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Research Article

On Some Inequalities of Uncertainty Principles Type in Quantum Calculus

Ahmed Fitouhi,[1] Néji Bettaibi,[2] Rym H. Bettaieb,[2] and Wafa Binous[3]

[1] *Faculté des Sciences de Tunis, Université de Tunis El Manar, 1060 Tunis, Tunisia*
[2] *Institut Préparatoire aux Études d'Ingénieur de Monastir, Université de Monastir, 5000 Monastir, Tunisia*
[3] *Institut de Biotechnologie, Université de Jendouba, 9000 Béja, Tunisia*

Correspondence should be addressed to Néji Bettaibi, neji.bettaibi@ipein.rnu.tn

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The aim of this paper is to generalize the q-Heisenberg uncertainty principles studied by Bettaibi et al. (2007), to state local uncertainty principles for the q-Fourier-cosine, the q-Fourier-sine, and the q-Bessel-Fourier transforms, then to provide an inequality of Heisenberg-Weyl-type for the q-Bessel-Fourier transform.

1. Introduction

The uncertainty principle is a metatheorem in harmonic analysis that asserts, with the use of some inequalities, that a function and its Fourier transform cannot be sharply localized. We refer to the survey article by Folland and Sitaram [1] and the book of Havin and Jöricke [2] for various classical uncertainty principles of different nature which may be found in the literature.

In [3], the authors gave q-analogues of the Heisenberg uncertainty principle for the q-Fourier-cosine and the q-Fourier-sine transforms. One of the aims of this paper is to provide a generalization of their work next to state local uncertainty principles for various q-Fourier transforms.

This paper is organized as follows. In Section 2, we present some preliminaries results and notations that will be useful in the sequel. In Section 3, we prove a density theorem and a q-analogue of the Hausdorff-Young inequality. Then, we state a generalization of the q-Heisenberg uncertainty principle for the q-Fourier-cosine and the q-Fourier-sine transforms. In Section 4, we state local uncertainty principles for the q-Fourier-cosine, q-Fourier-sine, and

q-Bessel-Fourier transforms. Then, we give a Heisenberg-Weyl-type inequality for some q-Bessel-Fourier transform.

2. Notations and preliminaries

Throughout this paper, we assume $q \in]0,1[$. We recall some usual notions and notations used in the q-theory (see [4, 5]). We refer to the book by Gasper and Rahman [4] for the definitions, notations, and properties of the q-shifted factorials and the q-hypergeometric functions.

We write $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$, and

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \qquad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \tag{2.1}$$

The q-derivative of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if } x \neq 0, \tag{2.2}$$

$(D_q f)(0) = \lim_{k\to+\infty}(D_q f)(q^k)$, provided that the limit exists.

The q-Jackson integrals from 0 to a and from 0 to ∞, of a function f, are (see [6])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \qquad \int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{2.3}$$

provided that the sums converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [6])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.4}$$

The q -integration by parts rule is given, for suitable functions f and g, by

$$\int_a^b g(x)D_qf(x)d_qx = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_qg(x)d_qx. \tag{2.5}$$

Jackson (see [6]) defined a q-analogue of the Gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots. \tag{2.6}$$

The third Jackson q-Bessel function (see [7, 8]) is

$$J_\nu(z;q^2) = \frac{z^\nu}{(1-q^2)^\nu \Gamma_{q^2}(\nu+1)}\, {}_1\varphi_1(0; q^{2\nu+2}; q^2, q^2z^2), \tag{2.7}$$

and the q-trigonometric functions (q-cosine and q-sine) are defined by (see [9])

$$\begin{aligned} \cos(x;q^2) &= \frac{\Gamma_{q^2}(1/2)}{q(1+q^{-1})^{1/2}} x^{1/2} J_{-1/2}\left(\frac{1-q}{q}x;q^2\right) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)} \frac{x^{2n}}{[2n]_q!}, \\ \sin(x;q^2) &= \frac{\Gamma_{q^2}(1/2)}{(1+q^{-1})^{1/2}} x^{1/2} J_{1/2}\left(\frac{1-q}{q}x;q^2\right) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)} \frac{x^{2n+1}}{[2n+1]_q!}. \end{aligned} \tag{2.8}$$

They verify

$$D_q\cos(x;q^2) = -\frac{1}{q}\sin(qx;q^2), \qquad D_q\sin(x;q^2) = \cos(x;q^2). \tag{2.9}$$

We need the following spaces and norms.

(i) $\mathcal{S}_{*q}(\mathbb{R}_q)$ is the space of even functions f on $\mathbb{R}_q$ satisfying

$$\forall n,m\in\mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q;\,0\le k\le n} \left|(1+x^2)^m D_q^k f(x)\right| < +\infty. \tag{2.10}$$

(ii) $L_q^n(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$, $n\ge 1$, $\nu\ge -1/2$, is the set of all functions defined on $\mathbb{R}_{q,+}$ such that

$$\|f\|_{n,\nu,q} = \left\{\int_0^{\infty}\left|f(x)\right|^n x^{2\nu+1}d_qx\right\}^{1/n} < \infty. \tag{2.11}$$

(iii) $L_q^n(\mathbb{R}_{q,+}) = L_q^n(\mathbb{R}_{q,+}, d_qx)$, $n\ge 1$, and $\|\cdot\|_{n,q} = \|\cdot\|_{n,-1/2,q}$.

(iv) $L_q^{\infty}(\mathbb{R}_{q,+})$ is the set of all bounded functions on $\mathbb{R}_{q,+}$. We write $\|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_{q,+}}|f(x)|$.

