Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 391839, 9 pages
doi:10.1155/2009/391839

Research Article

A New Iterative Method for Common Fixed Points of a Finite Family of Nonexpansive Mappings

Suwicha Imnang and Suthep Suantai

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Correspondence should be addressed to Suthep Suantai, scmti005@chiangmai.ac.th

Received 16 December 2008; Accepted 9 April 2009

Recommended by Jie Xiao

Let X be a real uniformly convex Banach space and C a closed convex nonempty subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C. For a given $x_1 \in C$, let $\{x_n\}$ and $\{x_n^{(i)}\}$, $i = 1, 2, \dots, r$, be sequences defined $x_n^{(0)} = x_n$, $x_n^{(1)} = a_{n1}^{(1)} T_1 x_n^{(0)} + (1 - a_{n1}^{(1)}) x_n^{(0)}$, $x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + a_{n1}^{(2)} T_1 x_n + (1 - a_{n2}^{(2)} - a_{n1}^{(2)}) x_n$, $\dots$, $x_{n+1} = x_n^{(r)} = a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}) x_n$, $n \geq 1$, where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1, 2, \dots, r\}$, $n \in \mathbb{N}$ and $i = 1, 2, \dots, j$. In this paper, weak and strong convergence theorems of the sequence $\{x_n\}$ to a common fixed point of a finite family of nonexpansive mappings T_i $(i = 1, 2, \dots, r)$ are established under some certain control conditions.

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, $y \in C$. Let $T_i : C \to C$, $i = 1, 2, \dots, r$, be nonexpansive mappings. Let $\mathrm{Fix}(T_i)$ denote the fixed points set of T_i, that is, $\mathrm{Fix}(T_i) := \{x \in C : T_i x = x\}$, and let $F := \bigcap_{i=1}^r \mathrm{Fix}(T_i)$.

For a given $x_1 \in C$, and a fixed $r \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(0)}\}, \{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(r)}\}$ by

$$x_n^{(0)} = x_n,$$

$$x_n^{(1)} = a_{n1}^{(1)} T_1 x_n^{(0)} + \left(1 - a_{n1}^{(1)}\right) x_n^{(0)}$$

$$x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + a_{n1}^{(2)} T_1 x_n + \left(1 - a_{n2}^{(2)} - a_{n1}^{(2)}\right) x_n,$$
$$\vdots$$
$$x_{n+1} = x_n^{(r)} = a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n, \quad n \geq 1, \tag{1.1}$$

where $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\dots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\dots,j$. If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\dots,r-1\}$ and $i = 1,2,\dots,j$, then (1.1) reduces to the iterative scheme

$$x_{n+1} = S_n x_n, \quad n \geq 1, \tag{1.2}$$

where $S_n := a_{nr}^{(r)} T_r + a_{n(r-1)}^{(r)} T_{r-1} + \cdots + a_{n1}^{(r)} T_1 + (1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}) I$, $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\dots,r$ and $n \in \mathbb{N}$.

If $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1,2,\dots,r-1\}$, $i = 1,2,\dots,j$ and $a_{ni}^{(r)} := \alpha_i$, for all $n \in \mathbb{N}$ for all $i = 1,2,\dots,r$, then (1.1) reduces to the iterative scheme defined by Liu et al. [1]

$$x_{n+1} = S x_n, \quad n \geq 1, \tag{1.3}$$

where $S := \alpha_r T_r + \alpha_{r-1} T_{r-1} + \cdots + \alpha_1 T_1 + (1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1) I$, $\alpha_i \geq 0$ for all $i = 2,3,\dots,r$ and $1 - \alpha_r - \alpha_{r-1} - \cdots - \alpha_1 > 0$. They showed that $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of T_i, $i = 1,2,\dots,r$, in Banach spaces, provided that T_i, $i = 1,2,\dots,r$ satisfy condition A. The result improves the corresponding results of Kirk [2], Maiti and Saha [3] and Sentor and Dotson [4].

