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Research Article

An Application of Differential Subordination

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We apply the general theory of differential subordination to obtain certian interesting criteria for p-valent starlikeness and strong starlikeness. Some applications of these results are also discussed.

1. Introduction

Let $\mathcal{A}_p$ $(p \in \mathbb{N} = \{1, 2, 3, \ldots\})$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m} \tag{1.1}$$

which are analytic in the open unit disk $\Delta := \{z : |z| < 1\}$.

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.2}$$

which are analytic in Δ. If $p(z) \in \mathcal{P}$ satisfies $\Re p(z) > 0$ $(z \in \Delta)$, then we say that $p(z)$ is a Carathéodory function.

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \Delta), \tag{1.3}$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta). \tag{1.4}$$

We denote this subordination by

$$f \prec g, \qquad f(z) \prec g(z) \quad (z \in \Delta). \tag{1.5}$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{or} \quad f(\Delta) \subset g(\Delta). \tag{1.6}$$

For $-1 \le b < a \le 1$ and $0 < \gamma \le 1$, a function $f \in \mathcal{A}_p$ is said to be in the class $S_p^*(\gamma, a, b)$ if it satisfies

$$\frac{zf'(z)}{f(z)} \prec p\left(\frac{1+az}{1+bz}\right)^{\gamma}. \tag{1.7}$$

Also, we write $S_p^*(\gamma, 1, -1) = SS_p^*(\gamma)$, the class of strongly starlike p-valent functions of order γ in Δ. $S_p^*(1, a, b) = S_p^*(a, b)$, the class of Janowski starlike p-valent function, $S_p^*(1, -1) = S_p^*$, the class of p-valent starlike function, and $S_p^*(1-2\gamma, 1) = S_p^*(\gamma)$ $(0 \le \gamma < 1)$, the class of p-valent starlike function of order γ.

For Carathéodory functions, Miller [1] obtained certain sufficient conditions applying the differential inequalities. Recently, Nunokawa et al. [2] have given some improvement of result by Miller [1]. Recently Ravichandran and Jayamala [3] studied some subordination results for Carathéodory functions. In this paper by extending the result of Ravichandran and Jayamala [3], we find sufficient conditions for the subordination $p(z) \prec q(z)$ to hold for given $q(z)$ and criteria for p-valent starlikeness. Our results include results obtained by Nunokawa et al. [2]. We also give some criteria for p-valently starlikeness and strong starlikeness.

To prove our result we need the following lemma due to Miller and Mocanu [4].

Lemma 1.1 (see [4, Theorem 3.4h, page 132]). *Let $q(z)$ be analytic and univalent in the unit disk Δ and $\theta(\omega)$ and let $\phi(\omega)$ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \qquad h(z) = \theta(q(z)) + Q(z). \tag{1.8}$$

Suppose that

(i) *$Q(z)$ is starlike univalent in Δ,*

(ii) $\Re\{zh'(z)/Q(z)\} = \Re\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$ *for $z \in \Delta$.*

If $p(z)$ is analytic in Δ with, $p(0) = q(0)$, $p(\Delta) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.9}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2. Application of Differential Subordination

By making use of Lemma 1.1, we first prove the following theorem.

Theorem 2.1. *Let* $0 \neq \alpha \in \mathbb{C}$ *and* λ *be a positive real number. Let* $q(z)$ *be convex univalent in* Δ *and* $\Re((1-\alpha) \setminus \alpha + m(q(z))^{m-1}) > 0, m \in \mathbb{N} \setminus \{1\}$. *If* $p \in \mathcal{P}$ *satisfies*

$$(1-\alpha)p(z) + \alpha\big(p(z)\big)^m + \alpha\lambda z p'(z) \prec h(z), \tag{2.1}$$

where

$$h(z) = (1-\alpha)q(z) + \alpha\big(q(z)\big)^m + \alpha\lambda z q'(z), \tag{2.2}$$

then

$$p(z) \prec q(z), \tag{2.3}$$

and $q(z)$ *is the best dominant of* (2.1).

