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Research Article

A Note on Locally Inverse Semigroup Algebras

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Let R be a commutative ring and S a finite locally inverse semigroup. It is proved that the semigroup algebra $R[S]$ is isomorphic to the direct product of Munn algebras $\mathcal{M}\ (R[G_J], m_J, n_J; P_J)$ with $J \in S/\mathcal{J}$, where m_J is the number of $\mathcal{R}$-classes in J, n_J the number of $\mathcal{L}$-classes in J, and G_J a maximum subgroup of J. As applications, we obtain the sufficient and necessary conditions for the semigroup algebra of a finite locally inverse semigroup to be semisimple.

1. Main results

A regular semigroup S is called a *locally inverse semigroup* if for all idempotent $e \in S$, the local submonoid eSe is an inverse semigroup under the multiplication of S. Inverse semigroups are locally inverse semigroups. Inverse semigroup algebras are a class of semigroup algebras which is widely investigated. One of fundamentally important results is that a finite inverse semigroup algebra is the direct product of full matrix algebras over group algebras of the maximum subgroups of this finite inverse semigroup. Consider that all local submonoids of a locally inverse semigroup are inverse semigroups, it is a very natural problem whether a finite locally inverse semigroup algebra has a similar representation to inverse semigroup algebras. This is the main topic of this note.

Let $\mathcal{A}$ be an R-algebra. Let m and n be positive integers, and let P be a fixed $n \times m$ matrix over $\mathcal{A}$. Let $\mathcal{M} := \mathcal{M}(\mathcal{A}; m, n; P)$ be the vector space of all $m \times n$ matrices over $\mathcal{A}$. Define a product $\circ$ in $\mathcal{M}$ by

$$A \circ B = APB \quad (A, B \in \mathcal{M}), \tag{1.1}$$

where APB is the usual matrix product of A, P, and B. Then $\mathcal{M}$ is an algebra over R. Following [1], we call $\mathcal{M}$ the Munn $m \times n$ *matrix algebra over* $\mathcal{A}$ *with sandwich matrix* P.

By a *semisimple semigroup*, we mean a semigroup each of whose principal factor is either a completely 0-simple semigroup or a completely simple semigroup. It is well known that a finite regular semigroup is semisimple. The Rees theorem tells us that any completely 0-simple semigroup (completely simple semigroup) is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G, I, \Lambda; P)$ ($\mathcal{M}(G, I, \Lambda; P)$), and vice versa (for Rees matrix semigroups, refer to [1]). In what follows, by the phrase "Let $S = \bigcup_{J\in S/\mathcal{J}} \mathcal{M}^0(G_J; I_J, \Lambda_J; P_J)$ be a finite regular semigroup," we mean that S is a finite regular semigroup in which the principal factor of S determined by the $\mathcal{J}$-class J is isomorphic to the Rees matrix semigroup $\mathcal{M}^0(G_J; I_J, \Lambda_J; P_J)$ or $\mathcal{M}(G_J; I_J, \Lambda_J; P_J)$ for any $J \in S/\mathcal{J}$.

The following is the main result of this paper.

Theorem 1.1. *Let $S = \bigcup_{J\in S/\mathcal{J}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semigroup algebra $R[S]$ is isomorphic to the direct product of $\mathcal{M}(R[G_J]; |I_J|, |\Lambda_J|; P_J)$ with $J \in S/\mathcal{J}$.*

Based on Theorem 1.1 and [1, Lemma 5.17, page 162, and Lemma 5.18, page 163], the following corollary is straightforward.

Corollary 1.2. *Let $S = \bigcup_{J\in S/\mathcal{J}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semigroup algebra $R[S]$ has an identity if and only if $|I_J| = |\Lambda_J|$ and P_J is invertible in the full matrix algebra $M_{|I_J|}(R[G_J])$ for all $J \in S/\mathcal{J}$.*

Reference [1, Lemma 5.18, page 163] told us that $\mathcal{M}(R[G_J], m_J, n_J; P_J)$ is isomorphic to the full matrix algebra $M_{n_J}(R[G_J])$ if $\mathcal{M}(R[G_J], m_J, n_J; P_J)$ has an identity. Now, we have the following.

Corollary 1.3. *Let $S = \bigcup_{J\in S/\mathcal{J}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. If $R[S]$ has an identity, then $R[S]$ is isomorphic to the direct product of the full matrix algebras $M_{|I_J|}(R[G_J])$ with $J \in S/\mathcal{J}$.*

The following corollary is a consequence of Corollary 1.3.

