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Research Article

Commutator Length of Finitely Generated Linear Groups

Mahboubeh Alizadeh Sanati

Department of Sciences, University of Golestan, P.O. Box 49165-386 Gorgan, Golestan, Iran

Correspondence should be addressed to Mahboubeh Alizadeh Sanati, malizadeh@gau.ac.ir

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The commutator length "cl(G)" of a group G is the least natural number c such that every element of the derived subgroup of G is a product of c commutators. We give an upper bound for cl(G) when G is a d-generator nilpotent-by-abelian-by-finite group. Then, we give an upper bound for the commutator length of a soluble-by-finite linear group over $\mathbf{C}$ that depends only on d and the degree of linearity. For such a group G, we prove that cl(G) is less than $k(k+1)/2 + 12d^3 + o(d^2)$, where k is the minimum number of generators of (upper) triangular subgroup of G and $o(d^2)$ is a quadratic polynomial in d. Finally we show that if G is a soluble-by-finite group of Prüffer rank r then $\mathrm{cl}(G) \leq r(r+1)/2 + 12r^3 + o(r^2)$, where $o(r^2)$ is a quadratic polynomial in r.

1. Introduction and Motivation

We recall that if g is a nonidentity element of the commutator subgroup of an arbitrary nonabelian group G, denote by cl(g)that the least integer such that g can be written as a product of cl(g) commutators. The *commutator length* of G is defined as

$$\mathrm{cl}(G) = \sup\{\mathrm{cl}(g) \mid g \in G\}. \tag{1.1}$$

It is assumed that the commutator length of the identity element is zero.

In the present paper, we study cl(G) when G is a soluble-by-finite linear group. In general, the commutator length of a linear group need not be finite. For example, the group G, as a subgroup $\mathrm{GL}(2,\mathbf{Q})$, generated by

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

is free and cl(G) is infinite [1]. Also, the *rank* of a group is d if the minimum number of its generators is d. In this case, G is called d-generator, too.

In 1966, Stroud [2] proved that the commutator length of a free nilpotent group of rank n is at most n. After about twenty years, analogous result was obtained for free metabelian nilpotent groups of rank n by Allambergenov and Roman'kov [3]. Using geometric methods, Goldstein and Turner [4] and Culler [5], extended by Bardakov [1] using permutational method, obtained an algorithm for computing the commutator length of elements of free groups. Many other people have worked on the following: "when is $\mathrm{cl}(G)$ finite, if G is free soluble group of rank k and solvability length d?" In some special cases the answer is positive; one can see [6–8].

In this paper, we have restricted our attention to groups which are nilpotent by abelian by finite. In the case where G is a linear group over a field of characteristic zero, the upper bound depends only on the number of generators of the group and the degree of linearity. Recall that the *Prüffer rank* of a group G is the least cardinal r such that every finitely generated subgroup of G can be generated by r elements. Our main results are as follows.

Theorem A. *Let $G = \langle x_1, \dots, x_d\rangle$ be a group with a nilpotent-by-abelian normal k-generator subgroup H of finite index. Then,*

$$\mathrm{cl}(G) \leq \frac{k(k+1)}{2} + 12d^3 + o\left(d^2\right), \tag{1.3}$$

where $o(d^2)$ is a quadratic polynomial in d.

Theorem B. *Let $G = \mathrm{GL}(n, F)$ be a d-generator linear group, where* $\mathrm{char}\, F = 0$ *and G does not contain a nonabelian free group. Then,*

$$\mathrm{cl}(G) \leq \frac{m(m+1)}{2} + 12d^3 + o\left(d^2\right), \tag{1.4}$$

where m is the number of generators of a triangularizable normal subgroup T of G, and $o(d^2)$ is a quadratic polynomial in d. Moreover, the number m is bounded by a function of n and d.

Corollary 1.1. *Let G be a soluble-by-finite group of finite Prüffer rank r. Then,*

$$\mathrm{cl}(G) \leq \frac{r(r+1)}{2} + 12r^3 + o\left(r^2\right), \tag{1.5}$$

where $o(r^2)$ is a quadratic polynomial in r.

Corollary 1.2. *Let G be a linear group of finite Prüffer rank r. Then,*

$$\mathrm{cl}(G) \leq \frac{r(r+1)}{2} + 12r^3 + o\left(r^2\right), \tag{1.6}$$

where $o(r^2)$ is a quadratic polynomial in r.

