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Research Article

Connectedness Degrees in *L*-Fuzzy Topological Spaces

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The notion of separatedness degrees of L-fuzzy subsets is introduced in L-fuzzy topological spaces by means of L-fuzzy closure operators. Furthermore, the notion of connectedness degrees of L-fuzzy subsets is introduced. Many properties of connectedness in general topology are generalized to L-fuzzy topological spaces.

1. Introduction

Since Chang [1] introduced fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology. In a Chang I-topology, the open sets are fuzzy, but the topology comprising those open sets is a crisp subset of I^X. However, in a completely different direction, Höhle [2] presented a notion of fuzzy topology being viewed as an I-fuzzy subset of 2^X. Then Kubiak [3] and Šostak [4] independently extended Höhle's fuzzy topology to L-subsets of L^X, which is called L-fuzzy topology (see [5, 6]). From a logical point of view, Ying [7] studied Höhle's topology and called it fuzzifying topology.

Connectivity is one of the most important notions in general topology. It has been generalized to L-topology in terms of many forms (see [8–17], etc.). In a fuzzifying topological space, Ying [18] introduced a definition of connectivity and Fang [19] proved Fan's theorem. In a $[0,1]$-fuzzy topological space $(X, \mathcal{T})$, Šostak introduced a notion of connectedness degree by means of the level $[0,1]$-topological spaces $(X, \mathcal{T}_\alpha)$ [20, 21], that is, it can be viewed as connectivity in a $[0,1]$-topological space. Although a definition of connectivity was also presented by Yue and Fang [22] in $[0,1]$-fuzzy topological spaces, it was defined for whole L-fuzzy topological space not for arbitrary L-fuzzy subset.

In this paper, we first introduce the notion of separatedness degrees in L-fuzzy topological spaces by means of L-fuzzy closure operators. Furthermore, we present the notion of connectedness degrees of L-fuzzy subsets, which is a generalization of Yue and

Fang's connectedness degree. Many properties of connectedness in general topology can be generalized to L-fuzzy topological spaces.

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ denotes a completely distributive DeMorgan algebra. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. The set of all nonzero co-prime elements of L is denoted by $J(L)$.

We say that a is wedge below b in L, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L : a \prec b\}$ for each $b \in L$. For any $b \in L$, define $\beta(b) = \{a \in L : a \prec b\}$. Some properties of β can be found in [23].

For a nonempty set X, the set of all nonzero coprime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ $(\lambda \in J(L))$. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively.

For any L-fuzzy set $A \in L^X$ and any $a \in L$, we use the following notations:

$$A_{[a]} = \{x \in X : A(x) \geq a\},$$
$$\underline{a}(x) = a, \quad \forall x \in X. \tag{2.1}$$

Definition 2.1 (see [3–5]). An L-fuzzy topology on a set X is a map $\mathcal{T} : L^X \to L$ such that

(LFT1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\perp}) = \top$;

(LFT2) for all $U, V \in L^X$, $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;

(LFT3) for all $U_j \in L^X$, $j \in J$, $\mathcal{T}(\bigvee_{j\in J} U_j) \geq \bigwedge_{j\in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the degree of closedness of U. The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

A mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is said to be continuous with respect to L-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if $\mathcal{T}_1(f_L^{\leftarrow}(U)) \geq \mathcal{T}_2(U)$ holds for all $U \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(U)(x) = U(f(x))$ [24].

Definition 2.2 (see [25]). An L-fuzzy closure operator on X is a mapping $\mathrm{Cl} : L^X \to L^{J(L^X)}$ satisfying the following conditions:

(LFC1) $\mathrm{Cl}(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} \mathrm{Cl}(A)(x_\mu)$, for all $x_\lambda \in J(L^X)$;

(LFC2) $\mathrm{Cl}(\underline{\perp})(x_\lambda) = \perp$ for any $x_\lambda \in J(L^X)$;

(LFC3) $\mathrm{Cl}(A)(x_\lambda) = \top$ for any $x_\lambda \leq A$;

(LFC4) $\mathrm{Cl}(A \vee B) = \mathrm{Cl}(A) \vee \mathrm{Cl}(B)$;

(LFC5) for all $a \in L \setminus \{\perp\}$, $(\mathrm{Cl}(\bigvee(\mathrm{Cl}(A))_{[a]}))_{[a]} \subseteq (\mathrm{Cl}(A))_{[a]}$.

$\mathrm{Cl}(A)(x_\lambda)$ is called the degree to which x_λ belongs to the closure of A.