Proof. Let

$$\theta(w) = (1-\alpha)w + \alpha w^m, \qquad \phi(w) = \alpha\lambda. \tag{2.4}$$

Then clearly $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ and $\phi(w) \neq 0$. Also let

$$\begin{aligned} Q(z) &= zq'(z)\phi\big(q(z)\big) = \alpha\lambda z q'(z), \\ h(z) &= \theta\big(q(z)\big) + Q(z) \\ &= (1-\alpha)q(z) + \alpha\big(q(z)\big)^m + \alpha\lambda z q'(z). \end{aligned} \tag{2.5}$$

Since $q(z)$ is convex univalent, $zq'(z)$ is starlike univalent. Therefore $Q(z)$ is starlike univalent in Δ, and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \frac{1}{\lambda}\Re\left\{\frac{1-\alpha}{\alpha} + m\big(q(z)\big)^{m-1} + \lambda\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \tag{2.6}$$

for $z \in \Delta$.

From (2.1)–(2.6) we see that

$$\theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big) = h(z). \tag{2.7}$$

Therefore, by applying Lemma 1.1, we conclude that $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1). The proof of the theorem is complete. □

By taking α as real and $q(z) = ((1+az)/(1+bz))^\gamma$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let* $-1 \leq b < a \leq 1$, $m \in \mathbb{N} \setminus \{1\}, 0 < \gamma \leq 1/(m-1)$, λ *be real number such that* $\lambda > 0$ *and* $0 < \alpha \leq 1$. *If* $p \in \mathcal{P}$ *satisfies*

$$(1-\alpha)p(z) + \alpha(p(z))^m + \alpha\lambda z p'(z) \prec h(z), \tag{2.8}$$

where

$$h(z) = (1-\alpha)\left(\frac{1+az}{1+bz}\right)^{\gamma} + \alpha\left(\frac{1+az}{1+bz}\right)^{m\gamma} + \frac{\alpha\lambda\gamma(a-b)z}{(1+az)^{1-\gamma}(1+bz)^{1+\gamma}}, \tag{2.9}$$

then

$$p(z) \prec \left(\frac{1+az}{1+bz}\right)^{\gamma}, \tag{2.10}$$

and $((1+az)/(1+bz))^{\gamma}$ *is the best dominant of* (2.8).

Corollary 2.3. *Let* $-1 \leq b < a \leq 1$, $\lambda > 0$. *If* $f \in \mathcal{A}_p$ *satisfies* $f(z) \neq 0$ *in* $0 < |z| < 1$ *and*

$$\frac{zf'(z)}{pf(z)}\left[1-\alpha+\frac{\alpha}{p}(1-\lambda p)\frac{zf'(z)}{f(z)} + \alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right] \prec h(z), \tag{2.11}$$

where

$$h(z) = (1-\alpha)\left(\frac{1+az}{1+bz}\right)^{\gamma} + \alpha\left(\frac{1+az}{1+bz}\right)^{2\gamma} + \frac{\alpha\lambda\gamma(a-b)z}{(1+az)^{1-\gamma}(1+bz)^{1+\gamma}}, \tag{2.12}$$

then

$$\frac{zf'(z)}{pf(z)} \prec \left(\frac{1+az}{1+bz}\right)^{\gamma}. \tag{2.13}$$

Proof. Let $p(z) = zf'(z)/pf(z)$, then $p \in \mathcal{P}$ and (2.11) can be written as

$$\begin{aligned}&(1-\alpha)p(z) + \alpha p^2(z) + \alpha\lambda z p'(z)\\ &\quad \prec (1-\alpha)\left(\frac{1+az}{1+bz}\right)^{\gamma} + \alpha\left(\frac{1+az}{1+bz}\right)^{2\gamma} + \frac{\alpha\lambda\gamma(a-b)z}{(1+az)^{1-\gamma}(1+bz)^{1+\gamma}}.\end{aligned} \tag{2.14}$$

Taking $m = 2$ in Corollary 2.2 and using (2.14), we have

$$\frac{zf'(z)}{pf(z)} \prec \left(\frac{1+az}{1+bz}\right)^{\gamma}. \tag{2.15}$$

□

By taking $p = \lambda = \gamma = a = 1$ and $b = -1$ in Corollary 2.3, we get the following result of Padmanabhan [5].

