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Research Article

Bifurcation Analysis for a Two-Dimensional Discrete-Time Hopfield Neural Network with Delays

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A bifurcation analysis is undertaken for a discrete-time Hopfield neural network with four delays. Conditions ensuring the asymptotic stability of the null solution are obtained with respect to two parameters of the system. Using techniques developed by Kuznetsov to a discrete-time system, we study the Neimark-Sacker bifurcation (also called Hopf bifurcation for maps) of the system. The direction and the stability of the Neimark-Sacker bifurcation are investigated by applying the normal form theory and the center manifold theorem.

1. Introduction

The investigation of dynamic behaviors for neural networks has been the subject of much recent activity since one of the models with electronic circuit implementation was proposed by Hopfield [1]. Since continuous-time Hopfield neural networks have been first considered in [2, 3], they have received much attention because of their applicability in problems of optimizations, signal processing, image processing, solving nonlinear algebraic equations, pattern recognitions, associative memories, and so on. The stability and the existence of periodic or quasiperiodic solutions of discrete-time Hopfield neural networks with or without delays have been considered in [4–8]. In [9], a bifurcation analysis has been studied for a two-dimensional discrete neural model with multidelays by applying the Euler method to continuous-time Hopfield neural networks with no self-connections.

In practice, due to the finite speeds of the switching and the transmission of signals in a network, time delays unavoidably exist in a working network, therefore, they should be incorporated into the model equations of the network. Clearly, introducing time delays into the model is more reasonable. In general, delay-differential equations exhibit much more complicated dynamics than the responding ordinary differential equations since a time delay could cause the change of stability of an equilibrium, and hence the bifurcation occurs. It is interesting to investigate the time delay how to affect the dynamics of a system, and it is important to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions when a Hopf bifurcation occurs.

In this paper, we consider the discrete-time Hopfield neural network of two neurons with four different delays:

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f_1(x_1(n-\tau_1)) + a_{12} f_2(x_2(n-\tau_2)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_3(x_1(n-\tau_3)) + a_{22} f_4(x_2(n-\tau_4)), \end{aligned} \tag{1.1}$$

where x_1 and x_2 denote the activities of neurons, $\beta \in (0,1)$ is the internal decay of the neurons, $\tau_i \geqslant 0$ $(i = 1,2,3,4)$ are delays, constants a_{11}, a_{12}, a_{21}, and a_{22} denote the connection weights, f_i $(i = 1,2,3,4) : \mathbb{R} \to \mathbb{R}$ are continuous transfer functions, and $f_i(0) = 0$.

Our purpose in this paper is using the techniques developed by Kuznetsov [10] to study the stability and the Neimark-Sacker bifurcation (also called Hopf bifurcation for maps) of the equilibrium $(0,0)$ of system (1.1). In Section 2, the conditions for the asymptotical stability of the equilibrium $(0,0)$ of (1.1) are established. Moreover, when the bifurcation parameter exceeds a critical value, the Neimark-Sacker bifurcation will occur. In the last section, we discuss the direction and stability of the Neimark-Sacker bifurcation by using the normal form theory and the center manifold theorem.

2. Stability and existence of Neimark-Scaker bifurcation

In this section, we first discuss the local stability of the equilibrium $(0,0)$ of system (1.1). The linearization of system (1.1) around $(0,0)$ is

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f_1{}'(0)(x_1(n-\tau_1)) + a_{12} f_2{}'(0)(x_2(n-\tau_2)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_3{}'(0)(x_1(n-\tau_3)) + a_{22} f_4{}'(0)(x_2(n-\tau_4)). \end{aligned} \tag{2.1}$$

Here we need the following hypothesis.

(H1) For $i = 1,2,3,4$, $f_i \in C^1(\mathbb{R})$ and $f_i(0) = 0$.

The Jacobian matrix of system (2.1) at the equilibrium $(0,0)$ leads us to the following characteristic equation:

$$\begin{vmatrix} \beta + a_{11} f_1{}'(0)e^{-\lambda\tau_1} - \lambda & a_{12} f_2{}'(0)e^{-\lambda\tau_2} \\ a_{21} f_3{}'(0)e^{-\lambda\tau_3} & \beta + a_{22} f_4{}'(0)e^{-\lambda\tau_4} - \lambda \end{vmatrix} = 0, \tag{2.2}$$

that is

$$\lambda^2 - 2(\beta + T)\lambda + \beta^2 + 2\beta T + D = 0, \tag{2.3}$$

where

$$D = a_{11}a_{22}f_1'(0)f_4'(0)e^{-\lambda(\tau_1+\tau_4)} - a_{12}a_{21}f_2'(0)f_3'(0)e^{-\lambda(\tau_2+\tau_3)},$$
$$T = \frac{1}{2}\left(a_{11}f_1'(0)e^{-\lambda\tau_1} + a_{22}f_4'(0)e^{-\lambda\tau_4}\right). \tag{2.4}$$

