Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 92439, 9 pages
doi:10.1155/2007/92439

Research Article

On Some Analytic Functions Defined by a Multiplier Transformation

Khalida Inayat Noor

Received 26 July 2007; Accepted 19 November 2007

Recommended by Heinrich Begehr

We introduce and study a new class of analytic functions defined in the unit disc using a certain multiplier transformation. Some inclusion results and other interesting properties of this class are investigated.

Copyright © 2007 Khalida Inayat Noor. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let $P_k(\eta)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \eta}{1-\eta} \right| d\theta \le k\pi, \tag{1.1}$$

where $z = re^{i\theta}$, $k \ge 2$, $0 \le \eta < 1$. For $\eta = 0$, we obtain the class P_k defined by Pinchuk [1], and for $\eta = 0$, $k = 2$, we have the class P of functions with positive real part, whereas $P_2(\eta) = P(\eta)$ is the class of functions with positive real part greater than η. We can write (1.1) as

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+(1-2\eta)ze^{-it}}{1-ze^{-it}}\, d\mu(t), \tag{1.2}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2, \qquad \int_0^{2\pi} |\, d\mu(t)\,| \le k. \tag{1.3}$$

We can also write (1.1), for $p \in P_k(\eta)$ in E, if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad p_1, p_2 \in P(\eta). \tag{1.4}$$

It is known [2] that the class $P_k(\eta)$ is a convex set. Let A be the class of functions f, defined by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.5}$$

which are analytic in E. By S, K, S^*, and C, we denote the subclasses of A which are univalent, close-to-convex, starlike, and convex in E, respectively. The class A is closed under the Hadamard product or convolution:

$$(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^{m+1}, \tag{1.6}$$

where

$$f(z) = \sum_{m=0}^{\infty} a_m z^{m+1}, \qquad g(z) = \sum_{m=0}^{\infty} b_m z^{m+1}. \tag{1.7}$$

We define the following.

Definition 1.1. Let $f \in$ A. Then, for $\alpha, \beta \geq 0$, $0 \leq \eta < \alpha + \beta \leq 1$, $k \geq 2$, and $z \in E$, $f \in Q_k(\alpha, \beta, \eta)$ if and only if

$$\left\{\alpha f'(z) + \beta\left(z f'(z)\right)'\right\} \in P_k(\eta). \tag{1.8}$$

We note that, for $\beta = 0$ and $k = 2$, $f' \in P(\eta) \subset P$ for $z \in E$ and this implies that f is univalent in E, see [3]. For any real number s, the multiplier transformations I_λ^s of functions $f \in A$ are defined by

$$f_\lambda^s(z) = I_\lambda^s f(z) = z + \sum_{m=2}^{\infty} \left(\frac{m+\lambda}{1+\lambda}\right)^s a_m z^m \quad (\lambda > -1). \tag{1.9}$$

It is obvious that $I_\lambda^s(I_\lambda^t f(z)) = I_\lambda^{s+t} f(z)$ for all real numbers s and t. The operator I_λ^s has been studied by several authors for different choices of s and λ, see [4–7]. It is worth noting that, for s any nonnegative integer and $\lambda = 0$, the operator I_λ^s is the differential operator defined by Sălăgeam [8]. Also the operator I_λ^s is related rather closely to the multiplier transformation discussed by Flett [9]. Using (1.9) and convolution, function $f_{\lambda,\mu}^s$ is defined as follows:

$$f_\lambda^s(z) * f_\lambda^s(z) = \frac{z}{(1-z)^\mu}, \quad z \in E, \ \mu > 0. \tag{1.10}$$

Motivated essentially by Choi et al. operator [10] and Noor integral operator [11–14], Cho and Kim [15] defined the operator $I^s_{\lambda,\mu} : A \to A$ as

$$I^s_{\lambda,\mu} f(z) = f^s_{\lambda,\mu}(z) * f(z), \tag{1.11}$$

where s is real, $\lambda > -1$, $\mu > 0$, and $f \in A$. In particular, $I^0_{0,2} f(z) = z f'(z)$, $I^1_{0,2} f(z) = f(z)$. From (1.10) and (1.11), we have

$$z\left(I^{s+1}_{\lambda,\mu} f(z)\right)' = (\lambda+1) I^s_{\lambda,\mu} f(z) - \lambda I^{s+1}_{\lambda,\mu} f(z), \tag{1.12}$$

$$z\left(I^{s}_{\lambda,\mu} f(z)\right)' = \mu I^s_{\lambda,\mu+1} f(z) - (\mu-1) I^{s}_{\lambda,\mu} f(z). \tag{1.13}$$

We now define the following.

