Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 91535, 8 pages
doi:10.1155/2007/91535

Research Article

Behavior of the Trinomial Arcs $B(n,k,r)$ When $0<\alpha<1$

Kaoutar Lamrini Uahabi and Mohammed Zaoui

Received 12 April 2007; Revised 14 July 2007; Accepted 12 August 2007

Recommended by Teodor Bulboaca

We deal with the family $B(n,k,r)$ of trinomial arcs defined as the set of roots of the trinomial equation $z^n=\alpha z^k+(1-\alpha)$, where $z=\rho e^{i\theta}$ is a complex number, n and k are two integers such that $0<k<n$, and α is a real number between 0 and 1. These arcs $B(n,k,r)$ are continuous arcs inside the unit disk, expressed in polar coordinates (ρ,θ). The question is to prove that $\rho(\theta)$ is a decreasing function, for each trinomial arc $B(n,k,r)$.

1. Introduction

Consider the trinomial equation

$$z^n=\alpha z^k+(1-\alpha), \tag{1.1}$$

where z is a complex number, n and k are two integers such that $k=1,2,\dots,n-1$, and α is a real number. The first discussion of the behavior of the roots of trinomial equations was in Fell [1]. She presented a description of the trajectories of these roots, called *trinomial arcs*. These arcs can be expressed in polar coordinates (ρ,θ) by a function $\rho(\theta)$ and are continuous in functions of α as α varies between 0 and 1, or between 1 and $+\infty$, or between $-\infty$ and 0. Fell [1] also studied the monotonicity of the function $\alpha(\theta)$ and gave a bound for the modulus of roots. However, she did not establish the monotonicity of ρ as a function of θ. Though the descriptive results of Fell [1] give us the information about the form and location of the trinomial arcs, nevertheless, these types of arcs are not suitably designed for further study. In this paper, we will restrict our attention to a family of trinomial arcs, solutions of (1.1) with $0<\alpha<1$, inside the unit disk $D_u=\{z:|z|\leq 1\}$. We begin by defining this family of trinomial curves denoted by $B(n,k,r)$, where n, k, and

r satisfy some conditions. Note that Dubuc and Zaoui [2] studied trinomial arcs denoted by B_m and which are part of this family of arcs $B(n,k,r)$. Next, we prove in this work that $\rho(\theta)$ is a differentiable function for these arcs. With a view to solving the problem of monotonicity of $\rho(\theta)$ for the trinomial arcs $B(n,k,r)$, two important intermediate results are shown. At last, this study allows us to prove that $\rho(\theta)$ is a decreasing function.

2. Study of the trinomial equation

In (1.1), fix n and k. For $z=\rho e^{i\theta}$ in (1.1), one has $\rho^n e^{in\theta}=\alpha\rho^k e^{ik\theta}+(1-\alpha)$. Separating real and imaginary parts, one gets $\rho^n \sin n\theta=\alpha\rho^k\sin k\theta$ and $\rho^n\cos n\theta=\alpha\rho^k\cos k\theta+(1-\alpha)$. So, when $\theta\neq l\pi/n$, where l is an integer, we get

$$\rho^{n-k}=\alpha\frac{\sin k\theta}{\sin n\theta}. \tag{2.1}$$

On the other hand, divide (1.1) by z^n and consider the imaginary part. When $\alpha\neq 0$ and $\theta\neq l\pi/(n-k)$, where l is an integer, we obtain that

$$\rho^k=\frac{(\alpha-1)}{\alpha}\frac{\sin n\theta}{\sin(n-k)\theta}. \tag{2.2}$$

Therefore, we have the next equation of the trajectories of roots of (1.1):

$$\rho^{n-k}\sin n\theta-\rho^n\sin(n-k)\theta=\sin k\theta. \tag{2.3}$$

In fact, Fell has studied in [1] the trinomial equation

$$\lambda z^n+(1-\lambda)z^k-1=0, \tag{2.4}$$

where z is a complex number, n and k are two integers such that $k=1,2,\dots,n-1$, and λ is a real number. Substituting into (2.4) the expression given for z^n by (1.1), we get $(z^k-1)[1-\lambda(1-\alpha)]=0$. So, $z^k=1$ or $\lambda(1-\alpha)=1$. When z is not a kth root of unity, it follows that $\alpha=1-1/\lambda$. Hence, in order to pass from (1.1) to (2.4), we can set $\alpha=1-1/\lambda$. It stems easily from this equality that the case $0\le\alpha\le 1$ of (1.1) corresponds to the case $1\le\lambda<+\infty$ of (2.4).

