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Research Article

A New Iteration Process for Approximation of Common Fixed Points for Finite Families of Total Asymptotically Nonexpansive Mappings

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Let E be a real Banach space, and K a closed convex nonempty subset of E. Let $T_1, T_2, \ldots, T_m : K \to K$ be m total asymptotically nonexpansive mappings. A simple iterative sequence $\{x_n\}_{n\geq 1}$ is constructed in E and necessary and sufficient conditions for this sequence to converge to a common fixed point of $\{Ti\}_{i=1}^m$ are given. Furthermore, in the case that E is a uniformly convex real Banach space, strong convergence of the sequence $\{x_n\}_{n=1}^\infty$ to a common fixed point of the family $\{Ti\}_{i=1}^m$ is proved. Our recursion formula is much simpler and much more applicable than those recently announced by several authors for the same problem.

1. Introduction

Let K be a nonempty subset of a normed real linear space E. A mapping $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0,\infty)$ with $\lim_{n\to\infty}\mu_n = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq (1+\mu_n)\|x - y\| \quad \forall n \geq 1. \tag{1.1}$$

The mapping T is called *uniformly L-Lipschitzian* if there exists a constant $L \geq 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq L\|x - y\| \quad \forall n \geq 1. \tag{1.2}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. They proved that if K is a nonempty closed convex bounded subset of a uniformly convex real Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

A mapping T is said to be *asymptotically nonexpansive in the intermediate sense* (see, e.g., [2]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \left(\|T^n x - T^n y\| - \|x-y\|\right) \le 0. \tag{1.3}$$

Observe that if we define

$$a_n := \sup_{x,y\in K} \left(\|T^n x - T^n y\| - \|x-y\|\right), \qquad \sigma_n = \max\{0, a_n\}, \tag{1.4}$$

then $\sigma_n \to 0$ as $n \to \infty$ and (1.3) reduces to

$$\|T^n x - T^n y\| \le \|x-y\| + \sigma_n, \quad \forall x,y \in K,\ n \ge 1. \tag{1.5}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [3] that if K is a nonempty closed convex bounded subset of a uniformly convex real Banach space E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings (see, e.g., [4]).

Sahu [5], introduced the class of nearly Lipschitzian mappings. Let K be a nonempty subset of a normed space E and let $\{a_n\}_{n\ge1}$ be a sequence in $[0,+\infty)$ such that $\lim_{n\to\infty} a_n = 0$. A mapping $T : K \to K$ is called *nearly Lipschitzian* with respect to $\{a_n\}_{n\ge1}$ if for each $n \in \mathbb{N}$, there exists $k_n \ge 0$ such that

$$\|T^n x - T^n y\| \le k_n\left(\|x-y\| + a_n\right) \quad \forall x,y \in K. \tag{1.6}$$

Define

$$\eta(T^n) := \sup\left\{ \frac{\|T^n x - T^n y\|}{\|x-y\| + a_n} : x,y \in K,\ x \ne y \right\}. \tag{1.7}$$

Observe that for any sequence $\{k_n\}_{n\ge1}$ satisfying (1.6), $\eta(T^n) \le k_n$ for all $n \in \mathbb{N}$ and that

$$\|T^n x - T^n y\| \le \eta(T^n)\left(\|x-y\| + a_n\right) \quad \forall x,y \in K,\ n \in \mathbb{N}. \tag{1.8}$$

$\eta(T^n)$ is called the *nearly Lipschitz constant.* A nearly Lipschitzian mapping T is said to be

(i) *nearly contraction* if $\eta(T^n) < 1$ for all $n \in \mathbb{N}$;

(ii) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$;

(iii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\eta(T^n) = 1$;

(iv) *nearly uniform L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$;

(v) *nearly uniform k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Example 1.1. Let $E = \mathbb{R}$, $K = [0,1]$. Define $T : K \to K$ by

$$Tx = \begin{cases} \dfrac{1}{2}, & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\ 0, & \text{if } x \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{1.9}$$

It is obvious that T is not continuous, and thus, not Lipschitz. However, T is nearly nonexpansive. In fact, for a real sequence $\{a_n\}_{n\geq 1}$ with $a_1 = 1/2$ and $a_n \to 0$ as $n \to \infty$, we have

$$\begin{gathered} \|Tx - Ty\| \leq \|x - y\| + a_1 \quad \forall x, y \in K, \\ \|T^n x - T^n y\| \leq \|x - y\| + a_n \quad \forall x, y \in K,\ n \geq 2. \end{gathered} \tag{1.10}$$

This is because $T^n x = 1/2$ for all $x \in [0,1]$, $n \geq 2$.

