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Research Article

Some Estimates of Schrödinger-Type Operators with Certain Nonnegative Potentials

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We consider the Schrödinger-type operator $H = (-\Delta)^2 + V^2$, where the nonnegative potential V belongs to the reverse Hölder class B_{q_1} for $q_1 \geq n/2$, $n \geq 5$. The L^p estimates of the operator $\nabla^4 H^{-1}$ related to H are obtained when $V \in B_{q_1}$ and $1 < p \leq q_1/2$. We also obtain the weak-type estimates of the operator $\nabla^4 H^{-1}$ under the same condition of V.

1. Introduction

In recent years, there has been considerable activity in the study of Schrödinger operators (see [1–4]). In this paper, we consider the Schrödinger-type operator

$$H = (-\Delta)^2 + V^2 \quad \text{on } \mathbb{R}^n,\ n \geq 5, \tag{1.1}$$

where the potential V belongs to B_{q_1} for $q_1 \geq n/2$. We are interested in the L^p boundedness of the operator $\nabla^4 H^{-1}$, where the potential V satisfies weaker condition than that in [5, Theorem 1, (2)]. The estimates of some other operators related to Schrödinger-type operators can be found in [2, 5].

Note that a nonnegative locally L^q integrable function V on $\mathbb{R}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x) dx\right) \tag{1.2}$$

holds for every ball B in $\mathbb{R}^n$.

It follows from [3] that the B_q class has a property of "self-improvement", that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$.

We now give the main results for the operator $\nabla^4 H^{-1}$ in this paper.

Theorem 1.1. *Suppose $V \in B_{q_1}$, $q_1 \geq n/2$. Then for $1 < p \leq q_1/2$ there exists a positive constant C_p such that*

$$\left\|\nabla^4 H^{-1} f\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.3}$$

By the proof of Theorem 1.1, we obtain the following weak-type estimate.

Theorem 1.2. *Suppose $V \in B_{q_1}$, $q_1 \geq n/2$. Then for $1 < p \leq q_1/2$ there exists a positive constant C_1 such that*

$$\left|\left\{x \in \mathbb{R}^n : \left|\nabla^4 H^{-1} f(x)\right| \geq \lambda\right\}\right| \leq \frac{C_1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.4}$$

Under a stronger condition on the potential V, Sugano [5] has obtained the following proposition.

Proposition 1.3. *Suppose $V \in B_{n/2}$ and there exists a constant C such that $V(x) \leq Cm(x,V)^2$. Then for $1 < p < \infty$ there exists a positive constant C_p such that*

$$\left\|\nabla^4 H^{-1} f\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.5}$$

As a direct consequence of our L^p estimates, we have the following corollary.

Corollary 1.4. *Suppose $V \in B_{q_1}$ for $q_1 \geq n/2$. Assume that $(-\Delta)^2 u + V^2 u = f$ in $\mathbb{R}^n$. Then*

$$\left\|\nabla^4 u\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p \leq \frac{q_1}{2}. \tag{1.6}$$

Throughout this paper, unless otherwise indicated, we will use C to denote constants, which are not necessarily the same at each occurrence. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$ such that $c \leq A/B \leq C$.

2. The auxiliary function $m(x,V)$ and estimates of fundamental solution

In this section, we firstly recall the definition of the auxiliary function $m(x,V)$ and some lemmas about the auxiliary function $m(x,V)$ which have been proven in [3].

Lemma 2.1. *If $V \in B_q$, $q > 1$, then the measure $V(x)dx$ satisfies the doubling condition, that is, there exists $C > 0$ such that*

$$\int_{B(x,2r)} V(y)dy \leq C \int_{B(x,r)} V(y)dy \tag{2.1}$$

holds for all balls $B(x,r)$ in $\mathbb{R}^n$.