3. Generalization of the Heisenberg uncertainty principle

The q-Fourier-cosine and the q-Fourier-sine transforms are defined as (see [8, 9])

$$\mathcal{F}_q(f)(x) = c_q\int_0^{\infty} f(t)\cos(xt;q^2)d_qt, \qquad {}_q\mathcal{F}(f)(x) = c_q\int_0^{\infty} f(t)\sin(xt;q^2)d_qt, \tag{3.1}$$

where

$$c_q = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)}. \tag{3.2}$$

Letting $q\uparrow 1$ subject to the condition $(\mathrm{Log}(1-q)/\mathrm{Log}(q))\in\mathbb{Z}$ gives, at least formally, the classical Fourier transforms (see [3, 10]). In the remainder of the present paper, we assume that this condition holds.

It was shown in [8, 9] that we have the following result.

Proposition 3.1. (1) *For* $f \in L_q^1(\mathbb{R}_{q,+})$*, one has* $\mathcal{F}_q(f) \in L_q^\infty(\mathbb{R}_{q,+})$ *and*

$$\left\|\mathcal{F}_q(f)\right\|_{\infty,q} \leq \frac{\left(1+q^{-1}\right)^{1/2}}{\Gamma_{q^2}(1/2)(q;q)_\infty^2}\|f\|_{1,q}. \tag{3.3}$$

(2) $\mathcal{F}_q$ *is an isomorphism of* $L_q^2(\mathbb{R}_{q,+})$ *(resp.,* $S_{*,q}(\mathbb{R}_q)$*) onto itself. Moreover, one has* $\mathcal{F}_q^{-1} = \mathcal{F}_q$ *and the following Plancherel formula:*

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q}, \quad f \in L_q^2(\mathbb{R}_{q,+}). \tag{3.4}$$

Similarly, it was shown in [3, 8] that the q-Fourier-sine transform verifies the following properties.

Proposition 3.2. (1) *For* $f \in L_q^1(\mathbb{R}_{q,+})$*, one has* ${}_q\mathcal{F}(f) \in L_q^\infty(\mathbb{R}_{q,+})$ *and*

$$\|{}_q\mathcal{F}(f)\|_{\infty,q} \leq \frac{\left(1+q^{-1}\right)^{1/2}}{\Gamma_{q^2}(1/2)(q;q)_\infty^2}\|f\|_{1,q}. \tag{3.5}$$

(2) ${}_q\mathcal{F}$ *is an isomorphism of* $L_q^2(\mathbb{R}_{q,+})$ *onto itself; its inverse is given by* ${}_q\mathcal{F}^{-1} = (1/q^2)_q\mathcal{F}$*. One has the following Plancherel formula:*

$$\|{}_q\mathcal{F}(f)\|_{2,q} = q\|f\|_{2,q}, \quad f \in L_q^2(\mathbb{R}_{q,+}). \tag{3.6}$$

Let us now state the following useful density result.

Proposition 3.3. *For all* $n \geq 1$*,* $S_{*,q}(\mathbb{R}_q)$ *is dense in* $L_q^n(\mathbb{R}_{q,+})$*.*

Proof. Let $n \geq 1$ and $f \in L_q^n(\mathbb{R}_{q,+})$. For $p \in \mathbb{N}$, put $f_p = f\cdot\chi_{[q^p,q^{-p}]}$, where $\chi_{[q^p,q^{-p}]}$ is the characteristic function of $[q^p, q^{-p}]$.

It is clear that for all $p \in \mathbb{N}$, $f_p \in S_{*,q}(\mathbb{R}_q)$ and $|f - f_p|^n \leq |f|^n$. So, the Lebesgue theorem implies that $(f_p)_p$ converges to f in $L_q^n(\mathbb{R}_{q,+})$. □

Remark 3.4. Using the density of $S_{*,q}(\mathbb{R}_q)$ in $L_q^n(\mathbb{R}_{q,+})$ ($n \geq 1$), one can see that the q-Fourier-cosine (resp., q-Fourier-sine) transform has a unique continuous extension on $L_q^n(\mathbb{R}_{q,+})$, that will also be denoted as $\mathcal{F}_q$ (resp., ${}_q\mathcal{F}$). We have the following q-analogue of the Hausdorff-Young inequality.

Theorem 3.5. *Let* $n \in]1,2]$ *(resp.,* $n = 1$*) and* $m = n/(n-1)$ *(resp.,* $m = \infty$*) be the dual exponent of* n*. For all* f *in* $L_q^n(\mathbb{R}_{q,+})$*, the functions* $\mathcal{F}_q(f)$ *and* ${}_q\mathcal{F}(f)$ *belong to* $L_q^m(\mathbb{R}_{q,+})$*, and one has*

$$\|\mathcal{F}_q(f)\|_{m,q} \leq C_1\|f\|_{n,q}, \qquad \|{}_q\mathcal{F}(f)\|_{m,q} \leq C_2\|f\|_{n,q}, \tag{3.7}$$

where

$$C_1 = \left(\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)(q;q)_\infty^2} \right)^{1-2((n-1)/n)}, \qquad C_2 = \left(\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)(q;q)_\infty^2} \right)^{1-2((n-1)/n)} q^{2((n-1)/n)}. \tag{3.8}$$

Proof. The result is a direct consequence of [11, Theorem 1.3.4, page 35], and Propositions 3.1 and 3.2, by taking $S_{*,q}(\mathbb{R}_q)$ as a set of simple functions. □

The following lemma gives relations between the two Fourier q-trigonometric transforms.