Remark 1.2. If K is a bounded domain of an asymptotically nonexpansive mapping T, then T is nearly nonexpansive. In fact, for all $x, y \in K$ and $n \in \mathbb{N}$, we have

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \leq \|x - y\| + \operatorname{diam}(K)\mu_n. \tag{1.11}$$

Furthermore, we easily observe that every nearly nonexpansive mapping is nearly asymptotically nonexpansive with $\eta(T^n) \equiv 1$ for all $n \in \mathbb{N}$.

Remark 1.3. If K is a bounded domain of a nearly asymptotically nonexpansive mapping T, then T is asymptotically nonexpansive in the intermediate sense. To see this, let T be a nearly asymptotically nonexpansive mapping. Then,

$$\|T^n x - T^n y\| \leq \eta(T^n)(\|x - y\| + a_n) \quad \forall x, y \in K,\ n \geq 1, \tag{1.12}$$

which implies that

$$\sup_{x,y\in K}(\|T^n x - T^n y\| - \|x - y\|) \leq (\eta(T^n) - 1)\operatorname{diam}(K) + \eta(T^n)a_n, \quad n \geq 1. \tag{1.13}$$

Hence,

$$\limsup_{n\to\infty}\sup_{x,y\in K}(\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.14}$$

We observe from Remarks 1.2 and 1.3 that the classes of nearly nonexpansive mappings and nearly asymptotically nonexpansive mappings are intermediate classes between the class of asymptotically nonexpansive mappings and that of asymptotically nonexpansive in the intermediate sense mappings.

The main tool for approximation of fixed points of generalizations of nonexpansive mappings remains *iterative technique.* Several authors have studied approximation of fixed points of generalizations of nonexpansive mappings using Mann and Ishikawa iterative methods (see, e.g., [6–19]).

Bose [20] proved that if K is a nonempty closed convex bounded subset of a uniformly convex real Banach space E satisfying Opial's condition [21] (i.e., for all sequences $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for all $y \neq x$ in E) and $T : K \to K$ is an *asymptotically nonexpansive mapping,* then the sequence $\{T^n x\}$ converges *weakly* to a fixed point of T provided that T is *asymptotically regular at* $x \in K$; that is, the limit

$$\lim_{n\to\infty}\left\|T^n x - T^{n+1}x\right\| = 0 \tag{1.15}$$

holds. Passty [13] and also Xu and Noor [22] showed that the requirement of Opial's condition can be replaced by the Fréchet differentiability of the space norm. Furthermore, Tan and Xu [23, 24] established that the asymptotic regularity of T at a point x can be weakened to the so-called *weakly asymptotic regularity of* T *at* x, defined as follows: T is weakly asymptotic regular at $x \in K$ if

$$\omega - \lim_{n\to\infty}\left(T^n x - T^{n+1}x\right) = 0 \tag{1.16}$$

holds, where $\omega - \lim$ denotes the weak limit.

In [17, 18], Schu introduced a modified Mann iteration scheme for approximation of fixed points of asymptotically nonexpansive self-mappings defined on a nonempty closed convex and bounded subset K of a uniformly convex real Banach space E. He proved that the iterative sequence $\{x_n\}$ generated by

$$x_1 \in K, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{1.17}$$

converges *weakly* to some fixed point of T if Opial's condition holds, $\{\mu_n\} \subset [0,\infty)$ for all $n \geq 1$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\{\alpha_n\}$ is a real sequence such that $0 < a \leq \alpha_n \leq b < 1$, $n \geq 1$, for some positive constants a and b. Neither condition (1.15) nor condition (1.16) is required with Schu's scheme. Schu's result, however, does not apply, for instance, to L^p spaces with $p > 1$, $p \neq 2$ because none of these spaces satisfies Opial's condition.

Rhoades [15] obtained a *strong* convergence theorem for asymptotically nonexpansive mappings in uniformly convex real Banach spaces using the modified Ishikawa-type iteration method. Osilike and Aniagbosor proved in [12] that the results of [15, 17, 18] remain true without the boundedness requirement imposed on K, provided that $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Tan and Xu [25] extended the theorem of Schu [18] to uniformly convex Banach space with a Fréchet differentiable norm without assuming that the space satisfies Opial's condition. Thus, their result applies to L^p spaces with $1 < p < \infty$.