For $T \in (-1-\beta, 1-\beta)$, we let

$$\Omega_0 = \{(T,D) \in \mathbb{R}^2 : E_1 < 0,\ E_2 < 0,\ E_3 > 0\}, \tag{2.5}$$

where

$$\begin{aligned} E_1 &= 2(1-\beta)T - (1-\beta)^2 - D, \\ E_2 &= -2(1+\beta)T - (1+\beta)^2 - D, \\ E_3 &= -2\beta T + 1 - \beta^2 - D. \end{aligned} \tag{2.6}$$

Theorem 2.1. *Suppose that hypothesis* (*H*1) *is satisfied and* $(T,D) \in \Omega_0$. *Then the zero solution of (1.1) is asymptotically stable.*

Proof. The characteristic equation for the linearization of (1.1) at (0,0) is (2.3). We consider the following two cases.

Case 1 ($T^2 \geqslant D$). In this case, the root of characteristic equation (2.3) is given by

$$\lambda_1 = \beta + T + \sqrt{T^2 - D}, \tag{2.7}$$

$$\lambda_2 = \beta + T - \sqrt{T^2 - D}. \tag{2.8}$$

Obviously, the eigenvalues $\lambda_{1,2}$ in (2.7) are inside the unit circle if and only if

$$(T,D) \in \Omega_1 \cap \Omega_2, \tag{2.9}$$

where

$$\begin{aligned} \Omega_1 &:= \{(T,D) \in \mathbb{R}^2 : D > 2(1-\beta)T - (1-\beta)^2,\ T < 1-\beta,\ T^2 \geqslant D\}, \\ \Omega_2 &:= \{(T,D) \in \mathbb{R}^2 : D > -2(1+\beta)T - (1+\beta)^2,\ T > -1-\beta,\ T^2 \geqslant D\}. \end{aligned} \tag{2.10}$$

Case 2 ($T^2 < D$). In this case, the characteristic equation (2.3) has a pair of conjugate complex roots

$$\begin{aligned} \lambda_1 &= \beta + T + \sqrt{D - T^2}\,i, \\ \lambda_2 &= \beta + T - \sqrt{D - T^2}\,i. \end{aligned} \tag{2.11}$$

It is easy to verify that $|\lambda_{1,2}| < 1$ if and only if

$$(T,D) \in \Omega_3, \tag{2.12}$$

where

$$\Omega_3 := \{(T,D) \in \mathbb{R}^2 : D < -2\beta T + 1 - \beta^2,\ T^2 < D\}. \tag{2.13}$$

Combining Cases 1 and 2, we know that $\Omega_0 = (\Omega_1 \cap \Omega_2) \cup \Omega_3$. Thus, the eigenvalues $\lambda_{1,2}$ of characteristic equation (2.3) are inside the unit circle for $(T,D) \in \Omega_0$. This implies that the zero solution of (1.1) is asymptotically stable. This completes the proof of Theorem 2.1. □

Now, we choose D as the bifurcation parameter to study the Neimark-Scaker bifurcation of $(0,0)$. For $T^2 < D$, let

$$\lambda(D) = \beta + T + \sqrt{D - T^2}\,i, \tag{2.14}$$

then the eigenvalues in (2.3) are conjugate complex pair $\lambda(D)$ and $\overline{\lambda(D)}$. The modulus of the eigenvalue is

$$|\lambda| = \sqrt{(\beta+T)^2 + (D - T^2)} = \sqrt{\beta^2 + 2\beta T + D}. \tag{2.15}$$

Then, $|\lambda| = 1$ if and only if

$$D = D^* := -2\beta T + 1 - \beta^2. \tag{2.16}$$

Obviously, for $T^2 < D < D^*$, we have

$$|\lambda| < 1. \tag{2.17}$$

Since the modulus of eigenvalue $|\lambda(D^*)| = 1$, we know that D^* is a critical value which destroys the stability of $(0,0)$. The following lemma is useful for the study of the bifurcation of $(0,0)$.

Lemma 2.2. *Suppose that (H1) is satisfied and $-\beta < T < 1 - \beta$, then*
(i) $((d/dD)|\lambda(D)|)|_{D=D^*} > 0$,
(ii) $\lambda^k(D^*) \neq 1$ *for* $k = 1,2,3,4$,
where $\lambda(D)$ and D^* are given by (2.14) and (2.16), respectively.

Proof. From the assumption $T \in (-\beta, 1-\beta)$, it is easy to see that $T^2 < D^*$. By a direct calculation, we obtain from (2.15) and (2.16) that

$$\left(\frac{d}{dD}|\lambda(D)|\right)\Bigg|_{D=D^*} = \frac{1}{2}\frac{1}{\sqrt{\beta^2 + 2\beta T + D}}\Bigg|_{D=D^*} = \frac{1}{2} > 0, \tag{2.18}$$

so (i) is true.