Lemma 2.3 (see [25]). *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and let* Cl *be the L-fuzzy closure operator induced by $\mathcal{T}$. Then for all $x_\lambda \in J(L^X)$, for all $A \in L^X$,*

$$\begin{aligned} \mathrm{Cl}(A)(x_\lambda) &= \bigwedge\left\{ (\mathcal{T}(D'))' : D \in L^X,\ x_\lambda \not\leq D \geq A \right\} \\ &= \bigwedge_{x_\lambda \not\leq D \geq A} (\mathcal{T}(D'))'. \end{aligned} \tag{2.2}$$

Definition 2.4 (see [17, 23]). In an L-topological space (X, τ), two L-fuzzy sets A, B are called separated if $A^- \wedge B = A \wedge B^- = \underline{\perp}$, where A^- denotes the closure of A.

Definition 2.5 (see [17, 23]). In an L-topological space (X, τ), an L-fuzzy set D is called connected if D can not be represented as a union of two separated non-null L-fuzzy sets.

3. Separatedness Degrees in L-Fuzzy Topological Spaces

In this section, in order to generalize Definition 2.5 to L-fuzzy topological spaces, we will introduce the concept of separatedness degrees in L-fuzzy topological spaces by means of L-fuzzy closure operators.

Definition 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $A, B \in L^X$. Define

$$\begin{aligned} \mathrm{Sep}(A,B) &= \left(\bigwedge\{ (\mathrm{Cl}(B)(x_\lambda))' : x_\lambda \leq A \} \right) \wedge \left(\bigwedge\left\{ (\mathrm{Cl}(A)(y_\mu))' : y_\mu \leq B \right\} \right) \\ &= \left(\bigwedge_{x_\lambda \leq A} (\mathrm{Cl}(B)(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \leq B} (\mathrm{Cl}(A)(y_\mu))' \right). \end{aligned} \tag{3.1}$$

Then $\mathrm{Sep}(A, B)$ is said to be the separatedness degree of A and B.

The following result is obvious.

Proposition 3.2. *Let $\mathcal{T} : L^X \to \{\perp, \top\}$ be an L-topology on X and $A, B \in L^X$. Then $\mathrm{Sep}(A, B) = \top$ if and only if A and B are separated in $(X, \mathcal{T})$.*

Lemma 3.3. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $A, B \in L^X$. If $A \wedge B \neq \underline{\perp}$, then* $\mathrm{Sep}(A, B) = \perp$.

Proof. From $A \wedge B \neq \underline{\perp}$, we can take $z_\gamma \in J(L^X)$ such that $z_\gamma \leq A \wedge B$. Thus we have

$$\begin{aligned} \mathrm{Sep}(A,B) &= \left(\bigwedge_{x_\lambda \leq A} (\mathrm{Cl}(B)(x_\lambda))' \right) \wedge \left(\bigwedge_{x_\lambda \leq B} (\mathrm{Cl}(A)(x_\lambda))' \right) \\ &\leq (\mathrm{Cl}(B)(z_\gamma))' \wedge (\mathrm{Cl}(A)(z_\gamma))' = \top' \wedge \top' = \perp. \end{aligned} \tag{3.2}$$

□

Lemma 3.4. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space, and $A,B,C,D\in L^X$. If $C\leq A$ and $D\leq B$, then* $\mathrm{Sep}(A,B)\leq\mathrm{Sep}(C,D)$.

Proof. If $C\leq A$ and $D\leq B$, then $\mathrm{Cl}(C)\leq\mathrm{Cl}(A)$ and $\mathrm{Cl}(D)\leq\mathrm{Cl}(B)$. Hence we have

$$\begin{aligned}\mathrm{Sep}(A,B)&=\left(\bigwedge_{x_\lambda\leq A}(\mathrm{Cl}(B)(x_\lambda))'\right)\wedge\left(\bigwedge_{y_\mu\leq B}(\mathrm{Cl}(A)(y_\mu))'\right)\\&\leq\left(\bigwedge_{x_\lambda\leq A}(\mathrm{Cl}(D)(x_\lambda))'\right)\wedge\left(\bigwedge_{y_\mu\leq B}(\mathrm{Cl}(C)(y_\mu))'\right)\\&\leq\left(\bigwedge_{x_\lambda\leq C}(\mathrm{Cl}(D)(x_\lambda))'\right)\wedge\left(\bigwedge_{y_\mu\leq D}(\mathrm{Cl}(C)(y_\mu))'\right)\\&=\mathrm{Sep}(C,D).\end{aligned}\tag{3.3}$$