Chang et al. [26] established *weak* convergence theorems for asymptotically nonexpansive mappings in Banach spaces without assuming any of the following conditions: (i) E satisfies the Opial's condition; (ii) T is asymptotically regular or weakly asymptotically regular; (iii) K is bounded. Their results improve and generalize the corresponding results of Bose [20], Górnicki [27], Passty [13], Schu [18],Tan and Xu [23–25], Xu and Noor [22], and many others.

G. E. Kim and T. H. Kim [4] studied the strong convergence of the Mann and Ishikawa-type iteration methods *with errors* for mappings which are asymptotically nonexpansive in the intermediate sense in real Banach spaces.

In all the above papers, the mapping T remains *a self-mapping* of nonempty closed convex subset K of a uniformly convex real Banach space E. If, however, the domain $D(T)$ of T is a *proper* subset of E, then the Mann and Ishikawa-type iterative processes and Schu's modifications of type (1.17) may fail to be well defined.

Chidume et al. [28] proved convergence theorems for asymptotically nonexpansive *nonself-mappings* in Banach spaces and extended the corresponding results of [12, 15, 17, 18, 26].

Alber et al. [29] introduced a more general class of asymptotically nonexpansive mappings called *total asymptotically nonexpansive mappings* and studied methods of approximation of fixed points of mappings belonging to this class.

Definition 1.4. A mapping $T : K \to K$ is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n, \quad n \geq 1. \tag{1.18}$$

Remark 1.5. If $\phi(\lambda) = \lambda$, then (1.18) reduces to

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + l_n, \quad n \geq 1. \tag{1.19}$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1.18) the class of mappings that includes the class of nonexpansive mappings. If $\mu_n = 0$ and $l_n = \sigma_n = \max\{0, a_n\}$, where $a_n := \sup_{x,y \in K}(\|T^n x - T^n y\| - \|x - y\|)$ for all $n \geq 1$, then (1.18) reduces to (1.5) which has been studied as mappings asymptotically nonexpansive in the intermediate sense.

The idea of Definition 1.4 is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorems applicable to all these classes of nonlinear mappings.

Another class of nonlinear mappings introduced as a further generalization of nonexpansive mappings with nonempty fixed point sets is the class of asymptotically quasi-nonexpansive mappings which properly contains the class of asymptotically nonexpansive operators with nonempty fixed point sets (see, e.g., [8, 16, 30–33]).

A mapping T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\|, \quad \forall x \in D(T), \; x^* \in F(T). \tag{1.20}$$

T is called *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \in [0, \infty)$ with $\lim_{n\to\infty}\mu_n = 0$ such that for all $x \in D(T)$ and $x^* \in F(T)$,

$$\|T^n x - x^*\| \leq \left(1 + \mu_n\right)\|x - x^*\|, \quad n \geq 1. \tag{1.21}$$

T is said to be *asymptotically quasi-nonexpansive in intermediate sense* if it is continuous and

$$\limsup_{n\to\infty}\left\{\sup_{x\in D(T),\, x^*\in F(T)}\left[\|T^n x - x^*\| - \|x - x^*\|\right]\right\} \leq 0. \tag{1.22}$$

Remark 1.6. Observe that if we define

$$a_n^* := \sup_{x\in D(T, x^*\in F(T)}\left[\|T^n x - x^*\| - \|x - x^*\|\right], \quad \sigma_n^* = \max\{0, a_n^*\}, \tag{1.23}$$

then $\sigma_n^* \to 0$ as $n \to \infty$ and (1.22) reduces to

$$\|T^n x - x^*\| \leq \|x - x^*\| + \sigma_n^*, \quad \forall x, y \in K, \; n \geq 1. \tag{1.24}$$

Existence theorems for common fixed points of certain families of nonlinear mappings have been established by various authors (see, e.g., [2, 34–37]).

Within the past 30 years or so, research on iterative approximation of common fixed points of generalizations of nonlinear nonexpansive mappings surged. Considerable research efforts have been devoted to developing iterative methods for approximating common fixed points (when they exist) of finite families of this class of mappings (see, e.g., [33, 38–46]).

In [16], Shahzad and Udomene established necessary and sufficient conditions for convergence of Ishikawa-type iteration sequences involving two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings in arbitrary real Banach spaces. They also established a sufficient condition for the convergence of the Ishikawa-type iteration sequences involving two uniformly continuous asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings in real uniformly convex Banach spaces.

Recently, Chidume and Ofoedu [47] introduced an iterative scheme for approximation of a common fixed point of a finite family of total asymptotically nonexpansive mappings in Banach spaces. More precisely, they proved the following theorems.