2. Theorem A

One begins with known results on commutator length that one needs in the proof of Theorem A.

Lemma 2.1 (Hartley [6]). *Let $G = \langle x_1, \ldots, x_d \rangle$ and let H be a nilpotent normal subgroup of G. Suppose that $H = \langle y_1^G, \ldots, y_k^G \rangle$ is generated by the conjugacy classes in G of elements $y_1, \ldots, y_k$. Then, every element of $[H, G]$ can be expressed in the form*

$$\prod_{i=1}^{d} [a_i, x_i] \prod_{j=1}^{k} [b_j, y_j] , \tag{2.1}$$

where a_i's and b_j's are in H and the product is taken for definiteness in order of increasing suffices.

Lemma 2.2 (Stroud [2]). *Let $G = \langle x_1, \ldots, x_d \rangle$ be a nilpotent group. Then, every element of G' can be expressed as the product of d commutators $[x_1, g_1], \ldots, [x_d, g_d]$ for suitable $g_i \in G$ $(1 \leq i \leq d)$.*

Now, we can prove the following lemma that is based on an argument of Rhemtulla [8]. Note that a weaker bound can be obtained from the theorem of Akhavan-Malayeri [9].

Lemma 2.3. *Let $G = \langle x_1, \ldots, x_d \rangle$ be a nilpotent-by-abelian group, then*

$$\mathrm{cl}(G) \leq \frac{d(d+1)}{2} . \tag{2.2}$$

Proof. We know that $G' = \langle [x_r, x_s]^G \mid 1 \leq r, s \leq d \rangle$ is nilpotent, since G is nilpotent by abelian. So, by Hartley's theorem, every element of $[G', G]$ can be expressed in the form $\prod_{i=1}^{d} [a_i, x_i] \prod_{j=1}^{k} [b_j, y_j]$, where a_i's and b_j's are in G' and $y_j \in \{[x_r, x_s] : 1 \leq r, s \leq d\}$ and $k = d(d-1)/2$. But $G/[G', G]$ is always nilpotent and by Stroud's theorem every element of G' has the form $\prod_{i=1}^{d} [x_i, g_i]$ modulo $[G', G]$ for some $g_i \in G$. Hence, every element of $g \in G'$ can be expressed in the form

$$g = \prod_{i=1}^{d} [x_i, g_i] \left(\prod_{i=1}^{d} [a_i, x_i] \prod_{j=1}^{k} [b_j, y_j] \right). \tag{2.3}$$

We can write $\prod_{i=1}^{d} [x_i, g_i] \prod_{i=1}^{d} [a_i, x_i] = [x_d, g_d] \cdots [x_1, g_1][a_1, x_1] \cdots [a_d, x_d]$. By severally useing the equality $[a, b][c, a] = [cb^{-1}, a]^{b^a}$, one sees that every element of G' has the following form:

$$g = \prod_{i=1}^{d} [a_i g_i^{-1}, x_i]^{a_i^{x_i} c_i} \prod_{i=1}^{k} [b_j, y_j] , \tag{2.4}$$

for suitable elements $c_i \in G'$. Hence, the assertion holds. □

The proof of Theorem A requires the following lemma that has been recently proved by Nikolov and Segal [10].

Lemma 2.4 (Nikolov and Segal [10]). *Let G be any d-generator finite group and let H be a normal subgroup of G. There exists a function $\alpha = \alpha(d)$ such that every element of $[H, G]$ is equal to a product of $\alpha = \alpha(d)$ commutators $[u, v]$ with $u \in H$ and $v \in G$. In particular* $\mathrm{cl}(G) \leq 12d^3 + o(d^2)$, *where $o(d^2)$ is a quadratic polynomial in d.*