In this work, we are interested in the case $0\le\alpha\le 1$, so we have

$$\operatorname{sign}(\sin n\theta)=\operatorname{sign}(\sin k\theta)=-\operatorname{sign}(\sin(n-k)\theta). \tag{2.5}$$

Definition 2.1. An angle θ which fulfills (2.5) will be called an (n,k)-feasible angle for the trinomial equation (1.1) with $0\le\alpha\le 1$.

Moreover, in view of the next lemma of [3], the trajectories of roots of (1.1) with $0\le\alpha\le 1$ are inside the unit disk.

Lemma 2.2. *For any (n,k)-feasible angle θ for (1.1) with $0\le\alpha\le 1$, the function of ρ, $-\rho^n\{\sin(n-k)\theta/\sin k\theta\}+\rho^{n-k}\{\sin n\theta/\sin k\theta\}-1$, is increasing and vanishes for one and only one positive value of ρ, which is not larger than* 1.

Remark 2.3. The upper and lower half-planes are symmetrical. So, we will restrict our study of trinomial arcs to the upper half-plane.

3. Description and definition of trinomial arcs $B(n,k,r)$

Notice that for $\alpha = 0$, (1.1) has n roots: the nth roots of unity. Fell [1], in her Descriptive Claim II, pages 314-315, tells us that the trajectories of the n roots can be described as trajectories of particles starting at these n roots. As α changes from 0 to 1, they move continuously until $\alpha = 1$, $(n-k)$ of them have moved into $(n-k)$th roots of unity, and k of them have collapsed to 0. There are k trajectories going to 0, the k tangents being lines going through 0 and one kth root of -1. Consider $C = \{n\text{th roots of unity}\}$, $D = \{(n-k)\text{th roots of unity}\}$, and $E = \{k\text{th roots of } -1\}$. Let γ be in C and let δ be the unique nearest neighbor of γ in $D \cap E$. Fell [1] asserts that, in the case $\delta \in D \cap E$ with $0 \le \alpha \le 1$, there exists γ' in C such that δ is equidistant from γ and from γ'. There exists also α_0 in $[0,1]$ such that the trajectories of two particles starting at γ and γ' when $\alpha = 0$ are continuous arcs until the point of their meeting on the line segment $\theta = \arg(\delta)$ when $\alpha = \alpha_0$. When α moves from α_0 to 1, the two roots remain on the segment $\theta = \arg(\delta)$, one of them goes to 0 and the other tends to δ. Fell shows in [1] that all the trinomial arcs solutions of (1.1) in the case $0 \le \alpha \le 1$ with $\delta \in D \cap E$ are such that the feasible angles θ belong to intervals of length less than or equal to π/n and bounded on the one side by $\arg(\delta)$ where δ is both a kth root of -1 and an $(n-k)$th root of unity, and on the other side by $\arg(\gamma)$ where γ is an nth root of unity. There are so two types of arcs in this case; the first type is such that θ belongs to $[\arg(\gamma), \arg(\delta)]$ where $\gamma \in C$ and the second type is such that θ belongs to $[\arg(\delta), \arg(\gamma')]$ where $\gamma' \in C$, such that δ is equidistant from γ and from γ'.