Definition 1.2. Let $f \in A$. Then, for s real, $\lambda > 1$, $\mu > 0$,

$$f \in Q^s_k(\lambda,\mu,\alpha,\beta,\eta) \quad \text{iff } I^s_{\lambda,\mu} f(z) \in Q_k(\alpha,\beta,\eta) \text{ for } z \in E. \tag{1.14}$$

2. Preliminary results

LEMMA 2.1. *If $h(z)$ is analytic in E with $h(0) = 1$ and if λ_1 is a complex number satisfying* $\operatorname{Re}\lambda_1 \ge 0$ $(\lambda_1 \ne 0)$, *then* $\{h(z) + \lambda_1 z h'(z)\} \in P_k(\delta)$, $0 \le \delta < 1$, *implies* $h(z) \in P_k(\delta + (1-\delta)(2\gamma - 1)$ *and*

$$\gamma = \int_0^1 \left(1 + t^{\operatorname{Re}\lambda_1}\right)^{-1} dt, \tag{2.1}$$

where γ is an increasing function of $\operatorname{Re}\lambda_1$ *and* $1/2 \le \gamma < 1$. *The estimate is sharp.*

Proof. Let $h(z) = (k/4 + 1/2)h_1(z) - (k/4 - 1/2)h_2(z)$, $h(z)$ is analytic in E with $h(0) = 1$. Then, $h(z) + \lambda_1 z h'(z) = (k/4+1/2)[h_1(z) + \lambda_1 z h_1'(z)] - (k/4 - 1/2)[h_2(z) + \lambda_1 z h_2'(z)]$. Since $[h(z) + \lambda_1 z h'(z)] \in P_k(\delta)$, we use (1.4) to have $[h_i(z) + \lambda_1 z h_i'(z)] \in P(\delta)$, $i = 1,2$. We now apply a lemma in [16] to conclude that $h_i \in P(\delta_1)$, $i = 1,2$, and $\delta_1 = \delta + (1-\delta)(2\gamma - 1)$, where γ is given by (2.1) and it is an increasing function of $\operatorname{Re}\lambda_1$ with $1/2 \le \gamma < 1$. Consequently $h \in P_k(\delta_1)$ in E. □

LEMMA 2.2 [17]. *If $p(z)$ is analytic in E with $p(0) = 1$, then, for any function F, analytic in E, the function $p * F$ takes values in the convex hull of image of E under F.*

LEMMA 2.3. *Let $\beta_1 < 1$. If the function p is analytic in E, with $p(0) = 1$, then $p \in P_k(\beta_2)$, $\beta_2 = (2\beta_1 - 1) + 2(1-\beta_1)\ln 2$, $z \in E$. This result is sharp.*

Proof. The proof is immediate when we use (1.4) and a similar result for the class $P(\beta_2)$ in [18]. □

LEMMA 2.4. *For $\eta_1 \leq 1$ and $\eta_2 \leq 1, P_k(\eta_1)*P_k(\eta_2) \subset P_k(1-2(1-\eta_1)(1-\eta_2))$. This result is sharp.*

Proof. Let $H \in P_k(\eta_1)$, $p \in P_k(\eta_2)$. Then, using (1.4), we can write

$$(H*p)(z) = \left(\frac{k}{4}+\frac{1}{2}\right)[(H_1*p_1)(z)] - \left(\frac{k}{4}-\frac{1}{2}\right)[(H_2*p_2)(z)],$$
$$H_i \in P(\eta_1), \quad p_i \in P(\eta_2), \quad i=1,2. \tag{2.2}$$