In the following, we will deal with (ii). Clearly, $\lambda^k(D^*) = 1$ for some $k \in \{1,2,3,4\}$ if and only if the argument $\arg\lambda(D^*) \in \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. From $T^2 < D^*$, (2.16), and the expression

$$\lambda(D^*) = \beta + T + \sqrt{D^* - T^2}\,i, \tag{2.19}$$

we see that

$$|\lambda(D^*)| = 1, \qquad \operatorname{Re}\lambda(D^*) > 0, \qquad \operatorname{Im}\lambda(D^*) > 0, \tag{2.20}$$

it follows that the argument $\arg\lambda(D^*) \in \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$ is wrong. This means that $\lambda^k(D^*) \neq 1$ for $k = 1,2,3,4$. The proof of Lemma 2.2 is complete. □

THEOREM 2.3. *Suppose that (H1) is satisfied and* $T \in (-\beta, 1-\beta)$. *Then*

(i) *if* $D > D^*$, *then the equilibrium* $(0,0)$ *of (1.1) is unstable,*

(ii) *if* $T^2 < D < D^*$, *then the equilibrium* $(0,0)$ *of (1.1) is asymptotically stable,*

(iii) *the Neimark-Sacker bifurcation occurs at* $D = D^*$, *that is, system (1.1) has a unique closed invariant curve bifurcation from the equilibrium* $(0,0)$ *near* $D = D^*$,

where D^* *is given by (2.16).*

By Lemma 2.2 and the results in [11], we have the theorem, so we omit the proof.

3. Direction and stability of the Neimark-Scaker bifurcation

In this section, we will give an algorithm to study the direction and the stability of the Neimark-Scaker bifurcation by using the normal form method and the center manifold theory for discrete-time system developed by Kuznetsov [10]. We may assume the following.

(H2) For $i = 1,2,3,4$, $f_i \in C^{(3)}(\mathbb{R},\mathbb{R})$, $f_i(0) = f_i''(0) = 0$, and $f_i'(0)f_i'''(0) \neq 0$.
Now (1.1) can be rewritten as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \beta + a_{11}f_1'(0)e^{-\lambda\tau_1} & a_{12}f_2'(0)e^{-\lambda\tau_2} \\ a_{21}f_3'(0)e^{-\lambda\tau_3} & \beta + a_{22}f_4'(0)e^{-\lambda\tau_4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} F_1(x_1, D) \\ F_2(x_2, D) \end{pmatrix}, \tag{3.1}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$. We denote

$$M(D) := \begin{pmatrix} \beta + a_{11}f_1'(0)e^{-\lambda\tau_1} & a_{12}f_2'(0)e^{-\lambda\tau_2} \\ a_{21}f_3'(0)e^{-\lambda\tau_3} & \beta + a_{22}f_4'(0)e^{-\lambda\tau_4} \end{pmatrix}, \tag{3.2}$$

$$\begin{aligned} r_1 &:= T + \sqrt{D - T^2}\,i - a_{11}f_1'(0)e^{-\lambda\tau_1}, \\ r_2 &:= T + \sqrt{D - T^2}\,i - a_{22}f_4'(0)e^{-\lambda\tau_4}. \end{aligned} \tag{3.3}$$

Then, from the definition of T, we can obtain

$$\overline{r}_1 = -r_2. \tag{3.4}$$

Let $q(D) \in \mathbb{C}^2$ be an eigenvector of $M(D)$ corresponding to eigenvalue $\lambda(D)$ given by (2.14), then

$$M(D)q(D) = \lambda(D)q(D). \tag{3.5}$$

Again let $p(D) \in \mathbb{C}^2$ be an eigenvector of the transposed matrix $M^T(D)$ corresponding to its eigenvalue, then

$$M^T(D)p(D) = \overline{\lambda(D)}p(D). \tag{3.6}$$

By a direct calculation, we obtain

$$
\begin{aligned}
q &\sim \left(1, \frac{a_{21} f_3'(0) e^{-\lambda\tau_3}}{r_2}\right)^T, \\
p &\sim \left(1, \frac{a_{12} f_2'(0) e^{-\lambda\tau_2}}{\overline{r}_2}\right)^T,
\end{aligned}
\tag{3.7}
$$

where r_j $(j = 1,2)$ is given by (3.3). For the eigenvector $q = (1, a_{21} f_3'(0)e^{-\lambda\tau_3}/r_2)^T$, to normalize p, let

$$
p = \frac{\overline{r}_2}{\overline{r}_2 - r_2}\left(1, \frac{a_{12} f_2'(0) e^{-\lambda\tau_2}}{\overline{r}_2}\right)^T,
\tag{3.8}
$$