If $r = 2$ and $a_{n1}^{(2)} := 0$ for all $n \in \mathbb{N}$, then (1.1) reduces to a generalization of Mann and Ishikawa iteration given by Das and Debata [5] and Takahashi and Tamura [6]. This scheme dealts with two mappings:

$$x_n^{(1)} = a_{n1}^{(1)} T_1 x_n + \left(1 - a_{n1}^{(1)}\right) x_n,$$
$$x_{n+1} = x_n^{(2)} = a_{n2}^{(2)} T_2 x_n^{(1)} + \left(1 - a_{n2}^{(2)}\right) x_n, \quad n \geq 1, \tag{1.4}$$

where $\{a_{n1}^{(1)}\}, \{a_{n2}^{(2)}\}$ are appropriate sequences in $[0,1]$.

The purpose of this paper is to establish strong convergence theorems in a uniformly convex Banach space of the iterative sequence $\{x_n\}$ defined by (1.1) to a common fixed point of T_i $(i = 1,2,\dots,r)$ under some appropriate control conditions in the case that one of T_i $(i = 1,2,\dots,r)$ is completely continuous or semicompact or $\{T_i\}_{i=1}^r$ satisfies condition (B). Moreover, weak convergence theorem of the iterative scheme (1.1) to a common fixed point of T_i $(i = 1,2,\dots,r)$ is also established in a uniformly convex Banach spaces having the Opial's condition.

2. Preliminaries

In this section, we recall the well-known results and give a useful lemma that will be used in the next section.

Recall that a Banach space X is said to satisfy *Opial's condition* [7] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$. A finite family of mappings $T_i : C \to C$ $(i = 1, 2, \ldots, r)$ with $F := \bigcap_{i=1}^{r} \mathrm{Fix}(T_i) \neq \emptyset$ is said to satisfy *condition (B)* [8] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\max_{1\le i\le r}\{\|x - T_i x\|\} \ge f(d(x,F))$ for all $x \in C$, where $d(x,F) = \inf\{\|x - p\| : p \in F\}$.

Lemma 2.1 (see [9, Theorem 2]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\left\|\lambda x + (1-\lambda)y\right\|^p \le \lambda\|x\|^p + (1-\lambda)\left\|y\right\|^p - w_p(\lambda)g\left(\left\|x - y\right\|\right), \tag{2.1}$$

for all x, y in $B_r = \{x \in X : \|x\| \le r\}, \lambda \in [0,1]$, where

$$w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda). \tag{2.2}$$

Lemma 2.2 (see [10, Lemma 1.6]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ nonexpansive mapping. Then $I - T$ is demiclosed at 0, that is, if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in Fix(T)$.*

Lemma 2.3 (see [11, Lemma 2.7]). *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that $lim_{n\to\infty}\|x_n - u\|$ and $lim_{n\to\infty}\|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}, r > 0$. Then for each $n \in \mathbb{N}$, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|^2 \le \sum_{i=1}^{n}\alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|), \tag{2.3}$$

for all $x_i \in B_r$ and all $\alpha_i \in [0,1]$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

Proof. Clearly (2.3) holds for $n = 1, 2$, by Lemma 2.1. Next, suppose that (2.3) is true when $n = k - 1$. Let $x_i \in B_r$ and $\alpha_i \in [0,1]$, $i = 1, 2, \ldots, k$ with $\sum_{i=1}^{k}\alpha_i = 1$. Then $\alpha_{k-1}/(1 - \sum_{i=1}^{k-2}\alpha_i)x_{k-1} + \alpha_k/(1 - \sum_{i=1}^{k-2}\alpha_i)x_k \in B_r$. By Lemma 2.1, we obtain that

$$\left\|\frac{\alpha_{k-1}}{1 - \sum_{i=1}^{k-2}\alpha_i}x_{k-1} + \frac{\alpha_k}{1 - \sum_{i=1}^{k-2}\alpha_i}x_k\right\|^2 \le \frac{\alpha_{k-1}}{1 - \sum_{i=1}^{k-2}\alpha_i}\|x_{k-1}\|^2 + \frac{\alpha_k}{1 - \sum_{i=1}^{k-2}\alpha_i}\|x_k\|^2. \tag{2.4}$$