Theorem CO1

Let E be a real Banach space, let K be a nonempty closed convex subset of E, and $T_i : K \to K$, $i = 1,2,\ldots,m$ be m total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $\{l_{in}\}$, $n \geq 1$, $i = 1,2,\ldots,m$ such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be given by

$$\begin{aligned}
x_1 &\in E, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n x_n, \quad \text{if } m = 1,\ n \geq 1, \\
x_1 &\in E, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1^n y_{1n}, \\
y_{1n} &= (1-\alpha_n)x_n + \alpha_n T_2^n y_{2n}, \\
&\vdots \\
y_{(m-2)n} &= (1-\alpha_n)x_n + \alpha_n T_{m-1}^n y_{(m-1)n}, \\
y_{(m-1)n} &= (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad \text{if } m \geq 2,\ n \geq 1.
\end{aligned} \tag{1.25}$$

Suppose $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$ $i = 1,2,\ldots,m$ and suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i$, $i = 1,2,\ldots,m$. Then the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - p\|$ exists, $p \in F$. Moreover, the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1,2,\ldots,m$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{y\in F}\|x_n - y\|$, $n \geq 1$.

Theorem CO2

Let E be a uniformly convex real Banach space, let K be a nonempty closed convex subset of E, and $T_i : K \to K$, $i = 1,2,\ldots,m$ be m uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $\{l_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i = 1,2,\ldots,m$ and $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\alpha_{in}\} \subset [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ by (1.25). Suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i$, $i = 1,2,\ldots,m$; and that one of $T_1, T_2, \ldots, T_m$ is compact, then $\{x_n\}$ converges strongly to some $p \in F$.

It is our purpose in this paper to construct *a new iterative sequence* much simpler than (1.25) for approximation of common fixed points of finite families of total asymptotically nonexpansive mappings and give necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. A sufficient condition for convergence of the iteration process to a common fixed point of mappings under our setting is also established in uniformly convex real Banach spaces. Our theorems unify, extend and generalize the corresponding results of Alber et al. [29], Sahu [5], Shahzad and Udomene [16], and a host of other results recently announced for the approximation of common fixed points of finite families of several classes of nonlinear mappings. Our iteration process is also of independent interest.

2. Preliminary

In the sequel, we shall need the following lemmas.

Lemma 2.1. *Let* $\{a_n\}$, $\{\alpha_n\}$, *and* $\{b_n\}$ *be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1+\alpha_n)a_n + b_n. \tag{2.1}$$

Suppose that $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$. *Then* $\{a_n\}$ *is bounded and* $\lim_{n\to\infty} a_n$ *exists. Moreover, if in addition,* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (Zeidler [48, pages 484-485]). *Let* E *be a uniformly convex real Banach space and* $t \in (0,1)$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences of* E *such that*

$$\limsup_{n\to\infty}\|x_n\| \le r, \qquad \limsup_{n\to\infty}\|y_n\| \le r, \qquad \lim_{n\to\infty}\|(1-t)x_n + ty_n\| = r \tag{2.2}$$

hold for some $r \ge 0$, *then* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

3. Main Results

Let K be a nonempty closed convex subset of a real normed space E. Let $T_1, T_2, \dots, T_m : K \to K$ be m total asymptotically nonexpansive mappings. We define the iterative sequence $\{x_n\}$ by

$$x_1 \in K, \quad x_{n+1} = (1-\alpha_{0n})x_n + \sum_{i=1}^{m} \alpha_{in} T_i^n x_n, \quad n \ge 1, \tag{3.1}$$

where $\{\alpha_{in}\}_{n=1}^{\infty}$, $i = 0,1,2,\dots m$ are sequences in $(0,1)$ such that $\sum_{i=0}^{m} \alpha_{in} = 1$.

We now state and prove our main theorems.

Theorem 3.1. *Let* E *be a real Banach space, let* K *be a nonempty closed convex subset of* E, *and let* $T_i : K \to K$, $i = 1,2,\dots,m$, *be* m *total asymptotically nonexpansive mappings with sequences* $\{\mu_{in}\}$, $\{l_{in}\}$, $n \ge 1$, $i = 1,2,\dots,m$ *such that* $F := \bigcap_{i=1}^{m} F(T_i) \ne \emptyset$. *Let* $\{x_n\}$ *be given by* (3.1). *Suppose* $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$ $i = 1,2,\dots,m$ *and suppose that there exist* M_i, $M_i^* > 0$ *such that* $\phi_i(\lambda_i) \le M_i^* \lambda_i$ *for all* $\lambda_i \ge M_i$, $i = 1,2,\dots,m$. *Then the sequence* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty}\|x_n - p\|$ *exists,* $p \in F$.