Lemma 3.6. (1) *For* $f \in L_q^2(\mathbb{R}_{q,+})$ *such that* $D_q f \in L_q^2(\mathbb{R}_{q,+})$, *one has*

$$_q\mathcal{F}(D_q f)(\lambda) = -\frac{\lambda}{q} \mathcal{F}_q(f)\left(\frac{\lambda}{q}\right), \quad \lambda \in \mathbb{R}_{q,+}. \tag{3.9}$$

(2) *Additionally, if* $\lim_{n\to+\infty} f(q^n) = 0$, *then*

$$\mathcal{F}_q(D_q f)(\lambda) = \frac{\lambda}{q^2}\, {_q\mathcal{F}}(f)(\lambda), \quad \lambda \in \mathbb{R}_{q,+}. \tag{3.10}$$

Proof. The same steps as in the proof of [3, Lemma 2]; the q-integration by parts rule and the fact that

$$\int_0^\infty f(t)\, d_q t = \lim_{n\to+\infty} \int_{q^n}^{q^{-n}} f(t)\, d_q t \tag{3.11}$$

give the result. □

In [3], the authors proved the following q-analogues of the Heisenberg uncertainty principle.

Theorem 3.7. *Let* f *be in* $L_q^2(\mathbb{R}_{q,+})$ *such that* $D_q f$ *is in* $L_q^2(\mathbb{R}_{q,+})$. *Then,*

$$\|tf\|_{2,q}\, \|\lambda \mathcal{F}_q(f)\|_{2,q} \geq \frac{q}{q^{3/2}+1} \|f\|_{2,q}^2. \tag{3.12}$$

In addition, if $\lim_{n\to+\infty} f(q^n) = 0$, *one has*

$$\|tf\|_{2,q}\, \|\lambda\, {_q\mathcal{F}}(f)\|_{2,q} \geq \frac{q}{q^{-3/2}+1} \|f\|_{2,q}^2. \tag{3.13}$$

Now, we are in a position to generalize Theorem 3.7. One obvious way to generalize it is to replace the L_q^2 norms by L_q^n norms. This is the purpose of the following result.

Theorem 3.8. *For* $1 \leq n \leq 2$ *and* $f \in L_q^2(\mathbb{R}_{q,+})$, *one has*

$$\|f\|_{2,q}^2 \leq C_1' \|xf\|_{n,q}\, \|\lambda \mathcal{F}_q(f)\|_{n,q}, \tag{3.14}$$

$$\|f\|_{2,q}^2 \leq C_2' \|xf\|_{n,q}\, \|\lambda\, {_q\mathcal{F}}(f)\|_{n,q}, \tag{3.15}$$

where

$$C_1' = q^{-1+1/n}\left(1+q^{-(n+1)/n}\right)C_2, \qquad C_2' = q^{-1}\left(1+q^{-(n+1)/n}\right)C_1, \tag{3.16}$$

with C_1 *and* C_2 *being given by* (3.8).

Proof. The case $n=2$ has been dealt with in Theorem 3.7. Now, assume $1 \le n < 2$ and let m be the dual exponent of n. Let $f \in S_{*,q}(\mathbb{R}_q)$ such that $\lim_{t\to 0} f(t) = 0$. From the relation

$$D_q(f\overline{f})(t) = D_q f(t)\overline{f}(t) + f(qt)D_q\overline{f}(t), \tag{3.17}$$

the q-integration by parts rule, and the Hölder inequality, we have, since $t|f(t)|^2$ tends to 0 as t tends to ∞ in $\mathbb{R}_{q,+}$,

$$\begin{aligned}
\frac{1}{q}\int_0^\infty |f(t)|^2 d_qt &= \left|\int_0^\infty tD_q(f\overline{f})(t)d_qt\right| \\
&\le \int_0^\infty \left|tD_qf(t)\overline{f}(t)\right|d_qt + \int_0^\infty \left|tf(qt)D_q\overline{f}(t)\right|d_qt \\
&\le \left(\int_0^\infty \left|t\overline{f}(t)\right|^n d_qt\right)^{1/n}\left(\int_0^\infty \left|D_qf(t)\right|^m d_qt\right)^{1/m} \\
&\quad + \left(\int_0^\infty \left|tf(qt)\right|^n d_qt\right)^{1/n}\left(\int_0^\infty \left|D_q\overline{f}(t)\right|^m d_qt\right)^{1/m}.
\end{aligned} \tag{3.18}$$

However, the change of variable $u = qt$ gives

$$\left(\int_0^\infty \left|tf(qt)\right|^n d_qt\right)^{1/n} = q^{-(n+1)/n}\left(\int_0^\infty \left|tf(t)\right|^n d_qt\right)^{1/n}. \tag{3.19}$$

So,

$$\frac{1}{q}\int_0^\infty |f(t)|^2 d_qt \le \left(1+q^{-(n+1)/n}\right)\|tf\|_{n,q}\|D_q(f)\|_{m,q}. \tag{3.20}$$

On the other hand, we have $D_q(f) = \mathcal{F}_q[\mathcal{F}_q(D_q(f))] = q^{-2}{}_q\mathcal{F}[{}_q\mathcal{F}(D_q(f))]$ since $D_q(f)$ is in $L_q^2(\mathbb{R}_{q,+})$. Then, by using Lemma 3.6 and the q-analogue of the Hausdorff-Young inequality, we obtain

$$\begin{aligned}
\|D_q(f)\|_{m,q} &\le C_1\left\|\mathcal{F}_q(D_q(f))\right\|_{n,q} = \frac{C_1}{q^2}\left\|\lambda\, {}_q\mathcal{F}(f)\right\|_{n,q}, \\
\|D_q(f)\|_{m,q} &\le q^{-2}C_2\left\|{}_q\mathcal{F}(D_q(f))\right\|_{n,q} = q^{-2}C_2\left\|\frac{\lambda}{q}\mathcal{F}_q(f)\left(\frac{\lambda}{q}\right)\right\|_{n,q} = q^{-2+1/n}C_2\left\|\lambda\mathcal{F}_q(f)\right\|_{n,q}.
\end{aligned} \tag{3.21}$$