In [2], Dubuc and Zaoui studied a class of trinomial arcs denoted by B_m and defined as the set of roots of (1.1) with $0 \le \alpha \le 1$, $n = m$, $k = m-2$, where m is an odd integer larger than 2 and the feasible angles belong to the interval $[\pi - \pi/m, \pi]$. They showed in [2] that $\rho(\theta)$ is a decreasing function on $[\pi - \pi/m, \pi]$ for the arcs B_m. Because m is an odd integer, we can say that γ such that $\arg(\gamma) = \pi - \pi/m$ is an nth root of unity and δ such that $\arg(\delta) = \pi$ is both a kth root of -1 and an $(n-k)$th root of unity. Dubuc and Zaoui have thus solved the problem of monotonicity of $\rho(\theta)$, pointed out in [1**?**], for some particular trinomial arcs, namely B_m, solutions of (1.1) in the case $0 \le \alpha \le 1$ with $\delta \in D \cap E$ and $\theta \in [\arg(\gamma), \arg(\delta)]$. In this paper, our objective is to study the monotonicity of $\rho(\theta)$ for all trinomial arcs corresponding to this case. In fact, these arcs, denoted by $B(n,k,r)$, will be defined as follows.

Let d be the greatest common divider of k and n, we assume that k/d and n/d are odd numbers, then we define d trinomial arcs, $B(n,k,0), B(n,k,1), \dots, B(n,k,d-1)$. Let $r \in \{0,1,\dots,d-1\}$, then for any angle $\theta \in (\theta_0, \theta_1)$ with $\theta_1 = (2r+1)\pi/d$ and $\theta_1 - \theta_0 = \pi/n$, there exists a unique solution $\rho = \rho(\theta) \in (0,1)$ to the equation $\rho^n \sin(n-k)\theta - \rho^{n-k}\sin n\theta + \sin k\theta = 0$ given by (2.3). Any solution of (2.3) induces the solution $z = \rho e^{i\theta}$ of (1.1) with $\alpha = \rho^{n-k}\sin n\theta / \sin k\theta$.

Figure 3.1 provides such an arc.

Remark 3.1. When $\alpha = 0$, (1.1) becomes $z^n = 1$. So, its solutions are the nth roots of unity. In the case $\alpha = 1$, (1.1) becomes $z^k[z^{n-k} - 1] = 0$. Then, the n roots of (1.1) are the $(n-k)$th roots of unity, which are simple roots and 0, a root of multiplicity k.

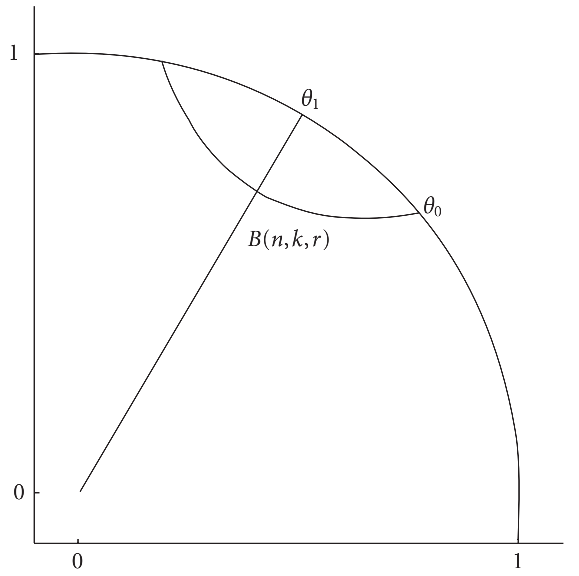

FIGURE 3.1. Trinomial arc defined for feasible angles $\theta \in (\theta_0, \theta_1)$.

When $n = 2$, the trajectories of roots of (1.1) with $0 < \alpha < 1$ are linear, then we define the continuous arcs $B(n,k,r)$ as follows.

Definition 3.2. Let n and k be two integers such that n is greater than or equal to 3 and $0 < k < n$. Let $d = \gcd(k,n)$ and $r \in \{0,1,\dots,d-1\}$, one assumes that k/d and n/d are odd numbers, then the continuous arc

$$B(n,k,r) = \{z = \rho(\theta)e^{i\theta} : 0 < (2r+1)\pi/d - \theta < \pi/n\} \tag{3.1}$$

is the set of roots of (1.1) with $0 < \alpha < 1$.