By the inductive hypothesis, there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\|\sum_{i=1}^{k-1}\beta_i y_i\right\|^2 \le \sum_{i=1}^{k-1}\beta_i\|y_i\|^2 - \beta_1\beta_2 g(\|y_1 - y_2\|) \tag{2.5}$$

for all $y_i \in B_r$ and all $\beta_i \in [0,1]$, $i = 1,2,\dots,k-1$ with $\sum_{i=1}^{k-1}\beta_i = 1$. It follows that

$$\begin{aligned}
\left\|\sum_{i=1}^{k}\alpha_i x_i\right\|^2 &= \left\|\sum_{i=1}^{k-2}\alpha_i x_i + \left(1 - \sum_{i=1}^{k-2}\alpha_i\right)\left(\frac{\alpha_{k-1}x_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k x_k}{1-\sum_{i=1}^{k-2}\alpha_i}\right)\right\|^2 \\
&\le \sum_{i=1}^{k-2}\alpha_i\|x_i\|^2 + \left(1 - \sum_{i=1}^{k-2}\alpha_i\right)\left\|\frac{\alpha_{k-1}x_{k-1}}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k x_k}{1-\sum_{i=1}^{k-2}\alpha_i}\right\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|) \\
&\le \sum_{i=1}^{k-2}\alpha_i\|x_i\|^2 + \left(1 - \sum_{i=1}^{k-2}\alpha_i\right)\left(\frac{\alpha_{k-1}\|x_{k-1}\|^2}{1-\sum_{i=1}^{k-2}\alpha_i} + \frac{\alpha_k \|x_k\|^2}{1-\sum_{i=1}^{k-2}\alpha_i}\right) - \alpha_1\alpha_2 g(\|x_1 - x_2\|) \\
&= \sum_{i=1}^{k}\alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|).
\end{aligned} \tag{2.6}$$

Hence, we have the lemma. □

3. Main Results

In this section, we prove weak and strong convergence theorems of the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

The next lemma is crucial for proving the main theorems.

Lemma 3.1. *Let X be a Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^{r}$ be a finite family of nonexpansive self-mappings of C. Let $a_{ni}^{(j)} \in [0,1]$ for all $j \in \{1,2,\dots,r\}$, $n \in \mathbb{N}$ and $i = 1,2,\dots,j$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.1). If $F \neq \emptyset$, then $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.*

Proof. Let $p \in F$. For each $n \ge 1$, we note that

$$\begin{aligned}
\left\|x_n^{(1)} - p\right\| &= \left\|a_{n1}^{(1)}T_1x_n + \left(1 - a_{n1}^{(1)}\right)x_n - p\right\| \\
&\le a_{n1}^{(1)}\|T_1x_n - p\| + \left(1 - a_{n1}^{(1)}\right)\|x_n - p\| \\
&\le a_{n1}^{(1)}\|x_n - p\| + \left(1 - a_{n1}^{(1)}\right)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned}\left\|x_n^{(2)}-p\right\| &= \left\|a_{n2}^{(2)}T_2x_n^{(1)}+a_{n1}^{(2)}T_1x_n+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)x_n-p\right\| \\ &\le a_{n2}^{(2)}\left\|T_2x_n^{(1)}-p\right\|+a_{n1}^{(2)}\|T_1x_n-p\|+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)\|x_n-p\| \\ &\le a_{n2}^{(2)}\left\|x_n^{(1)}-p\right\|+a_{n1}^{(2)}\|x_n-p\|+\left(1-a_{n2}^{(2)}-a_{n1}^{(2)}\right)\|x_n-p\| \\ &\le \|x_n-p\|.\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}\left\|x_n^{(3)}-p\right\| &= \left\|a_{n3}^{(3)}T_3x_n^{(2)}+a_{n2}^{(3)}T_2x_n^{(1)}+a_{n1}^{(3)}T_1x_n+\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)x_n-p\right\| \\ &\le a_{n3}^{(3)}\left\|T_3x_n^{(2)}-p\right\|+a_{n2}^{(3)}\left\|T_2x_n^{(1)}-p\right\|+a_{n1}^{(3)}\|T_1x_n-p\| \\ &\quad +\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)\|x_n-p\| \\ &\le a_{n3}^{(3)}\left\|x_n^{(2)}-p\right\|+a_{n2}^{(3)}\left\|x_n^{(1)}-p\right\|+a_{n1}^{(3)}\|x_n-p\| \\ &\quad +\left(1-a_{n3}^{(3)}-a_{n2}^{(3)}-a_{n1}^{(3)}\right)\|x_n-p\| \\ &\le \|x_n-p\|.\end{aligned} \tag{3.3}$$