Thus,

$$\begin{aligned}
\|f\|_{2,q}^2 &\le q^{-1}\left(1+q^{-(n+1)/n}\right)C_1\|tf\|_{n,q}\left\|\lambda\, {}_q\mathcal{F}(f)\right\|_{n,q}, \\
\|f\|_{2,q}^2 &\le q^{-1+1/n}\left(1+q^{-(n+1)/n}\right)C_2\|tf\|_{n,q}\left\|\lambda\mathcal{F}_q(f)\right\|_{n,q}.
\end{aligned} \tag{3.22}$$

Now, let $f \in L_q^2(\mathbb{R}_{q,+})$; it is easy to see that for all $p \in \mathbb{N}$, $f_p = f\chi_{[q^p,q^{-p}]} \in S_{*,q}(\mathbb{R}_q)$, $\lim_{t\to 0} f_p(t) = 0$, and $(f_p)_p$ converges to f in $L_q^2(\mathbb{R}_{q,+})$. Moreover, if the right-hand side of (3.14) (resp., (3.15)) is finite, then the functions tf and $\lambda\mathcal{F}_q(f)$ (resp., $\lambda\, {}_q\mathcal{F}(f)$) are in $L_q^n(\mathbb{R}_{q,+})$, and they are limits in $L_q^n(\mathbb{R}_{q,+})$ (as p tends to ∞) of tf_p and $\lambda\mathcal{F}_q(f_p)$ (resp., $\lambda\, {}_q\mathcal{F}(f_p)$), respectively. Finally, the substitution of f_p in (3.22) and a passage to the limit when p tends to ∞ complete the proof. □

4. Local uncertainty principles

In the literature, the first classical local inequalities were obtained by Faris (see [12]) in 1978, and they were generalized by Price (see [13, 14]) in 1983 and 1987. In this section, we will generalize Price's results by giving their q-analogues.

4.1. Local uncertainty principles for the q-Fourier trigonometric transforms

Theorem 4.1. *If* $0 < a < 1/2$, *there is a constant* $K = K(a,q)$ *such that for all bounded subset* E *of* $\mathbb{R}_{q,+}$ *and all* $f \in L_q^2(\mathbb{R}_{q,+})$, *one has*

$$\int_E |\mathcal{F}_q(f)(\lambda)|^2 d_q\lambda \leq K|E|^{2a} \|x^a f\|_{2,q}^2. \tag{4.1}$$

Here, $|E| = \int_0^\infty \chi_E(x) d_q x$ *and* $K = ((\widetilde{c}_q/\sqrt{[1-2a]_q})((1-2a)/2a))^{4a}(1/(1-2a)^2)$, *where* $\widetilde{c}_q = (1+q^{-1})^{1/2}/\Gamma_{q^2}(1/2)(q;q)_\infty^2$.

Proof. For $r > 0$, let $\chi_r = \chi_{[0,r]}$ be the characteristic function of $[0,r]$ and $\widetilde{\chi}_r = 1 - \chi_r$.

Then, for $r > 0$, we have, since $f\cdot\chi_r \in L_q^1(\mathbb{R}_{q,+})$,

$$\begin{aligned}\left(\int_E |\mathcal{F}_q(f)(\lambda)|^2 d_q\lambda\right)^{1/2} &= \|\mathcal{F}_q(f)\chi_E\|_{2,q} \leq \|\mathcal{F}_q(f\cdot\chi_r)\chi_E\|_{2,q} + \|\mathcal{F}_q(f\cdot\widetilde{\chi}_r)\chi_E\|_{2,q} \\ &\leq |E|^{1/2}\|\mathcal{F}_q(f\cdot\chi_r)\|_{\infty,q} + \|\mathcal{F}_q(f\cdot\widetilde{\chi}_r)\|_{2,q},\end{aligned} \tag{4.2}$$

and by the use of the Hölder inequality, we obtain

$$\begin{aligned}\|\mathcal{F}_q(f\cdot\chi_r)\|_{\infty,q} &\leq \widetilde{c}_q\|f\cdot\chi_r\|_{1,q} \\ &= \widetilde{c}_q\|x^{-a}\chi_r\cdot x^a f\|_{1,q} \leq \widetilde{c}_q\|x^{-a}\chi_r\|_{2,q}\|x^a f\|_{2,q} \leq \frac{\widetilde{c}_q}{\sqrt{[1-2a]_q}} r^{1/2-a}\|x^a f\|_{2,q}.\end{aligned} \tag{4.3}$$

On the other hand, since $f \in L_q^2(\mathbb{R}_{q,+})$, we have $f\cdot\widetilde{\chi}_r \in L_q^2(\mathbb{R}_{q,+})$, and by the Plancherel formula, we get

$$\|\mathcal{F}_q(f\cdot\widetilde{\chi}_r)\|_{2,q} = \|f\cdot\widetilde{\chi}_r\|_{2,q} = \|x^{-a}\widetilde{\chi}_r.x^a f\|_{2,q} \leq \|x^{-a}\widetilde{\chi}_r\|_{\infty,q}\|x^a f\|_{2,q} \leq r^{-a}\|x^a f\|_{2,q}. \tag{4.4}$$

So,

$$\left(\int_E |\mathcal{F}_q(f)(\lambda)|^2 d_q\lambda\right)^{1/2} \leq \left(\frac{\widetilde{c}_q}{\sqrt{[1-2a]_q}}|E|^{1/2} r^{1/2-a} + r^{-a}\right)\|x^a f\|_{2,q}. \tag{4.5}$$

The desired result is obtained by minimizing the right-hand side of the previous inequality over $r > 0$. □

Corollary 4.2. *For $0 < a < 1/2$ and $b > 0$, there is a constant $K_{a,b}$ such that for all $f \in L_q^2(\mathbb{R}_{q,+})$, one has*

$$\|f\|_{2,q}^{(a+b)} \leq K_{a,b}\left\|x^a f\right\|_{2,q}^{b}\left\|\lambda^b \mathcal{F}_q(f)\right\|_{2,q}^{a}. \tag{4.6}$$

Proof. For $r > 0$, put $E_r = [0, r[\, \cap\, \mathbb{R}_{q,+}$ and $\widetilde{E}_r = [r, +\infty[\, \cap\, \mathbb{R}_{q,+}$. It is easy to see that E_r is a bounded subset of $\mathbb{R}_{q,+}$ and $|E_r| \leq r$.