Now using a result from [19], we have, for $i=1,2$,

$$(H_i*p_i) \in P(\eta), \quad \eta = 1-2(1-\eta_1)(1-\eta_2). \tag{2.3}$$

This result is shown to be sharp in [19] and consequently $(H*p) \in P_k(\eta)$. □

3. Main results

THEOREM 3.1. *$Q_k^s(\lambda,\mu,\alpha,\beta,\eta) \subset Q_k^s(\lambda,\mu,1,0,\sigma)$ for*

$$\sigma = \sigma_1 + (1-\sigma_1)(2\sigma_2 - 1), \quad \sigma_1 = \frac{\eta}{\alpha+\beta},$$
$$\sigma_2 = \int_0^1 \left(1+t^{\beta/(\alpha+\beta)}\right)^{-1} dt, \quad \textit{with } \frac{1}{2} \leq \sigma_2 \leq 1. \tag{3.1}$$

Proof. Let $f \in Q_k^s(\lambda,\mu,\alpha,\beta,\eta)$. Then, by definition it follows that

$$\{\alpha(I_{\lambda,\mu}^s f)' + \beta(z(I_{\lambda,\mu}^s f)')'\} \in P_k(\eta), \quad z \in E. \tag{3.2}$$

Set $(I_{\lambda,\mu}^s f(z))' = p(z)$. Then p is analytic in E with $p(0)=1$ and for $z \in E$,

$$\left\{\frac{\alpha(I_{\lambda,\mu}^s f(z))' + \beta(z(I_{\lambda,\mu}^s f(z))')' - \eta}{\alpha+\beta-\eta}\right\}$$
$$= \left\{\frac{\alpha+\beta}{\alpha+\beta-\eta}p(z) + \frac{\beta}{\alpha+\beta-\eta}zp'(z) - \frac{\eta}{\alpha+\beta-\eta}\right\} \in P_k. \tag{3.3}$$

From (1.4) and (3.4), we have, for $i=1,2$,

$$\left[\frac{\alpha+\beta}{\alpha+\beta-\eta}p_i(z) + \frac{\beta}{\alpha+\beta-\eta}zp_i'(z) - \frac{\eta}{\alpha+\beta-\eta}\right] = h_i(z) \in P. \tag{3.4}$$

By putting $\sigma_1 = \eta/(\alpha+\beta)$, we see that

$$p_i(z) + \frac{\beta}{\alpha+\beta}zp_i'(z) = (1-\sigma_1)h_i(z) + \sigma_1 = H_i(z) \in P(\sigma_1). \tag{3.5}$$

Now using Lemma 2.1, we obtain $p_i \in P(\sigma)$, where σ is given by (3.1). Therefore, $(I_{\lambda,\mu}^s f)' \in P_k(\sigma)$ and consequently $f \in Q_k^s(\lambda,\mu,1,0,\sigma)$ in E. □

Remark 3.2. By writing $\sigma_1 = \eta/(\alpha+\beta)$, $\alpha_1 = \alpha/(\alpha+\beta)$, we can deduce from Definition 1.2 that $f \in Q_k^s(\lambda,\mu,\alpha,\beta,\eta)$, if and only if, for $0 \le \alpha_1 \le 1$,

$$\left[\alpha_1 (I_{\lambda,\mu}^s f)' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f)')'\right] \in P_k(\sigma_1), \quad z \in E. \tag{3.6}$$

In this case, we say that $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ in E.

Theorem 3.3. *Let s be real, $\lambda > -1$, $\mu > 0$. Then,*

$$Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1) \subset Q_k^s(\lambda,\mu,\alpha_1,\delta_1) \subset Q_k^{s+1}(\lambda,\mu,\alpha_1,\delta_2), \tag{3.7}$$

where α_1 and σ_1 are as defined in Remark 3.2 and

$$\delta_1 = \sigma_1 + (1-\sigma_1)(2\eta_1 - 1), \quad \eta_1 = \int_0^1 (1+t^{1/\mu})^{-1} dt, \tag{3.8}$$

$$\delta_2 = \delta_1 + (1-\delta_1)(2\eta_2 - 1), \quad \eta_2 = \int_0^1 (1+t^{1/(\lambda+1)})^{-1} dt. \tag{3.9}$$

Proof. We first show that $Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1) \subset Q_k^s(\lambda,\mu,\alpha_1,\delta_1)$.