By continuing the above argument, we obtain that

$$\left\|x_n^{(i)}-p\right\| \le \|x_n-p\| \quad \forall i=1,2,\ldots,r. \tag{3.4}$$

In particular, we get $\|x_{n+1}-p\| \le \|x_n-p\|$ for all $n\in\mathbb{N}$, which implies that $\lim_{n\to\infty}\|x_n-p\|$ exists. □

Lemma 3.2. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F\neq\emptyset$ and $a_{ni}^{(j)}\in[0,1]$ for all $j\in\{1,2,\ldots,r\}$, $n\in\mathbb{N}$ and $i=1,2,\ldots,j$ such that $\sum_{i=1}^{j}a_{ni}^{(j)}$ are in $[0,1]$ for all $j\in\{1,2,\ldots,r\}$ and $n\in\mathbb{N}$. For a given $x_1\in C$, let $\{x_n\}$ be defined by (1.1). If $0<\liminf_{n\to\infty}a_{ni}^{(r)}\le\limsup_{n\to\infty}(a_{n(r)}^{(r)}+a_{n(r-1)}^{(r)}+\cdots+a_{n1}^{(r)})<1$, then*

(i) $\lim_{n\to\infty}\|T_ix_n^{(i-1)}-x_n\|=0$ *for all* $i=1,2,\ldots,r$,

(ii) $\lim_{n\to\infty}\|T_ix_n-x_n\|=0$ *for all* $i=1,2,\ldots,r$,

(iii) $\lim_{n\to\infty}\|x_n^{(i)}-x_n\|=0$ *for all* $i=1,2,\ldots,r$.

Proof. (i) Let $p\in F$, by Lemma 3.1, $\sup_n\|x_n-p\|<\infty$. Choose a number $s>0$ such that $\sup_n\|x_n-p\|<s$, it follows by (3.4) that $\{x_n^{(i)}-p\}$, $\{T_ix_n^{(i-1)}-p\}\subseteq B_s$, for all $i\in\{1,2,\ldots,r\}$. □

By Lemma 2.4, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \alpha_1 \alpha_2 g(\|x_1 - x_2\|), \tag{3.5}$$

for all $x_i \in B_s$, $\alpha_i \in [0,1]$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. By (3.4) and (3.5), we have for $i = 1,2,\ldots,r$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \Big\| a_{nr}^{(r)} T_r x_n^{(r-1)} + a_{n(r-1)}^{(r)} T_{r-1} x_n^{(r-2)} + \cdots + a_{n1}^{(r)} T_1 x_n \\
&\qquad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) x_n - p \Big\|^2 \\
&\leq a_{nr}^{(r)} \left\| T_r x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| T_{r-1} x_n^{(r-2)} - p \right\|^2 + \cdots \\
&\qquad + a_{n1}^{(r)} \|T_1 x_n - p\|^2 + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\qquad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&\leq a_{nr}^{(r)} \left\| x_n^{(r-1)} - p \right\|^2 + a_{n(r-1)}^{(r)} \left\| x_n^{(r-2)} - p \right\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\qquad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\qquad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&\leq a_{nr}^{(r)} \|x_n - p\|^2 + a_{n(r-1)}^{(r)} \|x_n - p\|^2 + \cdots + a_{n1}^{(r)} \|x_n - p\|^2 \\
&\qquad + \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) \|x_n - p\|^2 \\
&\qquad - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \\
&= \|x_n - p\|^2 - a_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right).
\end{aligned} \tag{3.6}$$