Then, from the Plancherel formula and Theorem 4.1, we have

$$\begin{aligned}\|f\|_{2,q}^2 &= \left\|\mathcal{F}_q(f)\right\|_{2,q}^2 \\ &= \int_{E_r}\left|\mathcal{F}_q(f)\right|^2(\lambda)d_q\lambda + \int_{\widetilde{E}_r}\left|\mathcal{F}_q(f)\right|^2(\lambda)d_q\lambda \\ &\leq Kr^{2a}\left\|x^a f\right\|_{2,q}^2 + r^{-2b}\left\|\lambda^b \mathcal{F}_q(f)\right\|_{2,q}^2.\end{aligned} \tag{4.7}$$

Choosing $r > 0$ so as to minimize the right-hand side of the inequality, we obtain $\|f\|_{2,q}^2 \leq (K_{a,b}\|x^a f\|_{2,q}^b \|\lambda^b \mathcal{F}_q(f)\|_{2,q}^a)^{2/(a+b)}$, with $K_{a,b} = ((a/b)^{b/(a+b)} + (b/a)^{a/(a+b)})^{(a+b)/2} K^{b/2}$, and K is the constant given in Theorem 4.1. □

In the same way, one can prove the following local uncertainty principle for the q-Fourier-sine transform.

Theorem 4.3. *If $0 < a < 1/2$, there is a constant $K' = K'(a,q)$ such that for all bounded subset E of $\mathbb{R}_{q,+}$ and all $f \in L_q^2(\mathbb{R}_{q,+})$, one has*

$$\int_E \left|{}_q\mathcal{F}(f)(\lambda)\right|^2 d_q\lambda \leq K'|E|^{2a}\left\|x^a f\right\|_{2,q}^2, \tag{4.8}$$

where $K' = ((\widetilde{c}_q/\sqrt{[1-2a]_q})((1-2a)/2qa))^{4a}[1 + 2qa/(1-2a)]^2$.

Corollary 4.4. *For $0 < a < 1/2$ and $b > 0$, there is a constant $K'_{a,b}$ such that for all $f \in L_q^2(\mathbb{R}_{q,+})$, one has*

$$\|f\|_{2,q}^{(a+b)} \leq K'_{a,b}\|x^a f\|_{2,q}^{b}\|\lambda^b {}_q\mathcal{F}(f)\|_{2,q}^{a}, \tag{4.9}$$

with $K'_{a,b} = ((a/b)^{b/(a+b)} + (b/a)^{a/(a+b)})^{(a+b)/2}(K')^{b/2} q^{-(a+b)}$.

Proof. The same steps of Corollary 4.2 give the result. □

Theorem 4.5. *If $a > 1/2$, there is a constant $K_1 = K_1(a,q)$ such that for all bounded subset E of $\mathbb{R}_{q,+}$ and $f \in L_q^2(\mathbb{R}_{q,+})$, one has*

$$\int_E \left|\mathcal{F}_q(f)(\lambda)\right|^2 d_q\lambda \leq K_1|E|\|f\|_{2,q}^{(2-1/a)}\left\|x^a f\right\|_{2,q}^{1/a}, \tag{4.10}$$

$$\int_E \left|\mathcal{F}_q(f)(\lambda)\right|^2 d_q\lambda \leq K_1|E|\|f\|_{2,q}^{(2-1/a)}\left\|x^a f\right\|_{2,q}^{1/a}. \tag{4.11}$$

The proof of this result needs the following lemmas.

Lemma 4.6. *Suppose $a > 1/2$, then for all $f \in L_q^2(\mathbb{R}_{q,+})$, such that $x^a f \in L_q^2(\mathbb{R}_{q,+})$,*

$$\|f\|_{1,q}^2 \leq K_2\left[\|f\|_{2,q}^2 + \left\|x^a f\right\|_{2,q}^2\right], \tag{4.12}$$

where $K_2 = K_2(a,q) = (1-q)((q^{2a}, q^{2a}, -q, -q^{2a-1}; q^{2a})_\infty / (q, q^{2a-1}, -q^{2a}, -1; q^{2a})_\infty)$.

Proof. From [15, Example 1], and the Hölder inequality, we have

$$\|f\|_{1,q}^2 = \left[\int_0^{+\infty} \left(1+x^{2a}\right)^{1/2} |f(x)| \left(1+x^{2a}\right)^{-1/2} d_q x\right]^2 \leq K_2\left[\|f\|_{2,q}^2 + \|x^a f\|_{2,q}^2\right], \tag{4.13}$$

where $K_2 = \int_0^{+\infty} \left(1+x^{2a}\right)^{-1} d_q x = (1-q)((q^{2a}, q^{2a}, -q, -q^{2a-1}; q^{2a})_\infty / (q, q^{2a-1}, -q^{2a}, -1; q^{2a})_\infty)$. □

Lemma 4.7. *Suppose $a > 1/2$, then for all $f \in L_q^2(\mathbb{R}_{q,+})$, such that $x^a f \in L_q^2(\mathbb{R}_{q,+})$, one has*

$$\|f\|_{1,q} \leq K_3 \|f\|_{2,q}^{(1-1/2a)} \left\|x^a f\right\|_{2,q}^{1/2a}, \tag{4.14}$$

where $K_3 = K_3(a,q) = \left[2aK_2(2aq-q)^{1/2a-1}\right]^{1/2}$.