This family of arcs $B(n,k,r)$ exists in view of the following lemma.

LEMMA 3.3. *Let n and k be two integers such that n is greater than or equal to* 3 *and* $0 < k < n$. *Let* $d = \gcd(k,n)$ *and* $r \in \{0,1,\dots,d-1\}$, *one assumes that k/d and n/d are odd numbers, then in the trinomial equation (1.1) with* $0 < \alpha < 1$, *any angle of the interval* (θ_0, θ_1), *with* $\theta_1 = (2r+1)\pi/d$ *and* $\theta_0 = \theta_1 - \pi/n$ *is feasible. In particular, for each trinomial arc* $B(n,k,r)$, *one has* $\sin n\theta > 0$, $\sin k\theta > 0$, *and* $\sin(n-k)\theta < 0$ *for any θ in the interval* (θ_0, θ_1).

Proof. Let θ be an angle such that $(2r+1)\pi/d - \pi/n < \theta < (2r+1)\pi/d$. So, we have $(2r+1)n\pi/d - \pi < n\theta < (2r+1)n\pi/d$. Because the number n/d is odd, it follows that $\sin n\theta > 0$. On the other side, we have $(2r+1)k\pi/d - k\pi/n < k\theta < (2r+1)k\pi/d$. Since k/d is an odd number and $k < n$, it yields that $\sin k\theta > 0$. Finally, we have $(2r+1)(n-k)\pi/d - (n-k)\pi/n < (n-k)\theta < (2r+1)(n-k)\pi/d$. As the number $(n-k)/d$ is even and $k > 0$, we obtain that $\sin(n-k)\theta < 0$. The conditions (2.5) are fulfilled. □

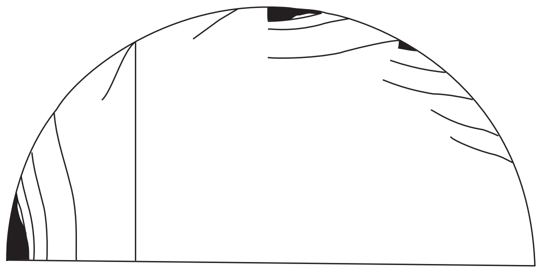

FIGURE 3.2. Trinomial arcs $B(n,k,r)$ inside the upper half unit disk.

4. Differentiability of the function $\rho(\theta)$ for the arcs $B(n,k,r)$

Now, we will prove that the derivative $d\rho/d\theta$ exists and it is well defined for the trinomial arcs $B(n,k,r)$.

PROPOSITION 4.1. *For each trinomial arc $B(n,k,r)$, the function $\rho(\theta)$ is differentiable for any feasible angle θ in the interval (θ_0,θ_1).*

Proof. Let $B(n,k,r)$ be a trinomial arc. By (2.2), we have $\rho^k(\theta) = (1-1/\alpha)\sin n\theta/\sin(n-k)\theta$. According to Lemma 3.3, the feasible angles θ are such that $\sin n\theta > 0$ and $\sin(n-k)\theta < 0$. If we put $f(\theta) = (1-1/\alpha)\sin n\theta/\sin(n-k)\theta$ and as $0 < \alpha = \alpha(\theta) < 1$, the denominator of $f(\theta)$ is never zero. The function $f(\theta)$ is well defined. In addition, f is differentiable and positive. So, the function $\rho(\theta) = [f(\theta)]^{1/k}$ is differentiable (since $\alpha(\theta)$ is differentiable, see Fell [1, pages 326–327]). Therefore, its derivative $d\rho/d\theta$ exists and it is well defined. □