Lemma 2.2. *For $0 < r < R < \infty$ and $V \in B_{q_1}$ for $q_1 \geq n/2$, there exists $C > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C\left(\frac{r}{R}\right)^{2-n/q_1} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy. \tag{2.2}$$

Assume that $V \in B_{q_1}$, $q_1 \geq n/2$. The auxiliary function $m(x,V)$ is defined by

$$\frac{1}{m(x,V)} \doteq \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^n. \tag{2.3}$$

Lemma 2.3. *If $r = 1/m(x,V)$, then*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy = 1. \tag{2.4}$$

Moreover,

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \sim 1, \quad \text{iff } r \sim \frac{1}{m(x,V)}. \tag{2.5}$$

Lemma 2.4. *There exists $l_0 > 0$ such that for any x and y in $\mathbb{R}^n$,*

$$\frac{1}{C}\left(1 + m(x,V)|x-y|\right)^{-l_0} \leq \frac{m(x,V)}{m(y,V)} \leq C\left(1 + m(x,V)|x-y|\right)^{l_0/(l_0+1)}. \tag{2.6}$$

In particular, $m(x,V) \sim m(y,V)$, if $|x-y| < C/m(x,V)$.

Lemma 2.5. *There exists $l_1 > 0$ such that*

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}} dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C\left(1 + Rm(x,V)\right)^{l_1}. \tag{2.7}$$

Lemma 2.6. *There exists $C > 0$, $c > 0$, and $l_0 > 0$ such that, for any $x, y \in \mathbb{R}^n$,*

$$\begin{aligned} c\left\{1 + |x-y|m(y,V)\right\}^{1/(l_0+1)} &\leq 1 + |x-y|m(x,V) \\ &\leq C\left\{1 + |x-y|m(y,V)\right\}^{l_0+1}. \end{aligned} \tag{2.8}$$

Refer to [3] for the proof of the above lemmas.

The next lemma has been obtained by Tao and Wang in [6].

Lemma 2.7. *Let* $q > s \geq 0$, $q \geq \max\{1, sn/\alpha\}$, $\alpha > 0$, *and* k*be sufficiently large, then there are positive constants* k_0, C, *and* C_k *such that*

$$\int_{|x-y|<r} \frac{V(y)^s}{|x-y|^{n-\alpha}} dy \leq C r^{\alpha-2s} \{1 + r m(x,V)\}^{sk_0},$$
$$\int_{\mathbb{R}^n} \frac{V(y)^s}{\{1 + m(x,V)|x-y|\}^k |x-y|^{n-\alpha}} dy \leq C_k m(x,V)^{2s-\alpha} \tag{2.9}$$

for any $r > 0$, $x \in \mathbb{R}^n$, *and* $V \in B_q$.

In order to prove Theorem 1.1, we need to give the estimates of the fundamental solution of H. Zhong has established the estimates of the fundamental solution of H in [2] when $V(x)$ is a nonnegative polynomial. Recently, Sugano [5] has obtained the polynomial decay estimates of the fundamental solution of H under a weaker condition on V in the following theorem.

Theorem 2.8. *Assume* $V \in B_{n/2}$ *and let* $\Gamma_H(x,y)$ *be the fundamental solution of* H. *For any positive integer* N, *there exists a constant* C_N *such that*

$$0 \leq \Gamma_H(x,y) \leq \frac{C_N}{(1 + m(x,V)|x-y|)^N} \frac{1}{|x-y|^{n-4}}. \tag{2.10}$$

3. Proof of the main results

In this section, we will prove Theorems 1.1 and 1.2.

Theorem 3.1. *Suppose* $V \in B_{q_1}$, $q_1 \geq n/2$. *Then for* $1 < p \leq q_1/2$ *there exists a positive constant* C_p *such that*

$$\left\| V^2 H^{-1} f \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.1}$$

Proof. Let $f \in L^p(\mathbb{R}^n)$ and

$$u(x) = \int_{\mathbb{R}^n} \Gamma_H(x,y) f(y) dy. \tag{3.2}$$

We need to show that

$$\left\| V^2 u \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.3}$$

Write

$$
\begin{aligned}
u(x) &= \int_{|x-y|<r} \Gamma(x,y) f(y)\,dy + \int_{|x-y|\geq r} \Gamma(x,y) f(y)\,dy \\
&= u_1(x) + u_2(x),
\end{aligned} \tag{3.4}
$$

where $r = 1/m(x,V)$.