Let $f \in Q_k^s(\lambda,\mu+1,\alpha_1,\sigma_1)$ and set

$$p(z) = \alpha_1\left[(I_{\lambda,\mu}^s f(z))'\right] + (1-\alpha_1)\left[(zI_{\lambda,\mu}^s f(z)')'\right]. \tag{3.10}$$

From (1.13) and (3.10), we have, for $z \in E$,

$$\left\{\alpha_1 (I_{\lambda,\mu+1}^s f(z))' + (1-\alpha_1)(z(I_{\lambda+\mu+1} f(z))')'\right\} = \left\{p(z) + \frac{1}{\mu} z p'(z)\right\} \in P_k(\sigma_1) \tag{3.11}$$

and, on using (1.4), it follows that $\operatorname{Re}\{p_i(z) + (1/\mu) z p_i'(z)\} > \sigma_1$, $z \in E$, $i = 1,2$.

Now, applying Lemma 2.1, we have $\operatorname{Re} p_i(z) > \delta_1$, $i = 1,2$, where δ_1 is given by (3.8). This implies $p \in P_k(\delta_1)$ for $z \in E$ and hence $f \in Q_k^s(\lambda,\mu,\alpha_1,\delta_1)$ in E. To prove $Q_k^s(\lambda,\mu,\alpha_1,\delta_1) \subset Q_k^{s+1}(\lambda,\mu,\alpha_1,\delta_2)$, we proceed as follows. Set

$$\left\{\alpha_1 (I_{\lambda,\mu}^{s+1} f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^{s+1} f(z))')'\right\} = h(z). \tag{3.12}$$

Then, using (1.12), we have

$$\left\{\alpha_1 (I_{\lambda,\mu}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'\right\} = \left\{h(z) + \frac{1}{\lambda+1} z h'(z)\right\} \in P_k(\delta_1). \tag{3.13}$$

With similar argument as detailed above, we obtain the required result. □

Theorem 3.4. *The class $Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ is closed under the convolution with a convex function. That is, if $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ and $\phi \in C$ for $z \in E$, then $(\phi * f) \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$.*

Proof. Let $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$. Consider

$$\begin{aligned}
&\alpha_1\left(I_{\lambda,\mu}^s(\phi * f)(z)\right)' + (1-\alpha_1)\left(z\left(I_{\lambda,\mu}^s(\phi * f)(z)\right)'\right)' \\
&\quad = \alpha_1\left(f_{\lambda,\mu}^s(z) * (\phi * f)(z)\right)' + (1-\alpha_1)\left(z\left(f_{\lambda,\mu}^s(z) * (\phi * f)(z)\right)'\right)' \\
&\quad = \alpha_1\left(\phi(z) * f_{\lambda,\mu}^s(z) * f(z)\right)' + (1-\alpha_1)\left(z\left(\phi(z) * f_{\lambda,\mu}^s(z) * f(z)\right)'\right)' \\
&\quad = \frac{\phi(z)}{z} * \left\{\alpha_1\left(I_{\lambda,\mu}^s f(z)\right)' + (1-\alpha_1)\left(z\left(I_{\lambda,\mu}^s f(z)\right)'\right)'\right\} \\
&\quad = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\phi(z)}{z} * h_1(z)\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\phi(z)}{z} * h_2(z)\right],
\end{aligned} \tag{3.14}$$

where $\phi(z)/z \in P(1/2)$ and $h_i \in P(\sigma_1)$. Using Lemma 2.2, we see that $[(\phi(z)/z) * h_i(z)] \in P(\sigma_1)$ and consequently $h \in P_k(\sigma_1)$, which implies that $\phi * f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$; the proof is complete. □

Corollary 3.5. *The class $Q_k^s(\lambda,\mu,\alpha_1,\sigma_1)$ is invariant under the following integral operators:*

(i) $f_1(z) = \int_0^z (f(t)/t)dt$,

(ii) $f_2(z) = (2/z)\int_0^z f(t)dt$ *(Libera's operator [20])*,

(iii) $f_3(z) = \int_0^z (f(t) - f(xt)/(t-xt))dt, |x| \le 1, x \ne 1$,

(iv) $f_4(z) = ((1+c)/z^c)\int_0^z t^{c-1} f(t)dt, \operatorname{Re} c > 0$.