5. Monotonicity of the function $\rho(\theta)$ for the arcs $B(n,k,r)$

In this section, our main interest is to show that $\rho(\theta)$ is a monotonic function, that is, the derivative $d\rho/d\theta$ is never zero, for each trinomial arc $B(n,k,r)$. Differentiating both sides of (2.3) with respect to θ, we obtain

$$\begin{aligned}&\left[(n-k)\rho^{n-k-1}\sin n\theta - n\rho^{n-1}\sin(n-k)\theta\right]\frac{d\rho}{d\theta}\\&\quad= k\cos k\theta + (n-k)\rho^n\cos(n-k)\theta - n\rho^{n-k}\cos n\theta.\end{aligned}\tag{5.1}$$

Supposing that $d\rho/d\theta = 0$, we will consider ρ^n and ρ^{n-k} as solutions of the system,

$$\begin{aligned}k\cos k\theta + (n-k)\rho^n\cos(n-k)\theta - n\rho^{n-k}\cos n\theta &= 0,\\ \rho^{n-k}\sin n\theta - \rho^n\sin(n-k)\theta - \sin k\theta &= 0.\end{aligned}\tag{5.2}$$

This system is equivalent to the following system:

$$\begin{aligned} R(\theta)\cdot\rho^{n-k} &= N_1(\theta), \\ R(\theta)\cdot\rho^{n} &= N_2(\theta), \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} R(\theta) &= (n-k)\sin k\theta - k\cos n\theta\sin(n-k)\theta, \\ N_1(\theta) &= (n-k)\sin n\theta - n\sin(n-k)\theta\cos k\theta, \\ N_2(\theta) &= (n-k)\sin n\theta\cos k\theta - n\sin(n-k)\theta. \end{aligned} \tag{5.4}$$

The difference of the two equalities of (5.3) leads to the equation:

$$R(\theta)\left[\rho^{n}-\rho^{n-k}\right] = U(\theta)[1-\cos k\theta] \tag{5.5}$$

with

$$U(\theta) = -\left[n\sin(n-k)\theta + (n-k)\sin n\theta\right]. \tag{5.6}$$

We now show that the hypothesis $d\rho/d\theta = 0$ leads to a contradiction. For that, we need the two following lemmas.

Lemma 5.1. *Let* $d = \gcd(k,n)$ *and* $r \in \{0,1,\dots,d-1\}$*, one assumes that* $0 < k < n$*,* k/d *and* n/d *are odd. Then, one assumes* $R(\theta) = (n-k)\sin k\theta - k\sin(n-k)\theta\cos n\theta > 0$ *for any feasible angle* θ *in the interval* (θ_0,θ_1)*, where* $\theta_1 = (2r+1)\pi/d$ *and* $\theta_0 = \theta_1 - \pi/n$*.*