Proof. Let $p \in F$. Then we have from (3.1) that

$$\begin{aligned}\|x_{n+1} - p\| &= \left\| \left(1 - \sum_{i=1}^{m} \alpha_{in}\right)(x_n - p) + \sum_{i=1}^{m} \alpha_{in}(T_i^n x_n - p) \right\| \\ &\le \left(1 - \sum_{i=1}^{m} \alpha_{in}\right)\|x_n - p\| + \sum_{i=1}^{m} \alpha_{in}\left[\|x_n - p\| + \mu_{in}\phi_i(\|x_n - p\|) + l_{in}\right].\end{aligned} \tag{3.2}$$

Since ϕ_i is an increasing function, it follows that $\phi_i(\lambda_i) \le \phi_i(M_i)$ whenever $\lambda_i \le M_i$ and (by hypothesis) $\phi_i(\lambda_i) \le M_i^* \lambda_i$ if $\lambda_i \ge M_i$. In either case, we have

$$\phi_i(\|x_n - p\|) \le \phi_i(M_i) + M_i^* \|x_n - p\| \tag{3.3}$$

for some $M_i > 0$, $M_i^* > 0$. Thus,

$$\|x_{n+1} - p\| \leq \left(1 + k_0 \sum_{i=1}^{m} \mu_{in}\right) \|x_n - p\| + k_0 \sum_{i=1}^{m} (\mu_{in} + l_{in}) \tag{3.4}$$

for some constant $k_0 > 0$. Hence,

$$\|x_{n+1} - p\| \leq (1 + \delta_n) \|x_n - p\| + \gamma_n, \quad n \geq 1, \tag{3.5}$$

where $\delta_n = k_0 \sum_{i=1}^{m} \mu_{in}$ and $\gamma_n = k_0 \sum_{i=1}^{m} (\mu_{in} + l_{in})$. Observe that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. So, from (3.5) and by Lemma 2.1 we obtain that the sequence $\{x_n\}$ is bounded and that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

3.1. Necessary and Sufficient Conditions for Convergence in Real Banach Spaces

Theorem 3.2. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K$, $i = 1, 2, \ldots, m$ be m continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $\{l_{in}\}$ $n \geq 1$, $i = 1, 2, \ldots, m$ such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be given by* (3.1). *Suppose $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i = 1, 2, \ldots, m$ and suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i$, $i = 1, 2, \ldots, m$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, \ldots, m$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{y\in F} \|x_n - y\|$, $n \geq 1$.*

Proof. It suffices to show that $\liminf_{n\to\infty} d(x_n, F) = 0$ implies that $\{x_n\}$ converges to a common fixed point of T_i, $i = 1, 2, \ldots, m$.

Necessity

Since (3.5) holds for all $p \in F$, we obtain from it that

$$d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F) + \gamma_n, \quad n \geq 1. \tag{3.6}$$

Lemma 2.1 then implies that $\lim_{n\to\infty} d(x_n, F)$ exists. But, $\liminf_{n\to\infty} d(x_n, F) = 0$. Hence, $\lim_{n\to\infty} d(x_n, F) = 0$.

Sufficiency

Next, we first show that $\{x_n\}$ is a Cauchy sequence in E. For all integer $m \geq 1$, we obtain from inequality (3.5) that

$$\begin{aligned} \|x_{n+m} - p\| &\leq \prod_{i=n}^{n+m-1} (1 + \delta_i) \|x_n - p\| + \left(\sum_{i=n}^{n+m-1} \gamma_i\right) \prod_{i=n}^{n+m-1} (1 + \delta_i) \\ &\leq \exp\left(\sum_{i=n}^{n+m-1} \delta_i\right) \|x_n - p\| + \left(\sum_{i=n}^{n+m-1} \gamma_i\right) \exp\left(\sum_{i=n}^{n+m-1} \delta_i\right), \end{aligned} \tag{3.7}$$

so that for all integers $m \geq 1$ and all $p \in F$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \left[1 + \exp\left(\sum_{i=n}^{n+m-1} \delta_i\right)\right]\|x_n - p\| + \left(\sum_{i=n}^{n+m-1} \gamma_i\right)\exp\left(\sum_{i=n}^{n+m-1} \delta_i\right).\end{aligned} \tag{3.8}$$

We therefore have that

$$\|x_{n+m} - x_n\| \leq D\|x_n - p\| + D\left(\sum_{i=n}^{\infty}\gamma_i\right) \tag{3.9}$$

for some constant $D > 0$. Taking infimum over $p \in F$ in (3.9) gives

$$\|x_{n+m} - x_n\| \leq Dd(x_n, F) + D\left(\sum_{i=n}^{\infty}\gamma_i\right). \tag{3.10}$$