One may write (see [21, 22])

$$\begin{aligned}
f_1(z) &= f(z) * \phi_1(z), \qquad & f_2(z) &= f(z) * \phi_2(z), \\
f_3(z) &= f(z) * \phi_3(z), \qquad & f_4(z) &= f(z) * \phi_4(z),
\end{aligned} \tag{3.15}$$

where ϕ_i, $i = 1,2,3,4$, are convex and

$$\begin{aligned}
\phi_1(z) &= -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \\
\phi_2(z) &= \frac{-2[z+\log(1-z)]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n, \\
\phi_3(z) &= \frac{1}{1-x}\log\left[\frac{1-xz}{1-z}\right] = \sum_{n=1}^{\infty} \frac{1-x^n}{(1-x)^n} z^n, \qquad |x| \le 1,\ x \ne 1, \\
\phi_4(z) &= \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \qquad \operatorname{Re} c > 0.
\end{aligned} \tag{3.16}$$

Now, the result follows by applying Theorem 3.4. Let μ_1 and μ_2 be linear operators defined on the class S as follows:

$$\mu_1(f(z)) = zf'(z), \qquad \mu_2(f(z)) = \frac{[f(z) + zf'(z)]}{2} \quad \text{(Livingston's operator [23])}. \tag{3.17}$$

Then, both of these operators can be written as a convolution operator [21], given by $\mu_i(f) = h_i * f$, $i = 1,2$, where

$$h_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}, \qquad h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - z^2/2}{(1-z)^2}. \tag{3.18}$$

It can easily be verified that the radius of convexity $r_c(h_1) = 2 - \sqrt{3}$ and $r_c(h_2) = 1/2$. These facts together with Theorem 3.4 yield the following.

Theorem 3.6. *Let $f \in Q_k^s(\lambda, \mu, \alpha_1, \sigma_1)$. Then,*

$$\begin{aligned} \mu_1(f) &= (f * h_1) \in Q_k^s(\lambda, \mu, \alpha_1, \sigma_1), \quad \text{for } |z| < 2 - \sqrt{3}, \\ \mu_2(f) &= (f * h_2) \in Q_k^s(\lambda, \mu, \alpha_1, \sigma_1), \quad \text{for } |z| < \frac{1}{2}. \end{aligned} \tag{3.19}$$

Theorem 3.7. *Let $0 \le \alpha_1 < \alpha_2$. Then, $Q_k^s(\lambda, \mu, \alpha_1, \sigma_1) \subset Q_k^s(\lambda, \mu, \alpha_2, \sigma_1)$.*

Proof. If $\alpha_1 = 0$, the result is obvious. Therefore, we assume that $\alpha_1 > 0$ and $f \in Q_k^s(\lambda, \mu, \alpha_2, \sigma_1)$. Let $(I_{\lambda,\mu}^s f(z))' = H_1(z)$. Then, by Theorem 3.1, $H_1 \in P_k(\sigma_1)$. Also, let

$$\left\{\alpha_1 (I_{\lambda,\varpi}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'\right\} = H_2(z), \quad H_2 \in P_k(\sigma_1) \text{ in } E. \tag{3.20}$$

Now,

$$\begin{aligned} \alpha_2 (I_{\lambda,\mu}^s f(z))' + (1-\alpha_2)(z(I_{\lambda,\mu}^s f(z))')' &= \frac{\alpha_2 - \alpha_1}{(1-\alpha_1)} H_1(z) + \frac{(1-\alpha_2)}{(1-\alpha_1)} H_2(z) \\ &= \frac{(\alpha_2 - \alpha_1)}{(1-\alpha_1)} H_1(z) + \left(1 - \frac{\alpha_2 - \alpha_1}{(1-\alpha_1)}\right) H_2(z). \end{aligned} \tag{3.21}$$