Therefore

$$\mathrm{a}_{ni}^{(r)} \left(1 - a_{n(r)}^{(r)} - a_{n(r-1)}^{(r)} - \cdots - a_{n1}^{(r)}\right) g\left(\left\| T_i x_n^{(i-1)} - x_n \right\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{3.7}$$

for all $i = 1,2,\ldots,r$. Since $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty} (a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, it implies by Lemma 3.1 that $\lim_{n\to\infty} g(\|T_i x_n^{(i-1)} - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_i x_n^{(i-1)} - x_n\| = 0$ for all $i = 1,2,\ldots,r$.

(ii) For $i \in \{1, 2, \ldots, r\}$, we have

$$\begin{aligned}\|T_i x_n - x_n\| &\le \left\|T_i x_n - T_i x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \left\|x_n - x_n^{(i-1)}\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\| \\ &\le \sum_{j=1}^{i-1} a_{nj}^{(i-1)} \left\|T_j x_n^{(j-1)} - x_n\right\| + \left\|T_i x_n^{(i-1)} - x_n\right\|.\end{aligned} \tag{3.8}$$

It follows from (i) that

$$\|T_i x_n - x_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.9}$$

(iii) For $i \in \{1, 2, \ldots, r\}$, it follows from (i) that

$$\left\|x_n^{(i)} - x_n\right\| \le \sum_{j=1}^{i} a_{nj}^{(i)} \left\|T_j x_n^{(j-1)} - x_n\right\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.10}$$

Theorem 3.3. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^\infty$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0, 1, \ldots, r)$ be defined by* (1.1). *If one of $\{T_i\}_{i=1}^r$ is completely continuous then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1, 2, \ldots, r$.*

Proof. Suppose that T_{i_0} is completely continuous where $i_0 \in \{1, 2, \ldots, r\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_{i_0} x_{n_k}\}$ converges. □

Let $\lim_{k\to\infty} T_{i_0} x_{n_k} = q$ for some $q \in C$. By Lemma 3.2 (ii), $\lim_{n\to\infty}\|T_{i_0} x_n - x_n\| = 0$. It follows that $\lim_{k\to\infty} x_{n_k} = q$. Again by Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, r$. It implies that $\lim_{k\to\infty} T_i x_{n_k} = q$. By continuity of T_i, we get $T_i q = q$, $i = 1, 2, \ldots, r$. So $q \in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - q\|$ exists, it follows that $\lim_{n\to\infty}\|x_n - q\| = 0$. By Lemma 3.2(iii), we have $\lim_{n\to\infty}\|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1, 2, \ldots, r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = q$ for all $j = 1, 2, \ldots, r$.

Theorem 3.4. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. Let the sequence $\{a_{ni}^{(j)}\}_{n=1}^\infty$ be as in Lemma 3.2. For a given $x_1 \in C$, let sequences $\{x_n\}$ and $\{x_n^{(i)}\}$ $(i = 0, 1, \ldots, r)$ be defined by* (1.1). *If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) then $\{x_n\}$ and $\{x_n^{(j)}\}$ converge strongly to a common fixed point of $\{T_i\}_{i=1}^r$ for all $j = 1, 2, \ldots, r$.*

Proof. Let $p \in F$. Then by Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists and $\|x_{n+1} - p\| \le \|x_n - p\|$ for all $n \ge 1$. This implies that $d(x_{n+1}, F) \le d(x_n, F)$ for all $n \ge 1$, therefore, we get $\lim_{n\to\infty} d(x_n, F)$ exists. By Lemma 3.2(ii), we have $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for each $i = 1, 2, \ldots, r$. It follows, by the condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing and $f(0) = 0$, therefore, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since