Corollary 1.4. *Let $S = \bigcup_{J\in S/\mathcal{J}} \mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$ be a finite locally inverse semigroup. Then the semigroup algebra $R[S]$ is semisimple if and only if for all $J \in S/\mathcal{J}$,*

(1) $|I_J| = |\Lambda_J|$;

(2) *P_J is invertible in the full matrix algebra $M_{|I_J|}(R[G_J])$;*

(3) *$R[G_J]$ is semisimple.*

2. Proof of Theorem 1.1

For our purpose, we have the Möbius inversion theorem [2].

Lemma 2.1. *Let $(P, \leq)$ be a locally finite partially ordered set (i.e., intervals are finite) in which each principal ideal has a maximum and G be an Abelian group. Suppose that $f : P \to G$ is a function and define $g : P \to G$ by $g(x) = \sum_{y\leq x} f(y)$. Then $f(x) = \sum_{y\leq x} g(y)\mu(x, y)$, where μ is a Möbius function.*

Now assume that S is a regular semigroup and $a, b \in S$. Define

$$a \leq b \Longleftrightarrow \text{there exist } e, \quad f \in E(S) \text{ such that } a = eb = bf. \tag{2.1}$$

Then $\leq$ is a partial order on S. Following [3], we call $\leq$ the *natural partial order* on S. Equivalently, $a \leq b$ if and only if for every (for some) $f \in E(R_b)$ $(f \in E(L_b))$, there exists $e \in E(R_a)$ $(e \in E(L_a))$ such that $e \leq f$ and $a = eb$ $(a = be)$. Moreover, Nambooripad [3, 4] proved that S is a locally inverse semigroup if and only if the natural partial order $\leq$ is compatible with respect to the multiplication of S.

Lemma 2.2. *Let S be a locally inverse semigroup and $a, b \in S$. Then for any $u \leq ab$, there exist $x \leq a$ and $y \leq b$ such that $u = xy$, $x \in R_u$ and $y \in L_u$.*

Proof. For any $e \in E(R_a)$, we have $ea = a$ and $eab = ab$. Let z be an inverse of ab. Clearly, $abz \in E(R_{ab})$. Note that $eabz = abz$. It is easy to check that $abze \in E(S)$, $abze \leq e$, and $abz\mathcal{R}abze$. Hence $abze\mathcal{R}ab$ and there exists $g \in E(S)$ such that $u = gab$ and $g \leq abze\,(\leq e)$. Thus $ga \leq a$. On the other hand, since $\mathcal{R}$ is a left congruence and since $abze\mathcal{R}ab$, we have $u = gab\mathcal{R}gabze = g$; while since $a\mathcal{R}e$, we have $ga\mathcal{R}ge = g$. These imply that $u\mathcal{R}ga$. Dually, we have $h \in E(S)$ such that $u = abh$, $bh \leq b$ and $u\mathcal{L}bh$. Since $u = gab = abh = uh = (ga)(bh)$, we know that ga and bh are the required elements x and y. □

Define a multiplication $\otimes$ on $S^0 = S \cup \{0\}$ by

$$x \otimes y = \begin{cases} xy & \text{if } x \neq 0,\ y \neq 0, \text{ and } y, xy \in J_x; \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where xy is the product of x and y in S. By the arguments of [4, page 9], $(S^0, \otimes)$ is a semigroup. We denote by $S^{\otimes}$ the semigroup $(S^0, \otimes)$. For any $J \in S/\mathcal{J}$, we denote $J^0 = J \cup \{0\}$. It is easy to check that $(J^0, \otimes)$ is a subsemigroup of $S^{\otimes}$, which is isomorphic to the principal factor of S determined by J. We will denote the semigroup $(J^0, \otimes)$ by $J^{\otimes}$. By the definition of $\otimes$, it is easy to see that in the semigroup $S^{\otimes}$,

(i) $J_x^{\otimes} \otimes J_x^{\otimes} \subseteq J_x^{\otimes}$ for all $x \in S$;

(ii) $J_x^{\otimes} \otimes J_y^{\otimes} = 0$ for all $x, y \in S$ such that $x \notin J_y$.