Proof. For $s \in \mathbb{R}_{q,+}$, define the function f_s by $f_s(x) = f(sx)$, $x \in \mathbb{R}_{q,+}$.

We have $\|f_s\|_{1,q} = s^{-1}\|f\|_{1,q}$, $\|x^a f_s\|_{2,q}^2 = s^{-2a-1}\|x^a f\|_{2,q}^2$.

Replacement of f by f_s in Lemma 4.6 gives

$$\|f\|_{1,q}^2 \leq K_2\left[s\|f\|_{2,q}^2 + s^{-2a+1}\|x^a f\|_{2,q}^2\right]. \tag{4.15}$$

Now, for all $r > 0$, put $\alpha(r) = \mathrm{Log}(r)/\mathrm{Log}(q) - E(\mathrm{Log}(r)/\mathrm{Log}(q))$. We have $s = (r/q^{\alpha(r)}) \in \mathbb{R}_{q,+}$ and $r \leq s < r/q$. Then, for all $r > 0$,

$$\|f\|_{1,q}^2 \leq K_2\left[\frac{r}{q}\|f\|_{2,q}^2 + r^{-2a+1}\left\|x^a f\right\|_{2,q}^2\right]. \tag{4.16}$$

The right-hand side of this inequality is minimized by choosing

$$r = (2a-1)^{1/2a} q^{1/2a} \|f\|_{2,q}^{-1/a} \left\|x^a f\right\|_{2,q}^{1/a}. \tag{4.17}$$

When this is done, we obtain the result. □

Proof of Theorem 4.5. Since the proofs of the two statements are similar, it is sufficient to prove (4.11).

Let E be a bounded subset of $\mathbb{R}_{q,+}$. When the right-hand side of the inequality (4.11) is finite, Lemma 4.6 implies that $f \in L_q^1(\mathbb{R}_{q,+})$; so $\mathcal{F}_q(f)$ is defined and bounded on $\mathbb{R}_{q,+}$. Using

Proposition 3.1, Lemma 4.7, and the fact that

$$\int_E |\mathcal{F}_q(f)(\lambda)|^2 d_q\lambda \le |E| \|\mathcal{F}_q(f)\|^2_{\infty,q}, \tag{4.18}$$

we obtain the result with $K_1 = ((1+q^{-1})/\Gamma^2_{q^2}(1/2)(q;q)^4_\infty)K_3^2$. □

Remark 4.8. By the same technique as in the proof of Corollary 4.2, we can show that Theorem 4.5 leads to inequalities (4.6) and (4.9) with some different constants.

4.2. Local uncertainty principles for the q-Bessel-Fourier transform

The q-Bessel-Fourier transform is defined (see [16]) for $f \in L^1_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$ by

$$\mathcal{F}_{\nu,q}(f)(\lambda) = c_{\nu,q}\int_0^\infty f(x) j_\nu(\lambda x; q^2) x^{2\nu+1} d_qx, \tag{4.19}$$

where

$$j_\nu(z;q^2) = (1-q^2)^\nu \Gamma_{q^2}(\nu+1)((1-q)q^{-1}z)^{-\nu} J_\nu((1-q)q^{-1}z;q^2) \tag{4.20}$$

is the normalized third Jackson q-Bessel function, and

$$c_{\nu,q} = \frac{(1+q^{-1})^{-\nu}}{\Gamma_{q^2}(\nu+1)}. \tag{4.21}$$

It was shown in [10] that for $\nu \ge -1/2$, we have the following result.

Theorem 4.9. (1) *For $f \in L^1_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$, one has $\mathcal{F}_{\nu,q}(f) \in L^\infty_q(\mathbb{R}_{q,+})$ and*

$$\|\mathcal{F}_{\nu,q}(f)\|_{\infty,q} \le \frac{c_{\nu,q}}{(q;q^2)^2_\infty}\|f\|_{1,\nu,q}. \tag{4.22}$$

(2) *$\mathcal{F}_{\nu,q}$ is an isomorphism of $L^2_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$ onto itself, $\mathcal{F}^{-1}_{\nu,q} = q^{4\nu+2}\mathcal{F}_{\nu,q}$, and one has the following Plancherel formula:*

$$\forall f \in L^2_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx), \quad \|\mathcal{F}_{\nu,q}\|_{2,\nu,q} = q^{2\nu+1}\|f\|_{2,\nu,q}. \tag{4.23}$$

The following result states a local uncertainty principle for the q-Bessel-Fourier transform.