Proof. Let θ be a feasible angle in $]\theta_0,\theta_1[$, with $\theta_1 = (2r+1)\pi/d$ and $\theta_0 = \theta_1 - \pi/n$. By first, remark that $\cos n\theta = 0$ if and only if $\theta = \theta_c = [2n(2r+1)/d - 1]\pi/2n$. Moreover, we have $\cos n\theta > 0$ for $\theta < \theta_c$ and $\cos n\theta < 0$ for $\theta > \theta_c$. By Lemma 3.3, we have $\sin k\theta > 0$ and $\sin(n-k)\theta < 0$. Then, we get $R(\theta) > 0$ for any θ in $]\theta_0,\theta_c]$. In the other case, that is when θ belongs to $]\theta_c,\theta_1[$, remarking that $R(\theta)$ can be expressed as $R(\theta) = (n-k)\sin n\theta\cos(n-k)\theta - n\sin(n-k)\theta\cos n\theta$, we will consider the function $K(\theta) = R(\theta)/\cos n\theta\cos(n-k)\theta = (n-k)\tan n\theta - n\tan(n-k)\theta$. In this case, we have $\cos n\theta < 0$. In addition, as $(2r+1)\pi/d - \pi/2n < \theta < (2r+1)\pi/d$, we get $(n-k)(2r+1)\pi/d - (n-k)\pi/2n < (n-k)\theta < (n-k)(2r+1)\pi/d$. Because $(n-k)(2r+1)/d$ is even, there exists a nonnegative integer q such that $2q = (n-k)(2r+1)/d$. Then, this double inequality becomes $2q\pi - (n-k)\pi/2n < (n-k)\theta < 2q\pi$. Since $(2q-1/2)\pi < 2q\pi - (n-k)\pi/2n$, we obtain that $\cos(n-k)\theta > 0$. The sign of $R(\theta)$ is so opposed to the sign of $K(\theta)$, which is derivable with $K'(\theta) = n(n-k)[\tan^2 n\theta - \tan^2(n-k)\theta]$. Since $\tan n\theta < 0$ and $\tan(n-k)\theta < 0$, the zeros of $K'(\theta)$ verify the equation $\tan n\theta = \tan(n-k)\theta$. Therefore, the unique solution of this equation is of the form $\theta = l\pi/k$ where l is an integer. However, $l\pi/k \in]\theta_c,\theta_1[$ if and only if $k(2r+1)/d - k/2n < l < k(2r+1)/d$. As $k(2r+1)/d$ is odd, there exists a nonnegative integer p such that $(2p+1) = k(2r+1)/d$. Hence, we get $(2p+1) - k/2n < l < (2p+1)$. Because $k < n$, it follows that $(2p+1/2) < l < (2p+1)$, which is not possible since l is an integer. We conclude that $K'(\theta)$ is never zero. Moreover, $K(\theta)$ goes to $-\infty$ as θ tends on the right to θ_c and $K(\theta_1) = 0$. This implies that $K(\theta) < 0$ and that $R(\theta) > 0$ for any θ in $]\theta_c,\theta_1[$. Therefore, $R(\theta) > 0$ for any θ in the interval $]\theta_0,\theta_1[$. □

Lemma 5.2. *Let $d = \gcd(k,n)$ and $r \in \{0,1,\dots,d-1\}$, one assumes that $0 < k < n$, k/d and n/d are odd. Then, one has $U(\theta) = -[n\sin(n-k)\theta + (n-k)\sin n\theta] > 0$ for any feasible angle θ in the interval (θ_0,θ_1), where $\theta_1 = (2r+1)\pi/d$ and $\theta_0 = \theta_1 - \pi/n$.*

Proof. Let θ be a feasible angle in $]\theta_0,\theta_1[$. The function $U(\theta)$ is derivable, with $U'(\theta) = -n(n-k)[\cos(n-k)\theta + \cos n\theta]$. The zeros of $U'(\theta)$ are of the form $\theta = (2l-1)\pi/k$ or of the form $\theta = (2l+1)\pi/(2n-k)$ where l is an integer. However, $(2l-1)\pi/k \in]\theta_0,\theta_1[$ if and only if $k(2r+1)/d + 1 - k/n < 2l < k(2r+1)/d + 1$. From the proof of the precedent lemma, we have $(2p+1) = k(2r+1)/d$, then, this double inequality becomes $p+1-k/2n < l < p+1$. As $k < n$, one has $(p+1/2) < p+1-k/2n$. Hence, $(p+1/2) < l < (p+1)$, which is impossible as l is an integer. On the other side, $(2l+1)\pi/(2n-k) \in]\theta_0,\theta_1[$ if and only if $(2n-k)(2r+1)/2d - 3/2 + k/2n < l < (2n-k)(2r+1)/2d - 1/2$. As $n(2r+1)/d$ is odd, there exists a nonnegative integer s such that $(2s+1) = n(2r+1)/d$. Moreover, since $(2p+1) = k(2r+1)/d$, it follows that $(2n-k)(2r+1)/2d - 1/2 = 2s - p$. Therefore, the double inequality above becomes $2s - p - 1 + k/2n < l < 2s - p$, then $2s - p - 1 < l < 2s - p$, which is not possible. It follows that $U'(\theta)$ is never zero. In addition, because $U(\theta_0) > 0$ and $U(\theta_1) = 0$, we deduce that $U(\theta) > 0$ for any angle θ in $]\theta_0,\theta_1[$. □

Using the two lemmas above, we can prove the next main result for the trinomial arcs $B(n,k,r)$.