Corollary 2.4. *Let $f \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f'(z)} \prec \frac{2\alpha(z^2+2z)+1-z^2}{(1-z)^2} \quad (0 < \alpha \le 1), \tag{2.16}$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0. \tag{2.17}$$

Theorem 2.5. *Let $\alpha, \beta, \xi, \eta \in \mathbb{C}$ and $\eta \neq 0$. Let $q(z)$ be convex univalent in Δ and satisfy*

$$\Re\left[\frac{1}{\eta}(\beta + 2\xi q(z))\right] > 0. \tag{2.18}$$

If $p \in \mathcal{P}$ satisfies

$$\alpha + \beta p(z) + \xi p^2(z) + \eta z p'(z) \prec \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z) = h(z), \tag{2.19}$$

then

$$p(z) \prec q(z), \tag{2.20}$$

and $q(z)$ is the best dominant of (2.19)

Proof. By setting $\theta(w) := \alpha + \beta w + \xi w^2$ and $\phi(w) := \eta$ it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ and that $\phi(w) \neq 0$ $(w \in \mathbb{C} \setminus \{0\})$.

Also, by letting

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \eta z q'(z), \\ h(z) &= \theta(q(z)) + Q(z) \\ &= \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z), \end{aligned} \tag{2.21}$$

we find that $Q(z)$ is starlike univalent in Δ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left[\frac{1}{\eta}(\beta + 2\xi q(z)) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right] > 0. \tag{2.22}$$

The differential subordination

$$\alpha + \beta p(z) + \xi p^2(z) + \eta z p'(z) \prec \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z) \tag{2.23}$$

becomes

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)). \tag{2.24}$$

Now, the result follows as an application of Lemma 1.1. □

Theorem 2.6. *Let α, β, ξ, η, and δ be complex numbers, $\delta \neq 0$. Let $0 \neq q(z)$ be univalent in Δ and satisfy the following conditions for $z \in \Delta$:*

(1) *let $Q(z) = \delta z q'(z)/q(z)$ be starlike,*

(2) $\Re\{(\beta/\delta)q(z) + (2\xi/\delta)q^2(z) - (\eta/\delta q(z)) + zQ'(z)/Q(z)\} > 0$.

If $p \in \mathcal{P}$ satisfies

$$\alpha + \beta p(z) + \xi(p(z))^2 + \frac{\eta}{p(z)} + \delta\frac{zp'(z)}{p(z)} \prec \alpha + \beta q(z) + \xi(q(z))^2 + \frac{\eta}{q(z)} + \delta\frac{zq'(z)}{q(z)}, \tag{2.25}$$

then

$$p(z) \prec q(z), \tag{2.26}$$

and $q(z)$ is the best dominant.

Proof. The proof of this theorem is much akin to the proof of Theorem 2.5 and hence can be omitted. □

Remark 2.7. By taking $\alpha = \beta = 0, \xi = (\lambda/\mu)\,\mu > 0, \lambda > -\mu/2, \eta = 1$, and $q(z) = (1+z)/(1-z)$ in Theorem 2.5 we get the result of Nunokawa et al. [2] which was proved by a different method.

Remark 2.8. For the choices of $\alpha = \beta = 0$ in Theorem 2.5, we get the result of [3, Theorem 1, page 192] and for $\alpha = \xi = \eta = 0$ in Theorem 2.6 we get the result of [3, Theorem 2, page 194].

Corollary 2.9. *Let $-1 \leq b < a \leq 1, 0 < \gamma \leq 1$ and $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies $f(z)f'(z) \neq 0$ in $0 < |z| < 1$, then*

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p\left(\frac{1+az}{1+bz}\right)^{\gamma} + \frac{\lambda\gamma(a-b)z}{(1+az)(1+bz)} \tag{2.27}$$

implies

$$f \in S_p^*(\gamma, a, b). \tag{2.28}$$

Also,

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{\gamma(a-b)z}{(1+az)(1+bz)} \tag{2.29}$$

implies

$$f \in S_p^*(\gamma, a, b). \tag{2.30}$$

Proof. By taking $\alpha = \xi = \eta = 0, \beta = p/\lambda$, $\delta = 1, p(z) = zf'(z)/pf(z)$, and $q(z) = ((1+az)/(1+bz))^{\gamma}$ in Theorem 2.6, we get the first part.