Proof of Theorem A. We know $\mathrm{cl}(G) \le \mathrm{cl}(G/H') + \mathrm{cl}(H)$. By Lemma 2.3, we conclude $\mathrm{cl}(G) \le \mathrm{cl}(G/H') + k(k+1)/2$. We now aim to find an upper bound for $\mathrm{cl}(G/H')$. Without loss of generality, we can assume that $H' = 1$. So H is abelian and by Lemma 3 from [7], every element of $[H, G]$ can be expressed as a product of at most d commutators, $\Pi_{i=1}^{d}[x_i, h_i]$, where $h_i \in H$. Hence, we can assume that $[H, G] = 1$. Then, H is central, and G is a central-by-finite group. Therefore, G' is finite. Since H is finitely generated abelian group, H^n is torsion free for some positive integer n. So $G' \cap H^n = 1$ and $G' \simeq (G/H^n)'$. But G/H^n is a finite group and by Segal's theorem, every element of its derived subgroup (and so G') is a product of at most $12d^3 + o(d^2)$ commutators, where $o(d^2)$ is a quadratic polynomial in d. That is, every element of G' modulo H' is a product of $12d^3 + o(d^2)$ commutators. Now, the proof is completed. □

Proof of Corollary 1.1. We can assume that G is generated by r elements. According to Robinson's theorem (see [11, Section 10.3]), G is nilpotent-by-abelian-by-finite group. Now, the result holds from Theorem A. □

Lemma 2.5 (Platonov [12]). *Let G be a subgroup of $\mathrm{GL}(n, F)$ of finite Prüffer rank r. Then, G is soluble by finite, and if $\mathrm{char}\, F = p > 0$ then G contains an abelian normal subgroup of finite index bounded in terms of r, n, and p.*

Proof of Corollary 1.2. Without loss of generality, we can assume that G is r-finitely generated. Then, according to Platonov's theorem G is soluble by finite. Now, the result follows from Corollary 1.1. □

In the case G is d-generator soluble group, we have the following result which was obtained by Akhavan-Malayeri [9, Theorem 1].

Theorem 2.6 (Akhavan-Malayeri [9]). *Let H be a nilpotent-by-abelian normal subgroup of finite index of a d-generator soluble group G. Then,*

$$\mathrm{cl}(G) \le \frac{k(k+1)}{2} + 72d^2 + 47d, \tag{2.5}$$

where k is the number of generators of H.

3. Theorem B

The next set of results deals with soluble-by-finite linear groups over a field of characteristic zero and one tries to find the number of generators of the upper triangular subgroups of these groups.

Lemma 3.1 (Wehrfritz [13]). *Let G be a soluble subgroup of $\mathrm{GL}(n, F)$ and let N be a unipotent (nilpotent) normal subgroup of G. Then, G contains a triangularizable normal subgroup T containing N such that the index of T in G is finite and divides*

$$f(n) = \left(\max_{\Sigma_{i=1}^{t} n_i = n} \left\{ \prod_{i=1}^{t} \left(n_i^2 (n_i^2 !) \right)^{n_i} n_i! \right\} \right)!. \tag{3.1}$$

Lemma 3.2 (Tits [14]). *If G is a linear group over a field of characteristic $p \ge 0$ that does not contain nonabelian free subgroups, then if $p = 0$, G contains a soluble normal subgroup of finite index and if $p > 0$, G contains a soluble normal locally finite subgroup.*

In Tits's article, it was proved that every finitely generated linear group which does not contain a nonabelian free group has a soluble subgroup of finite index. If one can calculate an upper bound for this index, then by using the method of the next lemma one can obtain an upper bound for the commutator length of a finitely generated linear group which does not contain a nonabelian free group, in general case. In the case char$F = 0$, one can use the following theorem.

Lemma 3.3 (Jordan [13]). *If* char $F = 0$, *and* G *is a soluble-by-finite subgroup of* $\mathrm{GL}(n, F)$, *then* G *contains a soluble normal subgroup of finite index at most* $\beta(n) = (\sqrt{8n} + 1)^{2n^2} - (\sqrt{8n} - 1)^{2n^2}$.

Proof of Theorem B. By Tits's theorem, G is soluble by finite and so by Jordan's theorem, G has a soluble normal subgroup H of finite index at most $\beta(n)$. Wehrfritz's theorem says that H has a nilpotent-by-abelian subgroup T such that the index of T in H divides $f(n)$. So we have

$$[G : T] = [G : H][H : T] \leq \beta(n) f(n) . \tag{3.2}$$

We consider $g(n) = \beta(n) \cdot f(n)$. We know by Nielsen-Schrier's theorem that the rank of T is at most $m = dg(n) + 1 - g(n)$. Now, the result holds by Theorem A. □

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