Since $H_1, H_2 \in P_k(\sigma_1)$ and $P_k(\sigma_1)$ is a convex set, see [2], we obtain the required result. □

Theorem 3.8. *Let $f_i \in Q_k^s(\lambda, \mu, \alpha_1, \zeta_i)$, $i = 1,2$, and let $\Psi = f_1 * f_2$. Then, $\Psi(z)/z \in Q_k^s(\lambda, \mu, 1, \zeta)$ for $z \in E$, where $\zeta = 1 - \delta(1-\delta_1)(1-\delta_2)(\ln 2 - 1)^2$ and*

$$\delta_i = \zeta_i + (1-\zeta_i)(2m-1). \tag{3.22}$$

Proof. Since $f_i \in Q_k^s(\lambda, \mu, \alpha_1, \zeta_i)$, it follows from Theorem 3.1 that $f_i \in Q_k^s(\lambda, \mu, 1, \delta_i)$, $\delta_i = \zeta_i + (1-\zeta_i)(2m-1)$, and

$$m = \int_0^1 \left(1 + t^{(1-\alpha)}\right)^{-1} dt. \tag{3.23}$$

Now,

$$\begin{aligned} \left(z(I_{\lambda,\mu}^s \Psi(z))'\right)' &= I_{\lambda,\mu}^s\left[(\Psi'(z) + z\psi''(z)\right] = \left(z(I_{\lambda,\mu}^s (f_1' * f_2)(z))\right)' \\ &= I_{\lambda,\mu}^s\left[(f_1'(z) * f_2'(z))\right] = (I_{\lambda,\mu}^s f_1(z))' * (I_{\lambda,\mu}^s f_2(z))'. \end{aligned} \tag{3.24}$$

Since $f_i \in Q_k^s(\lambda,\mu,1,\delta_i)$, it follows, by Lemma 2.4, that $\{\Psi'(z)+z\Psi''(z)\} \in Q_k^s(\lambda,\mu,1,\delta)$, where

$$\delta = 1-2(1-\delta_1)(1-\delta_2). \tag{3.25}$$

From (3.25) and Lemma 2.3, we have

$$\Psi'(z) \in Q_k^s(\lambda,\mu,1,\{1+4(1-\delta_1)(1-\delta_2)(\ln 2-1)\}). \tag{3.26}$$

From (3.26) and Lemma 2.3, again, we have

$$\frac{\Psi(z)}{z} \in Q_k^s(\lambda,\mu,1,\{1-\delta(1-\delta_1)(1-\delta_2)(\ln 2-1)^2\}), \quad z\in E. \tag{3.27}$$

We now consider the converse case of Theorem 3.1 as follows. □

Theorem 3.9. *Let* $f \in Q_k^s(\lambda,\mu,1,\sigma)$. *Then,* $f \in Q_k^s(\lambda,\mu,\alpha_1,\sigma)$, $0<\alpha_1\le 1$, *for* $|z|<r_{\alpha_1}$ $(\alpha_1 \ne 1/2)$, *where*

$$r_{\alpha_1} = \frac{1}{\{2(1-\alpha_1)+\sqrt{4\alpha_1^2-6\alpha_1+3}\}}. \tag{3.28}$$

This result is sharp.

Proof. Let $\phi_{\alpha_1}(z) = \alpha_1(I_{\lambda,\mu}^s f(z))' + (1-\alpha_1)(z(I_{\lambda,\mu}^s f(z))')'$. Then,

$$\phi_{\alpha_1}(z) = \frac{k_{\alpha_1}(z)}{z} * (I_{\lambda,\mu}^s f(z))', \quad \text{where } k_{\alpha_1}(z) = \alpha_1\frac{z}{1-z} + (1-\alpha_1)\frac{z}{(1-z)^2}. \tag{3.29}$$

It is known [23] that the function k_{α_1} is convex for $|z|<r_{\alpha_1}$, where r_{α_1} is given by (3.28) and this radius is sharp and consequently, for $|z|<r_{\alpha_1}$, by a well-known result, $k_{\alpha_1} \in P(1/2)$. Thus, using Lemma 2.2, and the given fact that $f \in Q_k^s(\lambda,\mu,1,\sigma)$, we obtain the required result. □

Acknowledgment

The author would like to thank Dr. S. M. Junaid Zaidi, (Rector CIIT), who generously supports scientific research in all aspects.