Theorem 5.3. *Let $d = \gcd(k,n)$ and $r \in \{0,1,\dots,d-1\}$, one assumes that $0 < k < n$, k/d and n/d are odd. For every trinomial arc $B(n,k,r)$, $\rho(\theta)$ is a decreasing function on the interval (θ_0,θ_1), where $\theta_1 = (2r+1)\pi/d$ and $\theta_1 - \theta_0 = \pi/n$.*

Proof. Let $B(n,k,r)$ be a trinomial arc. Lemmas 5.1 and 5.2 imply, respectively, that $R(\theta) > 0$ and $U(\theta) > 0$ for any θ in the interval $]\theta_0,\theta_1[$. Therefore, the relation $R(\theta)[\rho^n - \rho^{n-k}] = U(\theta)[1 - \cos k\theta]$ given by (5.5) implies that $\rho^n - \rho^{n-k} > 0$, which is impossible as $\rho < 1$. Thus, we have proved that for each trinomial arc $B(n,k,r)$, we have $d\rho/d\theta \neq 0$, that is, $\rho(\theta)$ is a monotonic function, for any angle θ in $]\theta_0,\theta_1[$.

Moreover, putting $\theta = \theta_0 = (2r+1)\pi/d - \pi/n$ in (2.3), we have $\rho^{n-k}(\theta_0)\sin n\theta_0 - \rho^n(\theta_0)\sin(n-k)\theta_0 - \sin k\theta_0 = 0$. Using the facts that $(2s+1) = n(2r+1)/d$ and $(2p+1) = k(2r+1)/d$, we get $\sin n\theta_0 = 0$ and $\sin(n-k)\theta_0 = -\sin k\theta_0$. So, it follows that $[\rho^n(\theta_0) - 1]\sin k\theta_0 = 0$. Hence, because $2\pi p < k\theta_0 < (2p+1)\pi$, it yields that $\sin k\theta_0 \neq 0$ and that $\rho(\theta_0) = 1$. Since $\rho(\theta)$ is less than or equal to 1 for any feasible angle θ, we deduce that $\rho(\theta)$ is a decreasing function on the interval (θ_0,θ_1). □

6. Conclusion

In this work, we have studied the behavior of the family of trinomial arcs $B(n,k,r)$, composed of all solutions of (1.1) in the case $0 < \alpha < 1$ with the feasible angles θ in the interval $[\arg(\gamma),\arg(\delta)]$, where γ is an nth root of unity and δ is both a kth root of -1 and an $(n-k)$th root of unity. The problem of monotonicity of the trinomial arcs is completely solved in this case. During the description and definition of $B(n,k,r)$, we have evoked another type of trinomial arcs, defined as the solutions of (1.1) in the case $0 < \alpha < 1$ with the feasible angles θ in the interval $[\arg(\delta),\arg(\gamma')]$, where δ is both a kth root of -1 and an $(n-k)$th root of unity and γ' is an nth root of unity. A later study of the behavior of this family of arcs would be interesting.

References

[1] H. Fell, "The geometry of zeros of trinomial equations," *Rendiconti del Circolo Matematico di Palermo*, vol. 29, no. 2, pp. 303–336, 1980.
[2] S. Dubuc and M. Zaoui, "Sur la quasi-convexité des arcs trinomiaux," *Rendiconti del Circolo Matematico di Palermo*, vol. 45, no. 3, pp. 493–514, 1996.
[3] S. Dubuc and A. Malik, "Convex hull of powers of a complex number, trinomial equations and the Farey sequence," *Numerical Algorithms*, vol. 2, no. 1, pp. 1–32, 1992.

Kaoutar Lamrini Uahabi: Forces Armées Royales Boulevard 49, Apartment no. 9,
Nador 62000, Morocco
Email address: lamrinika@yahoo.fr

Mohammed Zaoui: Department of Mathematics, Faculty of Sciences, Mohamed First University,
P.O. Box 524, Oujda 60000, Morocco
Email address: zaouimoh@menara.ma