Now, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=1}^{\infty}\gamma_i < \infty$, given $\epsilon > 0$, there exists an integer $N_1 > 0$ such that for all $n \geq N_1$, $d(x_n, F) < \epsilon/2(D+1)$ and $\sum_{i=n}^{\infty}\gamma_i < \epsilon/2(D+1)$. So for all integers $n \geq N_1$, $m \geq 1$, we obtain from (3.10) that

$$\|x_{n+m} - x_n\| < \epsilon. \tag{3.11}$$

Hence, $\{x_n\}$ is a Cauchy sequence in E, and since E is complete there exists $l^* \in E$ such that $x_n \to l^*$ as $n \to \infty$. We now show that l^* is a common fixed point of T_i, $i = 1, 2, \ldots, m$, that is, we show that $l^* \in F$. Suppose for contradiction that $l^* \in F^c$ (where F^c denotes the complement of F). Since F is a closed subset of E (recall each T_i, $i = 1, 2, \ldots, m$ is continuous), we have that $d(l^*, F) > 0$. But, for all $p \in F$, we have

$$\|l^* - p\| \leq \|l^* - x_n\| + \|x_n - p\|. \tag{3.12}$$

This implies

$$d(l^*, F) \leq \|x_n - l^*\| + d(x_n, F), \tag{3.13}$$

so that as $n \to \infty$ we obtain $d(l^*, F) = 0$ which contradicts $d(l^*, F) > 0$. Thus, l^* is a common fixed point of T_i, $i = 1, 2, \ldots, m$. This completes the proof. □

Remark 3.3. If $T_1, T_2, \ldots, T_m$ are asymptotically nonexpansive mappings, then $l_{in} = 0$ for all $n \geq 1$, $i = 1, 2, \ldots, m$ and $\phi_i(\lambda_i) = \lambda_i$ so that the assumption that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^*\lambda_i$ for all $\lambda_i \geq M_i$, $i = 1, 2, \ldots, m$ in the above theorems is no longer needed.

Thus, we have the following corollary.

Corollary 3.4. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K$, $i = 1,2,\dots,m$ be m continuous asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $n \geq 1$, $i = 1,2,\dots,m$ such that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be given by (3.1). Suppose $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $i = 1,2,\dots,m$. Then the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - p\|$ exists, $p \in F$. Moreover, $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1,2,\dots,m$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

3.2. Convergence Theorem in Real Uniformly Convex Banach Spaces

Theorem 3.5. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1,2,\dots,m$ be m uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $\{l_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i = 1,2,\dots,m$ and $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. From arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ by (3.1). Suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i$, $i = 1,2,\dots,m$. Then $\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0$, $i = 1,2,\dots,m$.*

Proof. Let $p \in F$. Then, by Theorem 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = r$. If $r = 0$, then by continuity of T_i, $i = 1,2,\dots,m$, we are done. Now suppose $r > 0$. We show that $\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0, i = 1,2,\dots,m$. We observe that

$$\begin{aligned} \|T_i^n x_n - p\| &\leq \|x_n - p\| + \mu_{in}\phi_i(\|x_n - p\|) + l_{in} \\ &\leq \|x_n - p\| + \mu_{in}\phi_i(M_i) + \mu_{in}M_i^*\|x_n - p\| + l_{in} \quad (i = 1,2,\dots,m). \end{aligned} \tag{3.14}$$

so that taking lim sup on both sides of this inequality, we obtain

$$\limsup_{n\to\infty}\|T_i^n x_n - p\| \leq r \quad (i = 1,2,\dots,m). \tag{3.15}$$

Let $\{t_n\}_{n\geq 1} \in (0,1)$ be such that $t_n \to 0$ as $n \to \infty$ and define

$$y_{in} := (1 - t_n)x_n + t_n T_i^n x_n \quad (i = 1,2,\dots,m). \tag{3.16}$$

Then,

$$\|y_{in} - x_n\| \leq t_n\|T_i^n x_n - x_n\| \leq t_n M \quad (i = 1,2,\dots,m) \tag{3.17}$$

for some $M > 0$. Thus,

$$\lim_{n\to\infty}\|y_{in} - x_n\| = 0 \quad (i = 1,2,\dots,m). \tag{3.18}$$

Furthermore,

$$\big|\|y_{in} - p\| - \|x_n - p\|\big| \leq \|y_{in} - x_n\| \quad (i = 1,2,\dots,m). \tag{3.19}$$