References

[1] B. Pinchuk, "Functions of bounded boundary rotation," *Israel Journal of Mathematics*, vol. 10, pp. 6–16, 1971.

[2] K. I. Noor, "On subclasses of close-to-convex functions of higher order," *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 2, pp. 279–289, 1992.

[3] W. Kaplan, "Close-to-convex schlicht functions," *The Michigan Mathematical Journal*, vol. 1, pp. 169–185 (1953), 1952.

[4] B. A. Uralegaddi and C. Somanatha, "Certain classes of univalent functions," in *Current Topics in Analytic Function Theory*, H. M. Srivastava and S. Owa, Eds., pp. 371–374, World Scientific, River Edge, NJ, USA, 1992.

[5] S. Owa and H. M. Srivastava, "Some applications of the generalized Libera integral operator," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 62, no. 4, pp. 125–128, 1986.

[6] I. B. Jung, Y. C. Kim, and H. M. Srivastava, "The Hardy space of analytic functions associated with certain one-parameter families of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 176, no. 1, pp. 138–147, 1993.

[7] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.

[8] G. S. Sălăgean, "Subclasses of univalent functions," in *Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Math*, pp. 362–372, Springer, Berlin, Germany, 1983.

[9] T. M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities," *Journal of Mathematical Analysis and Applications*, vol. 38, pp. 746–765, 1972.

[10] J. H. Choi, M. Saigo, and H. M. Srivastava, "Some inclusion properties of a certain family of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 432–445, 2002.

[11] J. L. Liu, "The Noor integral and strongly starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 441–447, 2001.

[12] J. L. Liu and K. I. Noor, "Some properties of Noor integral operator," *Journal of Natural Geometry*, vol. 21, no. 1-2, pp. 81–90, 2002.

[13] K. I. Noor, "On new classes of integral operators," *Journal of Natural Geometry*, vol. 16, no. 1-2, pp. 71–80, 1999.

[14] K. I. Noor and M. A. Noor, "On integral operators," *Journal of Mathematical Analysis and Applications*, vol. 238, no. 2, pp. 341–352, 1999.

[15] N. E. Cho and J. A. Kim, "Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation," *Computers & Mathematics with Applications*, vol. 52, no. 3-4, pp. 323–330, 2006.

[16] S. Ponnusamy, "Differential subordination and Bazilevič functions," *Indian Academy of Sciences. Proceedings. Mathematical Sciences*, vol. 105, no. 2, pp. 169–186, 1995.

[17] R. Singh and S. Singh, "Convolution properties of a class of starlike functions," *Proceedings of the American Mathematical Society*, vol. 106, no. 1, pp. 145–152, 1989.

[18] S. Ponnusamy and V. Singh, "Convolution properties of some classes of analytic functions," Internal Report, SPIC Science Foundation, 1990.

[19] Y. Ling and S. Ding, "A new criterion for starlike functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 3, pp. 613–614, 1996.

[20] R. J. Libera, "Some classes of regular univalent functions," *Proceedings of the American Mathematical Society*, vol. 16, pp. 755–758, 1965.

[21] R. W. Barnard and C. Kellogg, "Applications of convolution operators to problems in univalent function theory," *The Michigan Mathematical Journal*, vol. 27, no. 1, pp. 81–94, 1980.

[22] St. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.

[23] R. V. Nikolaeva and L. G. Repnīna, "A certain generalization of theorems due to Livingston," *Ukrainskiĭ Matematicheskiĭ Zhurnal*, vol. 24, pp. 268–273, 1972.

Khalida Inayat Noor: Mathematics Department, COMSATS Institute of Information Technology, Sector H-8/1, Islamabad 44000, Pakistan
Email addresses: khalidanoor@hotmail.com; khalidainayat@comsats.edu.pk