Theorem 4.10. *For $\nu \ge -1/2$ and $0 < a < \nu+1$, there is a constant $K_{a,\nu} = K(a,\nu,q)$ such that for all $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$ and all bounded subset E of $\mathbb{R}_{q,+}$, one has*

$$\int_E |\mathcal{F}_{\nu,q}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \le K_{a,\nu}|E|_\nu^{a/(\nu+1)}\|x^a f\|^2_{2,\nu,q}. \tag{4.24}$$

Here, $|E|_\nu = \int_0^\infty \chi_E(x)x^{2\nu+1}d_qx$, $\tilde{c}_{\nu,q} = c_{\nu,q}/(q;q^2)^2_\infty$, and

$$K_{a,\nu} = \left(\frac{\tilde{c}_{\nu,q}}{\sqrt{[2\nu+2-2a]_q}}\right)^{2a/(\nu+1)} \left[\left(\frac{aq^{2\nu+1}}{\nu+1-a}\right)^{1-a/(\nu+1)} + q^{2\nu+1}\left(\frac{aq^{2\nu+1}}{\nu+1-a}\right)^{-a/(\nu+1)}\right]^2. \tag{4.25}$$

Proof. Let $\nu \geq -1/2$, $0 < a < \nu + 1$, $f \in L_q^2(\mathbb{R}_{q,+}, x^{2\nu+1} d_q x)$, and let E be a bounded subset of $\mathbb{R}_{q,+}$. For $r > 0$, we have, since $f.\chi_r \in L_q^1(\mathbb{R}_{q,+}, x^{2\nu+1} d_q x)$,

$$\begin{aligned}\left(\int_E \left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2 \lambda^{2\nu+1} d_q\lambda\right)^{1/2} &= \left\|\mathcal{F}_{\nu,q}(f)\chi_E\right\|_{2,\nu,q} \\ &\leq \left\|\mathcal{F}_{\nu,q}(f\cdot\chi_r)\chi_E\right\|_{2,\nu,q} + \left\|\mathcal{F}_{\nu,q}(f\cdot\widetilde{\chi}_r)\chi_E\right\|_{2,\nu,q} \\ &\leq |E|_\nu^{1/2}\left\|\mathcal{F}_{\nu,q}(f\cdot\chi_r)\right\|_{\infty,q} + \left\|\mathcal{F}_{\nu,q}(f\cdot\widetilde{\chi}_r)\right\|_{2,\nu,q}.\end{aligned} \tag{4.26}$$

However, by the use of the Hölder inequality, we obtain

$$\begin{aligned}\left\|\mathcal{F}_{\nu,q}(f\cdot\chi_r)\right\|_{\infty,q} &\leq \widetilde{c}_{\nu,q}\left\|f\cdot\chi_r\right\|_{1,q} \\ &= \widetilde{c}_q\left\|x^{-a}\chi_r . x^a f\right\|_{1,\nu,q} \\ &\leq \widetilde{c}_{\nu,q}\left\|x^{-a}\chi_r\right\|_{2,\nu,q}\left\|x^a f\right\|_{2,\nu,q}.\end{aligned} \tag{4.27}$$

Now, if k is the integer such that $q^k \leq r < q^{k-1}$, we get, since $a < \nu + 1$,

$$\left\|x^{-a}\chi_r\right\|_{2,\nu,q}^2 = \int_0^\infty x^{-2a}\chi_r(x)x^{2\nu+1}d_q x = \int_0^{q^k} x^{2\nu+1-2a}d_q x = \frac{q^{2k(\nu+1-a)}}{[2\nu+2-2a]_q} \leq \frac{r^{2(\nu+1-a)}}{[2\nu+2-2a]_q}. \tag{4.28}$$

Then,

$$\left\|\mathcal{F}_{\nu,q}(f\cdot\chi_r)\right\|_{\infty,q} \leq \frac{\widetilde{c}_{\nu,q}}{\sqrt{[2\nu+2-2a]_q}} r^{(\nu+1-a)}\left\|x^a f\right\|_{2,\nu,q}. \tag{4.29}$$

On the other hand, since $f \in L_q^2(\mathbb{R}_{q,+}, x^{2\nu+1} d_q x)$, we have $f.\widetilde{\chi}_r \in L_q^2(\mathbb{R}_{q,+}, x^{2\nu+1} d_q x)$, and by the Plancherel formula (4.23), we obtain

$$\begin{aligned}\left\|\mathcal{F}_{\nu,q}(f\cdot\widetilde{\chi}_r)\right\|_{2,\nu,q} &= q^{2\nu+1}\left\|f\cdot\widetilde{\chi}_r\right\|_{2,\nu,q} = q^{2\nu+1}\left\|x^{-a}\widetilde{\chi}_r\cdot x^a f\right\|_{2,\nu,q} \\ &\leq q^{2\nu+1}\left\|x^{-a}\widetilde{\chi}_r\right\|_{\infty,q}\left\|x^a f\right\|_{2,q} \leq q^{2\nu+1}r^{-a}\left\|x^a f\right\|_{2,\nu,q}.\end{aligned} \tag{4.30}$$

So,

$$\left(\int_E \left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2 \lambda^{2\nu+1} d_q\lambda\right)^{1/2} \leq \left(\frac{\widetilde{c}_{\nu,q}}{\sqrt{[2\nu+2-2a]_q}}|E|_\nu^{1/2} r^{(\nu+1-a)} + q^{2\nu+1}r^{-a}\right)\left\|x^a f\right\|_{2,\nu,q}. \tag{4.31}$$

By minimization of the right-hand side of the previous inequality over $r > 0$ and by easy computation, we obtain the desired result. □

Theorem 4.11. *For $\nu \geq -1/2$ and $a > \nu+1$, there exists a constant $K'_{a,\nu}$ such that for all bounded subset E of $\mathbb{R}_{q,+}$ and all f in $L^2_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$, one has*

$$\int_E \left|\mathcal{F}_{\nu,q}(f)(\lambda)\right|^2 \lambda^{2\nu+1} d_q\lambda \leq K'_{a,\nu}|E| \, \|f\|_{2,\nu,q}^{2(1-(\nu+1)/a)} \left\|x^a f\right\|_{2,\nu,q}^{2((\nu+1)/a)}. \tag{4.32}$$