This implies that

$$\lim_{n\to\infty}\|y_{in}-p\| = \lim_{n\to\infty}\|x_n-p\| = r \quad (i=1,2,\ldots,m). \tag{3.20}$$

But,

$$\|y_{in}-p\| = \|(1-t_n)(x_n-p)+t_n(T_i^n x_n-p)\| \quad (i=1,2,\ldots,m). \tag{3.21}$$

So,

$$\lim_{n\to\infty}\|(1-t_n)(x_n-p)+t_n(T_i^n x_n-p)\| = r, \quad (i=1,2,\ldots,m). \tag{3.22}$$

Hence, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n-T_i^n x_n\| = 0, \quad (i=1,2,\ldots,m). \tag{3.23}$$

This completes the proof. □

Theorem 3.6. *Let E be a uniformly convex real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K$, $i=1,2,\ldots,m$ be m uniformly continuous total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}$, $\{l_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i=1,2,\ldots,m$ and $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. From arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ by (1.24). Suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^*\lambda_i$ whenever $\lambda_i \geq M_i$, $i=1,2,\ldots,m$; and that one of $T_1, T_2, \ldots, T_m$ is compact, then $\{x_n\}$ converges strongly some $p \in F$.*

Proof. We obtain from Theorem 3.5 that

$$\lim_{n\to\infty}\|T_i^n x_n - x_n\| = 0, \quad i=1,2,\ldots,m. \tag{3.24}$$

Using the recursion formula (3.1), we observe that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\alpha_{0n}x_n + \sum_{i=1}^{m}\alpha_{in}T_i^n x_n - x_n\right\| \\ &= \left\|\sum_{i=1}^{m}\alpha_{in}(T_i^n x_n - x_n)\right\| \\ &\leq \sum_{i=1}^{m}\alpha_{in}\|T_i^n x_n - x_n\|.\end{aligned} \tag{3.25}$$

It then follows from (3.24) and (3.25) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.26}$$

Without loss of generality, let T_1 be compact. Since T_1 is continuous and compact, it is completely continuous. Thus, there exists a subsequence $\{T_1^{n_k}x_{n_k}\}$ of $\{T_1^n x_n\}$ such that $T_1^{n_k}x_{n_k} \to x^*$ as $k \to \infty$ for some $x^* \in E$. Thus $T_1^{n_k+1}x_{n_k} \to T_1x^*$ as $k \to \infty$ and from (3.24), we have that $\lim_{k\to\infty}x_{n_k} = x^*$. Also from (3.24) $T_2^{n_k}x_{n_k} \to x^*,\ T_3^{n_k}x_{n_k} \to x^*, \dots,\ T_m^{n_k}x_{n_k} \to x^*$ as $k \to \infty$. Thus, $T_2^{n_k+1}x_{n_k} \to T_2x^*,\ T_3^{n_k+1}x_{n_k} \to T_3x^*, \dots,\ T_m^{n_k+1}x_{n_k} \to T_mx^*$ as $k \to \infty$. Now, since from (3.26), $\|x_{n_k+1} - x_{n_k}\| \to 0$ as $k \to \infty$, it follows that $x_{n_k+1} \to x^*$ as $k \to \infty$. Next, we show that $x^* \in F$. Observe that

$$\begin{aligned} \|x^* - T_ix^*\| \le \|x^* - x_{n_k+1}\| &+ \left\|x_{n_k+1} - T_i^{n_k+1}x_{n_k+1}\right\| \\ &+ \left\|T_i^{n_k+1}x_{n_k+1} - T_i^{n_k+1}x_{n_k}\right\| + \left\|T_i^{n_k+1}x_{n_k} - T_ix^*\right\|. \end{aligned} \tag{3.27}$$

Taking limit as $k \to \infty$ and using the fact that $T_i\ \ (i = 1,2,\dots m)$ are uniformly continuous we have that $x^* = T_ix^*\ \ (i = 1,2,\dots m)$ and so $x^* \in F(T_i)\ \ (i = 1,2,\dots m)$. But by Theorem 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists, $p \in F$. Hence, $\{x_n\}$ converges strongly to $x^* \in F$. This completes the proof. □

In view of Remark 12, the following corollary is now obvious.