Proof of the second part follows, by setting $\alpha = \beta = \xi = \eta = 0, \delta = 1, p(z) = zf'(z)/pf(z)$, and $q(z) = ((1+az)/(1+bz))^{\gamma}$. □

For $\alpha = \xi = 0, \beta = 1$, $p(z) = zf'(z)/f(z)$, and $q(z) = (1+az)/(1-z), -1 < a \leq 1$ in Theorem 2.5, we have the following result.

Corollary 2.10. *If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$, $z \in \Delta$ and*

$$\frac{zf'(z)}{f(z)}\left[\left(1 - \eta\frac{zf'(z)}{f(z)}\right) + \eta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec h(z), \tag{2.31}$$

where

$$h(z) = \frac{1+az}{1-z} + \eta\frac{(1+a)z}{(1-z)^2}, \tag{2.32}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z}. \tag{2.33}$$

One notes that if $h(z) = u + iv$, then $h(\Delta)$ is the exterior of the parabola given by

$$v^2 = -\frac{(1+a)}{\eta}\left[u - \frac{2-2a-\eta(1+a)}{4}\right] \tag{2.34}$$

with its vertex as $((2-2a-\eta(1+a)/4), 0)$ (see [5, 6]).

By taking $\eta = a = 1$ in Corollary 2.10, we obtain the following.

Corollary 2.11. *If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$, $z \in \Delta$, and*

$$\frac{zf'(z)}{f(z)}\left[2 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right] \prec \frac{1+2z-z^2}{(1-z)^2}, \tag{2.35}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}. \tag{2.36}$$

Region $h(\Delta)$ has been shown shaded in Figure 1.

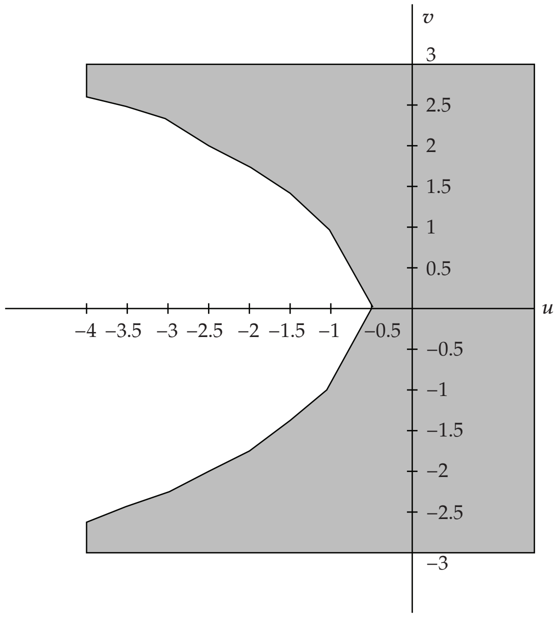

Figure 1: $\eta = a = 1$.

Letting $\alpha = \beta = 0, \xi = \eta = 1, p(z) = zf'(z)/f(z)$, and $q(z) = (1 + (1 - 2\gamma)z)/(1 - z)$ in Theorem 2.5, we get the following.

Corollary 2.12. *If* $f \in \mathcal{A}$ *satisfies* $f(z) \neq 0, 0 < |z| < 1$, *and*

$$\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec h(z), \tag{2.37}$$

where

$$h(z) = \frac{(1 - 2\gamma)^2 z^2 + 2(2 - 3\gamma)z + 1}{(1 - z)^2} \tag{2.38}$$

for some γ $(0 \leq \gamma < 1)$, *then*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma. \tag{2.39}$$

For the univalent function $h(z)$ given by (2.38), One now finds the image $h(\Delta)$ of the unit disk Δ.

Let $h = u + iv$, where u and v are real. One has

$$u = -\frac{(2-3\gamma) + (1+2\gamma^2 - 2\gamma)\cos\theta}{(1-\cos\theta)}, \qquad v = \frac{2\gamma(1-\gamma)\sin\theta}{1-\cos\theta}. \tag{2.40}$$

Elimination of θ yields

$$v^2 = -\frac{8\gamma^2(1-\gamma)}{3-2\gamma}\left[u - \frac{2\gamma^2+\gamma-1}{2}\right]. \tag{2.41}$$

Therefore, one concludes that

$$h(\Delta) = \left\{w = u + iv;\ v^2 > -\frac{8\gamma^2(1-\gamma)}{3-2\gamma}\left[u - \frac{2\gamma^2+\gamma-1}{2}\right]\right\}, \tag{2.42}$$

which properly contains the half plane $\Re w > (2\gamma^2 + \gamma - 1)/2$.