Proof. Since $a > \nu+1$, the same steps as in the proof of Theorem 4.5 and the relation (4.22) give the result with

$$K'_{a,\nu} = \frac{(q^{2a}, q^{2a}, -q^{2\nu+2}, -q^{2(a-\nu-1)}; q^{2a})_\infty}{(q^{2\nu+2}, q^{2(a-\nu-1)}, -q^{2a}, -1; q^{2a})_\infty} c'_{\nu,q},$$
$$c'_{\nu,q} = (1-q)\left(\frac{c_{\nu,q}}{(q;q^2)^2_\infty}\right)^2 \left(\frac{a}{\nu+1} - 1\right)^{(\nu+1)/a} \left(\frac{a}{a-\nu-1}\right) q^{-2(\nu+1)((a-\nu-1)/a)}. \tag{4.33}$$

□

Corollary 4.12. *For $\nu \geq -1/2$ and $a, b > 0$, there is a constant $K_{a,b,\nu} = K(a,b,\nu,q)$ such that for all $f \in L^2_q(\mathbb{R}_{q,+}, x^{2\nu+1}d_qx)$, one has*

$$\|f\|_{2,\nu,q}^{(a+b)} \leq K_{a,b,\nu} \|x^a f\|_{2,\nu,q}^{b} \|\lambda^b \mathcal{F}_{\nu,q}(f)\|_{2,\nu,q}^{a}, \tag{4.34}$$

with

$$K_{a,b,\nu} = \begin{cases} \left[\left(\dfrac{b}{a}\right)^{a/(a+b)} + \left(\dfrac{a}{b}\right)^{b/(a+b)}\right]^{(a+b)/2} (K_{a,\nu})^{b/2} \dfrac{q^{-(2\nu+1)(a+b)}}{([2\nu+2]_q)^{ab/2(\nu+1)}} \\ \qquad \text{if } a < \nu+1, \\ \left(\dfrac{K'_{a,\nu}}{[2\nu+2]_q}\right)^{ab/(2\nu+2)} \left(q^{-(4\nu+2)}\left[\left(\dfrac{b}{\nu+1}\right)^{(\nu+1)/(\nu+b+1)} + \left(\dfrac{b}{\nu+1}\right)^{-b/(\nu+b+1)}\right]\right)^{a(\nu+b+1)/2(\nu+1)} \\ \qquad \text{if } a > \nu+1, \end{cases} \tag{4.35}$$

where $K_{a,\nu}$ (resp., $K'_{a,\nu}$) is the constant given in Theorem 4.10 (resp., Theorem 4.11).

Proof. For $r > 0$, we put $E_r = [0, r[\,\cap\, \mathbb{R}_{q,+}$ and $\widetilde{E}_r = [r, +\infty[\,\cap\, \mathbb{R}_{q,+}$.

We have E_r is a bounded subset of $\mathbb{R}_{q,+}$ and $|E_r|_\nu \leq r^{2\nu+2}/[2\nu+2]_q$. Then, the Plancherel formula (4.23) and Theorems 4.10 and 4.11 lead to

$$\begin{aligned} q^{4\nu+2}\|f\|^2_{2,\nu,q} &= \left\|\mathcal{F}_{\nu,q}(f)\right\|^2_{2,\nu,q} = \int_{E_r} \left|\mathcal{F}_{\nu,q}(f)\right|^2(\lambda)\lambda^{2\nu+1}d_q\lambda + \int_{\widetilde{E}_r} \left|\mathcal{F}_{\nu,q}(f)\right|^2(\lambda)\lambda^{2\nu+1}d_q\lambda \\ &\leq \begin{cases} K_{a,\nu}\left|E_r\right|_\nu^{a/(\nu+1)} \left\|x^a f\right\|^2_{2,\nu,q} + r^{-2b}\left\|\lambda^b \mathcal{F}_{\nu,q}(f)\right\|^2_{2,\nu,q} & \text{if } a < \nu+1, \\ K'_{a,\nu}\left|E_r\right| \|f\|_{2,\nu,q}^{2(a-\nu-1)/a} \left\|x^a f\right\|_{2,\nu,q}^{2(\nu+1)/a} + r^{-2b}\left\|\lambda^b \mathcal{F}_{\nu,q}(f)\right\|^2_{2,\nu,q} & \text{if } a > \nu+1, \end{cases} \\ &\leq \begin{cases} \dfrac{K_{a,\nu}}{[2\nu+2]_q^{a/(\nu+1)}} r^{2a}\left\|x^a f\right\|^2_{2,\nu,q} + r^{-2b}\left\|\lambda^b \mathcal{F}_{\nu,q}(f)\right\|^2_{2,\nu,q} & \text{if } a < \nu+1, \\ K'_{a,\nu}\dfrac{r^{2\nu+2}}{[2\nu+2]_q} \|f\|_{2,\nu,q}^{2(a-\nu-1)/a} \left\|x^a f\right\|_{2,\nu,q}^{2(\nu+1)/a} + r^{-2b}\left\|\lambda^b \mathcal{F}_{\nu,q}(f)\right\|^2_{2,\nu,q} & \text{if } a > \nu+1. \end{cases} \end{aligned} \tag{4.36}$$

The desired result follows by minimizing the right expressions over $r > 0$. □

Remark that when $a = b = 1$, we obtain a Heisenberg-Weyl-type inequality for the q-Bessel-Fourier transform.

Corollary 4.13. *For $\nu \geq -1/2$, $\nu \neq 0$, one has for all $f \in L_q^2(\mathbb{R}_{q,+}, x^{2\nu+1} d_q x)$,*

$$\|f\|_{2,\nu,q}^2 \leq K_{1,1,\nu} \|xf\|_{2,\nu,q} \left\| \lambda \mathcal{F}_{\nu,q}(f) \right\|_{2,\nu,q}. \tag{4.37}$$

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