Because of the self-improvement of the B_{q_1} class, $V \in B_{q_0}$ for some $q_0 > q_1$, we have

$$
\begin{aligned}
|u_1(x)| &\leq C \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-4}}\,dy \\
&\leq C \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right)^{2/q_0} \left(\int_{|x-y|<r} |x-y|^{-(n-4)q'} dy \right)^{1/q'} \\
&= C r^{4-2n/q_0} \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right)^{2/q_0},
\end{aligned} \tag{3.5}
$$

where $1/q' + 2/q_0 = 1$.

Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left| V^2(y) u_1(y) \right|^{q_0/2} dy \\
&\quad \leq C \int_{\mathbb{R}^n} \left(\int_{|x-y|<r} |f(y)|^{q_0/2} dy \right) V(x)^{q_0} m(x,V)^{n-2q_0} dx \\
&\quad = C \int_{\mathbb{R}^n} |f(y)|^{q_0/2} \left(\int_{|x-y|<1/m(x,V)} V(x)^{q_0} m(x,V)^{n-2q_0} dx \right) dy.
\end{aligned} \tag{3.6}
$$

Now, let $R = 1/m(y,V)$. Then

$$
\begin{aligned}
\int_{|x-y|<1/m(x,V)} V(x)^{q_0} m(x,V)^{n-2q_0} dx &\leq C R^{2q_0-n} \int_{|x-y|<CR} V(x)^{q_0} dx \\
&\leq C R^{2q_0} \left(R^{-n} \int_{|x-y|<CR} V(x)\,dx \right)^{q_0} \\
&= C \left(\frac{1}{R^{n-2}} \int_{|x-y|<CR} V(x)\,dx \right)^{q_0} \\
&\leq C,
\end{aligned} \tag{3.7}
$$

where we used (1.2), Lemmas 2.3 and 2.4.

Hence, we have proved that for some $q_0 > q_1 \geq n/2$,

$$\int_{\mathbb{R}^n} \left|V^2(x)u_1(x)\right|^{q_0/2} dx \leq C\int_{\mathbb{R}^n} \left|f(x)\right|^{q_0/2} dx. \tag{3.8}$$

By choosing $s = 2$, $\alpha = 4$, and $r = 1/m(x,V)$ in Lemma 2.7, we immediately have

$$\int_{|x-y|<1/m(x,V)} \frac{V^2(x)}{|x-y|^{n-4}} dx \leq 4^{k_0}. \tag{3.9}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^n} \left|V^2(x)u_1(x)\right| dx &\leq C\int_{\mathbb{R}^n} \left|f(y)\right| \left(\int_{|x-y|<1/m(x,V)} \frac{V^2(x)}{|x-y|^{n-4}} dx\right) dy \\ &\leq C_{k_0}\int_{\mathbb{R}^n} \left|f(y)\right| dy.\end{aligned} \tag{3.10}$$

Therefore, by using interpolation we have

$$\left\|V^2 u_1\right\|_{L^{p_1}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1}(\mathbb{R}^n)} \quad \text{for } 1 \leq p_1 \leq \frac{q_0}{2}. \tag{3.11}$$

Then we deal with u_2.