Corollary 2.13. *Let* $-1 \le b < a \le 1$ *and* $\Re\ \beta \ge 0$. *If* $f \in \mathcal{A}_p$ *satisfies* $f'(z) \ne 0$ *in* $0 < |z| < 1$ *and*

$$(1-\beta)\frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} \prec h(z), \tag{2.43}$$

where

$$h(z) = \frac{b(pb-\beta a)z^2 + ((2p+1-\beta)b - (1+\beta)a)z + p - \beta}{p(1+bz)^2}, \tag{2.44}$$

then

$$f \in S_p^*(b,a). \tag{2.45}$$

Proof. If we let $p(z) = pf(z)/zf'(z)$, then $p \in \mathcal{P}$ and (2.43) can be expressed as

$$\beta p(z) + zp'(z) \prec \beta\left(\frac{1+az}{1+bz}\right) + \frac{(a-b)z}{(1+bz)^2}. \tag{2.46}$$

Hence, by taking $\alpha = \xi = 0, \eta = 1, q(z) = (1+az)/(1+bz)$ and $\Re\ \beta \ge 0$ in Theorem 2.5, we have $p(z) \prec (1+az)/(1+bz)$. So, $f(z) \in S_p^*(b,a)$. □

Setting $p = 1$ and $b = -1$ in Corollary 2.13, we get the following corollay.

Corollary 2.14. *Let* $-1 < a \le 1$ *and* $\Re\, \beta \ge 0$. *If* $f \in \mathcal{A}$ *satisfies* $f'(z) \ne 0$ *in* $0 < |z| < 1$ *and*

$$(1-\beta)\frac{f(z)}{zf(z)} + \frac{f(z)f''(z)}{(f'(z))^2} \prec h(z), \tag{2.47}$$

where

$$h(z) = \frac{(1+\beta a)z^2 + ((\beta-3)-(1+\beta)a)z + 1 - \beta}{(1-z)^2}, \tag{2.48}$$

then

$$\frac{f(z)}{zf'(z)} \prec \frac{1+az}{1-z}. \tag{2.49}$$

Remark 2.15. For the function $h(z)$ given by (2.48), we have

$$h(\Delta) = \left\{ w = u + iv;\, v^2 > a_0[u - b_0] \right\}, \tag{2.50}$$

which properly contains the half plane $\Re w > b_0$, where

$$\begin{aligned} a_0 &= (1+a)\beta^2, \\ b_0 &= \frac{5 + a + 2\beta(a-1)}{4}. \end{aligned} \tag{2.51}$$

By putting $p = a = \beta = 1$ and $b = -1$ in Corollary 2.13, we get the following result of Tuneski [7].

Corollary 2.16. *If* $f(z) \in \mathcal{A}$ *and*

$$\frac{f(z)f''(z)}{(f'(z))^2} \prec \frac{2z(z-2)}{(1-z)^2}, \tag{2.52}$$

then

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > 0. \tag{2.53}$$

Remark 2.17. By putting $0 = b < a \le 1, p = 1$, and $\beta = 0$ in Corollary 2.13, we get the result obtained by Singh [8], which refines the result of Silverman [9].

Corollary 2.18. *Let* $0 \neq \eta$ *and* $q(z)$ *be convex univalent in* Δ *with* $q(0) = 1$ *and satisfy* (2.18).
Let $f \in \mathcal{A}_p$ *and*

$$\psi(z) := \alpha + \frac{\beta}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} + \frac{\xi}{p^2}\left(\frac{f(z)}{z^p}\right)^{2\mu} + \eta\mu\left(\frac{f(z)}{z^p}\right)^{\mu}\left[\frac{zf'(z)}{pf(z)} - 1\right]. \tag{2.54}$$