then we have $\langle p,q\rangle = 1$, where $\langle\cdot,\cdot\rangle$ means the standard scalar product in $\mathbb{C}^2 : \langle p,q\rangle = \overline{p_1}q_1 + \overline{p_2}q_2$. Any vector $x \in \mathbb{R}^2$ can be represented for D near D^* as

$$
x = yq(D) + \overline{yq(D)},
\tag{3.9}
$$

for some complex y. Obviously,

$$
y = \langle p(D), x\rangle.
\tag{3.10}
$$

Thus, system (3.1) can be transformed for D near D^* into the following form:

$$
y \longrightarrow \lambda(D)y + g(y,\overline{y},D),
\tag{3.11}
$$

where $\lambda(D)$ can be written as $\lambda(D) = (1+\varphi(D))e^{i\theta(D)}$, ($\varphi(D)$ is a smooth function with $\varphi(D^*) = 0$), and

$$
g(y,\overline{y},D) = \sum_{k+l\geqslant 2} \frac{1}{k!\,l!} g_{kl}(D) y^k \overline{y}^l.
\tag{3.12}
$$

Form assumption (H2), we know that F_i $(i = 1,2)$ in (3.1) can be expanded as

$$
\begin{aligned}
F_1(\xi,D) &= \frac{a_{11}}{6} f_1'''(0)\xi_1^3 + \frac{a_{12}}{6} f_2'''(0)\xi_2^3 + O(\|\xi\|^4), \\
F_2(\xi,D) &= \frac{a_{21}}{6} f_3'''(0)\xi_3^3 + \frac{a_{22}}{6} f_4'''(0)\xi_4^3 + O(\|\xi\|^4).
\end{aligned}
\tag{3.13}
$$

It follows that

$$
\begin{aligned}
B_i(u,v) &:= \sum_{j,k=1}^{2} \left.\frac{\partial^2 F_i(\xi,D^*)}{\partial\xi_j\partial\xi_k}\right|_{\xi=0} u_j v_k = 0, \quad i = 1,2, \\
C_1(u,v,w) &:= \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 F_1(\xi,D^*)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0} u_j v_k w_l = a_{11} f_1'''(0)u_1v_1w_1 + a_{12} f_2'''(0)u_2v_2w_2, \\
C_2(u,v,w) &:= \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 F_2(\xi,D^*)}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0} u_j v_k w_l = a_{21} f_3'''(0)u_1v_1w_1 + a_{22} f_4'''(0)u_2v_2w_2.
\end{aligned}
\tag{3.14}
$$

By (3.12)–(3.14) and the formulae

$$\begin{aligned} g_{20}(D^*) &= \langle p, B(q,q)\rangle, & g_{11}(D^*) &= \langle p, B(q,\overline{q})\rangle, \\ g_{01}(D^*) &= \langle p, B(\overline{q},\overline{q})\rangle, & g_{21}(D^*) &= \langle p, C(q,q,\overline{q})\rangle, \end{aligned} \tag{3.15}$$

we obtain

$$\begin{gathered} g_{20}(D^*) = g_{11}(D^*) = g_{02}(D^*) = 0, \\ g_{21}(D^*) = \overline{p_1}C_1(q,q,\overline{q}) + \overline{p_2}C_2(q,q,\overline{q}), \end{gathered} \tag{3.16}$$

which, together with $e^{-i\theta(D^*)} = \overline{\lambda(D^*)}$ and the expression of D, implies that

$$\begin{aligned} &\operatorname{Re}\left(\frac{e^{-i\theta(D^*)}g_{21}}{2}\right) - \operatorname{Re}\left(\frac{(1-2e^{-i\theta(D^*)})e^{-2i\theta(D^*)}}{2(1-e^{-i\theta(D^*)})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2 \\ &\quad = \operatorname{Re}\left(\frac{e^{-i\theta(D^*)}}{2}g_{21}\right). \end{aligned} \tag{3.17}$$

From the above argument and the results [10, 12], we have the following result.

THEOREM 3.1. *Suppose that* (*H*2) *is satisfied and* $T \in (-\beta, 1-\beta)$. *Then the direction and the stability of the Neimark-Sacker bifurcation of (1.1) can be determined by the sign of* $\operatorname{Re}((e^{-i\theta(D^*)}/2)g_{21})$. *Indeed, if* $\operatorname{Re}((e^{-i\theta(D^*)}/2)g_{21}) < 0 (> 0)$, *then the Neimark-Sacker bifurcation of (1.1) at* $D = D^*$ *is supercritical (subcritical) and a unique closed invariant curve bifurcating from* $(0,0)$ *is asymptotically stable (unstable), where* D^* *is given by (2.16).*

The proof is similar to our above argument and we will omit it.

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