Thus $R_0[S^{\otimes}]$ is the direct sum of the contracted semigroup algebras $R_0[J^{\otimes}]$ with $J \in S/\mathcal{J}$. Note that $J^{\otimes}$ is isomorphic to some principal factor of S. We observe that $J^{\otimes}$ is a completely 0-simple semigroup since S is a semisimple semigroup, and thus $J^{\otimes}$ is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)$. By a result of [1], $R_0[\mathcal{M}^0(G_J, I_J, \Lambda_J; P_J)]$ is isomorphic to $\mathcal{M}(R[G_J], |I_J|, |\Lambda_J|; P_J)$. Consequently, to verify Theorem 1.1, we need only to prove that $R[S]$ is isomorphic to $R_0[S^{\otimes}]$.

For the convenience of description, we introduce the semigroup $\overline{S}$. Put $\overline{S} = \{\overline{x} \mid x \in S\} \cup \{0\}$. Define a multiplication on $\overline{S}$ as follows:

$$\overline{x} * \overline{y} = \overline{x \otimes y}, \tag{2.3}$$

where we will identify $\overline{0}$ with 0. It is easy to see that $\overline{S}$ is isomorphic to $S^{\otimes}$. Hence the contracted semigroup algebra $R_0[\overline{S}]$ is isomorphic to the contracted semigroup algebra $R_0[S^{\otimes}]$. For $J \in S/\mathcal{J}$, we denote $\overline{J} = \{\overline{x} \mid x \in J\} \cup \{0\}$. It is easy to check that $(\overline{J}, *)$ is a subsemigroup of $\overline{S}$ isomorphic to the semigroup $J^{\otimes}$. So, for any $J, K \in S/\mathcal{J}$, we have

$$\overline{J} * \overline{K} \begin{cases} \subseteq \overline{J} & \text{if } K = J, \\ = 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

For Theorem 1.1, it remains to prove the following lemma.

Lemma 2.3. $R[S] \cong R_0[\overline{S}]$.

Proof. We consider the mapping $\varphi : R[S] \to R_0[\overline{S}]$ given on the basis by $\varphi(s) = \sum_{t\leq s}\overline{t}$ $(s \in S)$. Clearly, φ is well defined. Of course, φ and $\overline{\bullet}$ may be regarded as the mappings of the ordered set $(S, \leq)$ into the additive group of $R_0[\overline{S}]$. Now, by applying the Möbius inversion theorem to the mappings φ and $\overline{\bullet}$, we have

$$\overline{s} = \sum_{t\leq s} \varphi(t)\mu(t,s) = \varphi\left(\sum_{t\leq s} t\mu(t,s)\right), \tag{2.5}$$

where μ is the Möbius function for $(S, \leq)$. Hence φ is surjective.

We will prove that φ is injective. For $\alpha_0 = \sum_{x\in S} p_x^0 x \in R[S]$, we denote by supp (α_0) the set $\{x \in S \mid p_x^0 \neq 0\}$ and by $M(\alpha_0)$ the set of maximal elements in the set $\operatorname{supp}(\alpha_0)$ with respect to the partial order $\leq$. In recurrence, we define $\alpha_n = \alpha_{n-1} - \sum_{x\in M(\alpha_{n-1})} p_x^{n-1}x$, where $\alpha_n = \sum_{x\in \operatorname{supp}(\alpha_n)} p_x^n x$. Let $\beta_n = \sum_{x\in\operatorname{supp}(\beta_n)} q_x^n x$ with $n = 0,1,2,\ldots$. If $\varphi(\alpha_n) = \varphi(\beta_n)$, then by the definition of φ, $\sum_{x\in M(\alpha_n)} p_x\overline{x} + \Gamma_{\alpha_n} = \varphi(\alpha_n) = \varphi(\beta_n) = \sum_{y\in M(\beta_n)} q_y^n\overline{y} + \Gamma_{\beta_n}$, where $\Gamma_{\alpha_n} = \sum_{x\in M(\alpha_n)}\sum_{y\in S, y<x} p_y^n\overline{y}$ and $\Gamma_{\beta_n} = \sum_{x\in M(\beta_n)}\sum_{y\in S,y<x} q_y^n\overline{y}$, and hence $\sum_{x\in M(\alpha_n)} p_x^n\overline{x} = \sum_{x\in M(\beta_n)} q_x^n\overline{x}$, thus $M(\alpha_n) = M(\beta_n)$ and $p_x^n = q_x^n$ for any $x \in M(\alpha_n)$. This can imply the following.