If

$$\psi(z) \prec \alpha + \beta q(z) + \xi q^2(z) + \eta z q'(z), \tag{2.55}$$

then

$$\frac{1}{p}\left(\frac{f(z)}{z^p}\right)^{\mu} \prec q(z), \tag{2.56}$$

and $q(z)$ *is the best dominant.*

Proof. By taking $p(z) = (1/p)(f(z)/z^p)^{\mu}$ in Theorem 2.5, we have the above corollary. □

Corollary 2.19. *Let* $0 \neq \lambda \in \mathbb{C}$ *and* $q(z)$ *be convex univalent in* Δ *with* $q(0) = 1$ *and satisfy*

$$\Re\left(\frac{\mu}{\lambda}\right) > 0. \tag{2.57}$$

(i) *If* $f \in \mathcal{A}$ *satisfies*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec q(z) + \frac{\lambda}{\mu} z q'(z), \tag{2.58}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z). \tag{2.59}$$

(ii) *If* $f \in \mathcal{A}$ *satisfies*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - \left(\frac{f(z)}{z}\right)^{\mu} \prec \frac{1}{\mu} z q'(z), \tag{2.60}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z), \tag{2.61}$$

and $q(z)$ *is the best dominant.*

Proof. Proof of the first part follows from Corollary 2.18, by taking $\beta = p = 1, \alpha = \xi = 0$, and $\eta = \lambda/\mu$.

The proof of the second part follows from Corollary 2.18, by taking $\alpha = \beta = \xi = 0, p = 1$, and $\eta = 1/\mu$. □

By taking $\lambda = \mu = n$ where n is a positive integer and $q(z) = A+(1-A)[-1-(2/z)\log(1-z)]$ in the first part of Corollary 2.19, we get the following result of Ponnusamy [10].

Corollary 2.20. *Let $f \in \mathcal{A}$, then for a positive integer n, one has that*

$$\Re\left\{(1-n)\left(\frac{f(z)}{z}\right)^n + nf'(z)\left(\frac{f(z)}{z}\right)^{n-1}\right\} > \beta \tag{2.62}$$

implies

$$\left(\frac{f(z)}{z}\right)^n \prec A + (1-A)\left(-1 - \frac{2}{z}\log(1-z)\right), \tag{2.63}$$

and $A + (1-A)[-1-(2/z)\log(1-z)]$ is the best dominant.

Remark 2.21. By taking $\mu = 1$ and $q(z) = 1 + (A/(1+\delta))z$ in Corollary 2.19 and $\mu = \lambda = 1$ and $q(z) = A/B + (1 - A/B)(\log(1+Bz)/Bz)$ we get the result of Ponnusamy and Juneja [11].

By taking $\beta = \xi = \eta = 0, \alpha = p = 1, \delta = 1/\mu, p(z) = (1/p)(f(z)/z^p)^\mu$, and $q(z) = e^{\mu A z}$ in Theorem 2.5, we get the following result obtained by Owa and Obradović [12].

Corollary 2.22. *Let $f \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)} \prec 1 + Az, \tag{2.64}$$

then

$$\left(\frac{f(z)}{z}\right)^\mu \prec e^{\mu A z}, \tag{2.65}$$

and $e^{\mu A z}$ is the best dominant.

We remark here that $q(z) = e^{\mu A z}$ is univalent if and only if $|\mu A| < \pi$.

Remark 2.23. For a special case when $p(z) = (1/p)(f(z)/z^p)^\mu, q(z) = 1/(1-z)^{2b}$ where $b \in \mathbb{C} \setminus \{0\}$ and $\beta = \xi = \eta = 0, \alpha = \mu = p = 1$, and $\delta = 1/b$ in Theorem 2.6, we have the result obtained by Srivastava and Lashin [13].

Corollary 2.24. *If $f \in \mathcal{A}$ satisfies*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^\mu - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec q(z) + \frac{\lambda}{\mu}zq'(z), \tag{2.66}$$

then

$$\left(\frac{z}{f(z)}\right)^{\mu} \prec q(z), \tag{2.67}$$

and q is the best dominant.

Proof. By taking $p(z) = (1/p)(z^p/f(z))^{\mu}$ and $\alpha = \xi = 0, \beta = p = 1$ and $\eta = \lambda/\mu$ in Theorem 2.5, we get the previous corollary. □

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