega)\phi(a_v)\gamma_0(a_v)^s(\gamma\chi)(\det a_v)d^\times a_v\, d^\times a_\Omega\, dk_\Omega. \end{aligned} \tag{6.40}$$

As $\phi_v=\operatorname{char}(\mathbb{Y}(\mathbb{O}_v))$, $M_v\cap\mathbb{Y}(\mathbb{O})=M(\mathbb{O}_v)$ (cf. Section 5),

$$\begin{aligned} &\int_{M(F_v)} f_T(m(a_v)m(a_\Omega)k_\Omega)\phi(a_v)\gamma_0^s(a_v)(\gamma\chi)(\det a_v)d^\times a_v \\ &\quad= \int_{M(\mathbb{O}_v)} f_T(m(a_v)m(a_\Omega)k_\Omega)\gamma_0^s(a_v)(\gamma\chi)(\det a_v)d^\times a_v \\ &\quad= \frac{L(s+1/2,\pi_v,\gamma_v\chi_v,\sigma)}{j_{T_v}(s)d_{H_v}(s)}f_T(m(a_\Omega)k_\Omega), \quad \text{by Theorem 5.4.} \end{aligned} \tag{6.41}$$

Here we are viewing $f_T(m(a_v)m(a_\Omega)k_\Omega)$ as a functional l_{T_v} on π_v by Example 5.1 in Section 5. Hence

$$I_{\Omega\cup\{v\}} = \frac{L(s+1/2,\pi_v,\gamma_v\chi_v,\sigma)}{j_{T_v}(s)d_{H_v}(s)}I_\Omega(s). \tag{6.42}$$

□

To complete the computation of our global integral, let

$$j_T^S(s)=\prod_{v\notin S}j_{T_v}(s), \qquad d_H^S(s)=\prod_{v\notin S}d_{H_v}(s). \tag{6.43}$$

Define partial L function of π as

$$L^S\left(s+\frac{1}{2},\pi,\gamma\chi,\sigma\right)=\prod_{v\notin S}L\left(s+\frac{1}{2},\pi_v,(\gamma_v\chi_v),\sigma\right). \tag{6.44}$$

Since $I(s)=\lim_\Omega I_\Omega(s)$, by Theorem 6.2, let Ω be a finite set of $\mathbf{v}$ approaching to $\mathbf{v}$ by adding one place each time, then the following holds.

Theorem 6.3. *Choose f,ϕ,Φ and $S\subset \mathbf{v}$ as in Section 6.1. Then for all $s\in\mathbb{C}$,*

$$I(s,\phi,\Phi,f)=\frac{R(s)}{j_T^S(s)d_H^S(s)}L^S\left(s+\frac{1}{2},\pi,\gamma\chi,\sigma\right), \tag{6.45}$$

where $R(s)=I_S(s)$ is a meromorphic function of s.

Proof. Argue as [6, Theorem 6.1], the partial L function is a meromorphic function. Also by the analytic property of Eisenstein series, $I(s,\phi,\Phi,f)$ itself is a meromorphic function, hence $R(s)=I_S(s)$ is a meromorphic function of s. □

Remark 6.4. We remark here that following [4, pages 118-119], under our assumption one can show that by choosing appropriate ϕ, Φ, f, we can let that $R(s)\neq 0$.

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