$\lim_{n\to\infty} d(x_n, F) = 0$, given any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \epsilon/2$ for all $n \geq n_0$. In particular, $d(x_{n_0}, F) < \epsilon/2$. Then there exists $q \in F$ such that $\|x_{n_0} - q\| < \epsilon/2$. For all $n \geq n_0$ and $m \geq 1$, it follows by Lemma 3.1 that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\| \leq \|x_{n_0} - q\| + \|x_{n_0} - q\| < \epsilon. \tag{3.11}$$

This shows that $\{x_n\}$ is a Cauchy sequence in C, hence it must converge to a point of C. Let $\lim_{n\to\infty} x_n = p^*$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and F is closed, we obtain $p^* \in F$. By Lemma 3.2(iii), $\lim_{n\to\infty}\|x_n^{(j)} - x_n\| = 0$ for each $j \in \{1, 2, \ldots, r\}$. It follows that $\lim_{n\to\infty} x_n^{(j)} = p^*$ for all $j = 1, 2, \ldots, r$. □

In Theorem 3.4, if $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, r-1\}$ and $i = 1, 2, \ldots, j$, we obtain the following result.

Corollary 3.5. *Let X be a uniformly convex Banach space and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(r)} \in [0,1]$ for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^r a_{ni}^{(r)}$ are in $[0,1]$ for all $n \in \mathbb{N}$. For a given $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.2). If the family $\{T_i\}_{i=1}^r$ satisfies condition (B) and $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Remark 3.6. In Corollary 3.5, if $a_{ni}^{(r)} = \alpha_i$, for all $n \in \mathbb{N}$ and for all $i = 1, 2, \ldots, r$, the iterative scheme (1.2) reduces to the iterative scheme (1.3) defined by Liu et al. [1] and we obtain strong convergence of the sequence $\{x_n\}$ defined by Liu et al. when $\{T_i\}_{i=1}^r$ satisfies condition (B) which is different from the condition (A) defined by Liu et al. and we note that the result of Senter and Dotson [4] is a special case of Theorem 3.4 when $r = 1$.

In the next result, we prove weak convergence for the iterative scheme (1.1) for a finite family of nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.7. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (1.1). If the sequence $\{a_{ni}^{(j)}\}_{n=1}^\infty$ is as in Lemma 3.2, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Proof. By Lemma 3.2(ii), $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, r$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality we may assume that $x_n \to u$ weakly as $n \to \infty$ for some $u \in C$. By Lemma 2.2, we have $u \in F$. Suppose that there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ that converge weakly to u and v, respectively. From Lemma 2.2, we have u, $v \in F$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - u\|$ and $\lim_{n\to\infty}\|x_n - v\|$ exist. It follows from Lemma 2.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. □

For $a_{ni}^{(j)} := 0$, for all $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, r-1\}$ and $i = 1, 2, \ldots, j$ in Theorem 3.7, we obtain the following result.

Corollary 3.8. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^r$ be a finite family of nonexpansive self-mappings of C with $F \neq \emptyset$ and $a_{ni}^{(r)} \in [0,1]$ for all $i = 1,2,\ldots,r$ and $n \in \mathbb{N}$ such that $\sum_{i=1}^r a_{ni}^{(r)}$ are in $[0,1]$ for all $n \in \mathbb{N}$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (1.2). If $0 < \liminf_{n\to\infty} a_{ni}^{(r)} \leq \limsup_{n\to\infty}(a_{n(r)}^{(r)} + a_{n(r-1)}^{(r)} + \cdots + a_{n1}^{(r)}) < 1$, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$.*

Remark 3.9. In Corollary 3.8, if $a_{ni}^{(r)} = \alpha_i$, for all $n \in \mathbb{N}$ and for all $i = 1,2,\ldots,r$, then we obtain weak convergence of the sequence $\{x_n\}$ defined by Liu et al. [1].

Acknowledgments

The authors would like to thank the Commission on Higher Education, the Thailand Research Fund, the Thaksin University, and the Graduate School of Chiang Mai University, Thailand for their financial support.

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