For $1 < p \leq q_0/2$, by the Hölder inequality,

$$\begin{aligned}\left|u_2(x)\right| &\leq C\int_{|x-y|\geq r} \frac{|f(y)|dy}{\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}} \\ &\leq C\left(\int_{|x-y|\geq r} \frac{|f(y)|^p dy}{\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}}\right)^{1/p} \\ &\quad\times\left(\int_{|x-y|\geq r} \frac{dy}{\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}}\right)^{1/p'} \\ &= Cr^{4/p'}\left(\int_{|x-y|\geq r} \frac{|f(y)|^p dy}{\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}}\right)^{1/p},\end{aligned} \tag{3.12}$$

where $r = 1/m(x,V)$ and we apply the second inequality for $s = 0$ and $\alpha = 4$ in Lemma 2.7 to the last step.

Thus, for $1 \le p \le q_0/2$,

$$\int_{\mathbb{R}^n} \left|V^2(x)u_2(x)\right|^p dx$$
$$\le C\int_{\mathbb{R}^n} \left|f(y)\right|^p \left(\int_{|x-y|\ge 1/m(x,V)} \frac{\left|V(x)\right|^{2p} dx}{m(x,V)^{4p-4}\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}}\right) dy. \tag{3.13}$$

Fix $y \in \mathbb{R}^n$ and let $R = 1/m(y,V)$. By Lemmas 2.4, 2.6, and 2.7,

$$\int_{|x-y|\ge 1/m(x,V)} \frac{\left|V(x)\right|^{2p} dx}{m(x,V)^{4p-4}\left(1+|x-y|m(x,V)\right)^N |x-y|^{n-4}}$$
$$\le C\int_{|x-y|\ge 1/m(x,V)} \frac{\left|V(x)\right|^{2p} dx}{R^{4-4p}\left(1+|x-y|R^{-1}\right)^{N_1} |x-y|^{n-4}} \quad \left(N_1 = \frac{N-4(p-1)l_0}{l_0+1}\right) \tag{3.14}$$
$$\le C_k \frac{1}{R^{4-4p}} m(y,V)^{4p-4}$$
$$\le C$$

if we choose N large enough.

From this, we have

$$\int_{\mathbb{R}^n} \left|V^2(x)u_2(x)\right|^p dx \le \int_{\mathbb{R}^n} \left|f(x)\right|^p dx \quad \text{for } 1 \le p \le \frac{q_0}{2}. \tag{3.15}$$

Thus the theorem is proved. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $V \in B_{q_1}$ for some $q_1 \ge n/2$. By Theorem 3.1, we have

$$\left\|V^2\left((-\Delta)^2 + V^2\right)^{-1} f\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 \le p \le \frac{q_1}{2}. \tag{3.16}$$

It follows that

$$\left\|(-\Delta)^2\left((-\Delta)^2 + V^2\right)^{-1} f\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 \le p \le \frac{q_1}{2}. \tag{3.17}$$

Because $\nabla^4(-\Delta)^{-2}$ is a Calderón-Zygmund operator, for $1 < p \le q_1/2$, we have

$$\left\|\nabla^4\left((-\Delta)^2 + V^2\right)^{-1} f\right\|_{L^p(\mathbb{R}^n)} \le C_p\left\|(-\Delta)^2\left((-\Delta)^2 + V^2\right)^{-1} f\right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.18}$$

□

Proof of Theorem 1.2. Note that $\nabla^4(-\Delta)^{-2}$ satisfies

$$\left|\left\{x \in \mathbb{R}^n : \left|\nabla^4(-\Delta)^{-2} f(x)\right| \geq \lambda\right\}\right| \leq \frac{C_1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}. \tag{3.19}$$

Thus, by the proof of Theorem 1.1,

$$\begin{aligned}\left|\left\{x \in \mathbb{R}^n : \left|\nabla^4\left((-\Delta)^2 + V^2\right)^{-1} f(x)\right| \geq \lambda\right\}\right| &\leq \frac{C_1}{\lambda}\left\|(-\Delta)^2\left((-\Delta)^2 + V^2\right)^{-1} f\right\|_{L^1(\mathbb{R}^n)} \\ &\leq \frac{C_1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{3.20}$$

□

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