Fact 2.4. If $\varphi(\alpha_n) = \varphi(\beta_n)$, then $M(\alpha_n) = M(\beta_n)$ and by the definition of φ, $\varphi(\alpha_{n+1}) = \varphi(\beta_{n+1})$.

By the definition of φ, the following facts are immediate.

Fact 2.5. $\alpha_n = \beta_n$ if and only if $M(\alpha_n) = M(\beta_n)$ and $\alpha_{n+1} = \beta_{n+1}$.

Fact 2.6. If $\varphi(\alpha_n) = \varphi(\beta_n)$ and $M(\alpha_n) = \operatorname{supp}(\alpha_n), M(\beta_n) = \operatorname{supp}(\beta_n)$, then $\alpha_n = \beta_n$.

Note that $|\operatorname{supp}(\alpha_0)| < \infty$ and $\operatorname{supp}(\alpha_{n+1}) \subsetneq \operatorname{supp}(\alpha_n)$. We thus have a smallest integer k such that $M(\alpha_k) = \operatorname{supp}(\alpha_k)$. Clearly, $\alpha_{k+1} = 0$. This means that k is the smallest integer t such that $\alpha_{t+1} = 0$. Similarly, there exists the smallest integer l such that $\beta_{l+1} = 0$ and $M(\beta_l) = \operatorname{supp}(\beta_l)$. Now, assume $\varphi(\alpha_0) = \varphi(\beta_0)$. By using Fact 2.4 repeatedly,

$$\varphi(\alpha_1) = \varphi(\beta_1), \qquad \varphi(\alpha_2) = \varphi(\beta_2), \ldots, \qquad \varphi(\alpha_{k+1}) = \varphi(\beta_{k+1}). \tag{2.6}$$

But $\varphi(\alpha_{k+1}) = 0$, we have $\varphi(\beta_{k+1}) = 0$ and by the definition of φ, $\beta_{k+1} = 0$. Thus $k+1 \geq l+1$ by the minimality of l, and $k \geq l$. Similarly, $l \geq k$. Therefore $k = l$. Since $\varphi(\alpha_k) = \varphi(\beta_k)$, by Fact 2.6, we have $\alpha_k = \beta_k$ since $M(\alpha_k) = \operatorname{supp}(\alpha_k)$ and $M(\beta_l) = \operatorname{supp}(\beta_l)$. Again by the hypothesis $\varphi(\alpha_0) = \varphi(\beta_0)$, and by Fact 2.4, $M(\alpha_0) = M(\beta_0)$; and by (2.6), $M(\alpha_1) = M(\beta_1)$, $M(\alpha_2) = M(\beta_2), \ldots, M(\alpha_k) = M(\beta_k)$. By Fact 2.5, $M(\alpha_{k-1}) = M(\beta_{k-1})$; and $\alpha_k = \beta_k$ imply $\alpha_{k-1} = \beta_{k-1}$; moreover, by using Fact 2.5 repeatedly, $\alpha_{k-2} = \beta_{k-2}, \ldots, \alpha_1 = \beta_1$ and $\alpha_0 = \beta_0$. We have now proved that φ is injective.

Finally, for any $s, t \in S$, by (2.4), we have

$$\overline{s} * \overline{t} = \begin{cases} \overline{st} & \text{if } s,t \in J_{st}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.7}$$

and by Lemma 2.2,

$$\varphi(s)*\varphi(t) = \left(\sum_{x\le s}\overline{x}\right)*\left(\sum_{y\le t}\overline{y}\right) = \sum_{x\in J_{st},x\le s}\ \sum_{y\in J_{st},y\le t}\overline{x}*\overline{y} = \sum_{x\in J_{st},x\le s}\ \sum_{y\in J_{st},y\le t}\overline{xy}. \tag{2.8}$$

Moreover, by Lemma 2.2, we have

$$\varphi(st) = \sum_{u\le st}\overline{u} = \sum_{x\in J_{st},x\le s}\ \sum_{y\in J_{st},y\le t}\overline{xy} = \sum_{x\le s,x\in J_{st}}\ \sum_{y\le t,y\in J_{st}}\overline{x}*\overline{y} = \varphi(s)*\varphi(t). \tag{2.9}$$

Thus φ is a homomorphism of $R[S]$ into $R_0[\overline{S}]$. Consequently, φ is an isomorphism of $R[S]$ onto $R_0[\overline{S}]$. □

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