Corollary 3.7. *Let E be a uniformly convex real Banach space, let K be a nonempty closed convex subset of E, and let $T_i : K \to K,\ i = 1,2,\dots,m$ be m asymptotically nonexpansive mappings with sequences $\{\mu_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\mu_{in} < \infty,\ i = 1,2,\dots,m$; $F = \bigcap_{i=1}^{m}F(T_i) \neq \emptyset$ and that one of $T_1,T_2,\dots,T_m$ is compact. From arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ by* (3.1)*. Then $\{x_n\}$ converges strongly to to some $p \in F$.*

Remark 3.8. Observe that the theorems of this paper remain true for mappings $T_1,T_2,\dots,T_m$ satisfying (1.5) provided that $\sum_{n=1}^{\infty}\sigma_n < \infty$. In this case, the requirement that there exist $M_i,\ M_i^* > 0$ such that $\phi_i(\lambda_i) \le M_i^*\lambda_i$ for all $\lambda_i \ge M_i,\ i = 1,2,\dots,m$ is not needed.

Remark 3.9. A prototype for $\phi : [0,\infty) \to [0,\infty)$ satisfying the conditions of our theorems is $\phi(\lambda) = \lambda^s,\ 0 < s \le 1$. Prototypes for the sequences $\{\alpha_{in}\}_{n=1}^{\infty}$ for all $i = 0,1,2,\dots,m$ in this paper are the following:

$$\alpha_{0n} := \frac{n}{2n+1}; \qquad \alpha_{in} := \frac{n+1}{m(2n+1)} \quad \forall n \in \mathbb{N},\ i = 1,2,\dots,m. \tag{3.28}$$

Remark 3.10. Addition of *bounded* (or the so called mean) error terms to the iteration process studied in this paper leads to no further generalization.

Definition 3.11. A mapping $T : K \to K$ is said to be *total asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\},\ n \ge 1$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in E,\ x^* \in F(T)$,

$$\|T^nx - x^*\| \le \|x - x^*\| + \mu_n\phi(\|x - x^*\|) + l_n, \quad n \ge 1. \tag{3.29}$$

Remark 3.12. If $\phi(\lambda) = \lambda$, then (3.29) reduces to

$$\|T^n x - x^*\| \le (1 + \mu_n)\|x - x^*\| + l_n, \quad n \ge 1. \tag{3.30}$$

In addition, if $l_n = 0$ for all $n \ge 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi-nonexpansive mappings studied by various authors. If $\mu_n = 0$ and $l_n = 0$ for all $n \ge 1$, we obtain from (3.30) the class of quasi-nonexpansive mappings. Observe that the class of total asymptotically nonexpansive mappings with nonempty fixed point sets belongs to the class of total asymptotically quasi-nonexpansive mappings. Moreover, if $\mu_n = 0$ and $l_n = \sigma_n^*$, then (3.29) reduces to (1.24).

It is trivial to observe that all the theorems of this paper carry over to the class of total asymptotically quasi-nonexpansive mappings with little or no modifications.

A subset K of a real normed linear space E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. It is well known (see, e.g., [28]) that every closed convex nonempty subset of a uniformly convex Banach space is a retract. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P ia a retraction, then $Py = y$ for all y in the range of P. The mapping p is called a sunny nonexpansive retraction if for all $x \in E$ and $t \in (0,1)$, $P((1-t)x + tP(x)) = P(x)$.

Definition 3.13. Let K be a nonempty closed and convex subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A *nonself* map $T : K \to E$ is said to be *total asymptotically nonexpansive* if there exist sequences $\{\mu_n\}_{n\ge1}$, $\{l_n\}_{n\ge1}$ in $[0,+\infty)$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le \|x - y\| + \mu_n\phi(\|x - y\|) + l_n, \quad n \ge 1. \tag{3.31}$$

Let $T_1, T_2, \dots, T_m : K \to E$ be m total asymptotically nonexpansive *nonself* maps; assuming existence of common fixed points of these operators, our theorems and method of proof easily carry over to this class of mappings using the iterative sequence $\{x_n\}$ defined by

$$x_1 \in K, \quad x_{n+1} = P\left(\alpha_{0n}x_n + \sum_{i=1}^{r}\alpha_{in}T_i(PT_i)^{n-1}x_n\right) \tag{3.32}$$

instead of (3.1) provided that the well definedness of P as a sunny nonexpansive retraction is guaranteed.

Remark 3.14. It is clear that the recursion formula (3.1) introduced and studied in this paper is much simpler than the recursion formulas (1.25) studied earlier for this problem.

Remark 3.15. Our theorems unify, extend, and generalize the corresponding results of Alber et al. [29], Sahu [5], Shahzad and Udomene [16], and a host of other results recently announced (see, e.g., [8, 16, 22, 28, 30, 39, 40, 44, 47, 49–58]) for the approximation of common fixed points of finite families of several classes of nonlinear mappings.

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