□

Lemma 3.5. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space, $A,B\in L^X$ and $a\in J(L)$. Then $(\mathrm{Sep}(A,B))'\not\geq a$ if and only if there exist $D,E\in L^X$ such that*

$$D\geq A,\quad E\geq B,\quad D\wedge B=E\wedge A=\underline{\perp},\quad (\mathcal{T}(D'))'\vee(\mathcal{T}(E'))'\not\geq a.\tag{3.4}$$

Proof. Suppose that $(\mathrm{Sep}(A,B))'\not\geq a$. Then $(\mathrm{Sep}(A,B))'\not\geq b$ for some $b\in\beta^*(a)$. This implies

$$\bigvee_{x_\lambda\leq A}\mathrm{Cl}(B)(x_\lambda)\vee\bigvee_{y_\mu\leq B}\mathrm{Cl}(A)(y_\mu)\not\geq b.\tag{3.5}$$

Further more, we have

$$\bigvee_{x_\lambda\leq A}\bigwedge_{x_\lambda\not\leq E\geq B}(\mathcal{T}(E'))'\vee\bigvee_{y_\mu\leq B}\bigwedge_{y_\mu\not\leq D\geq A}(\mathcal{T}(D'))'\not\geq b.\tag{3.6}$$

Hence for any $x_\lambda\leq A$ and for any $y_\mu\leq B$, there are $D_{y_\mu},E_{x_\lambda}\in L^X$ such that $x_\lambda\not\leq E_{x_\lambda}\geq B$, $y_\mu\not\leq D_{y_\mu}\geq A$ and $(\mathcal{T}(D'_{y_\mu}))'\vee(\mathcal{T}(E'_{x_\lambda}))'\not\geq b$. Let $E=\bigwedge_{x_\lambda\leq A}E_{x_\lambda}$ and $D=\bigwedge_{y_\mu\leq B}D_{y_\mu}$. Then, obviously, we have that $D\geq A$, $E\geq B$, $D\wedge B=E\wedge A=\underline{\perp}$ and

$$\begin{aligned}(\mathcal{T}(D'))'\vee(\mathcal{T}(E'))'&=\left(\mathcal{T}\left(\bigvee_{y_\mu\leq B}D'_{y_\mu}\right)\right)'\vee\left(\mathcal{T}\left(\bigvee_{x_\lambda\leq A}E'_{x_\lambda}\right)\right)'\\&\leq\bigvee_{y_\mu\leq B}\left(\mathcal{T}\left(D'_{y_\mu}\right)\right)'\vee\bigvee_{x_\lambda\leq A}(\mathcal{T}(E'_{x_\lambda}))'\not\geq a.\end{aligned}\tag{3.7}$$

Conversely if there exist $D, E \in L^X$ such that

$$D \geq A, \quad E \geq B, \quad D \wedge B = E \wedge A = \underline{\perp}, \quad (\mathcal{T}(D'))' \vee (\mathcal{T}(E'))' \ngeq a. \tag{3.8}$$

Then by

$$\begin{aligned}(\mathrm{Sep}(A,B))' &= \bigvee_{x_\lambda \leq A} \mathrm{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \mathrm{Cl}(A)(y_\mu) \\ &= \bigvee_{x_\lambda \leq A} \bigwedge_{x_\lambda \nleq G \geq B} (\mathcal{T}(G'))' \vee \bigvee_{y_\mu \leq B} \bigwedge_{y_\mu \nleq H \geq A} (\mathcal{T}(H'))' \\ &\leq (\mathcal{T}(D'))' \vee (\mathcal{T}(E'))'\end{aligned} \tag{3.9}$$

we can obtain that $(\mathrm{Sep}(A,B))' \ngeq a$. □

4. Connectedness Degrees in L-Fuzzy Topological Spaces

Definition 4.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Define

$$\begin{aligned}\mathrm{Con}(G) &= \bigwedge \left\{ (\mathrm{Sep}(A,B))' : A, B \in L^X \setminus \{\underline{\perp}\},\ G = A \vee B \right\} \\ &= \bigwedge \left\{ \bigvee_{x_\lambda \leq A} \mathrm{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \leq B} \mathrm{Cl}(A)(y_\mu) : A, B \in L^X \setminus \{\underline{\perp}\},\ G = A \vee B \right\}.\end{aligned} \tag{4.1}$$

Then $\mathrm{Con}(G)$ is said to be the connectedness degree of G.

The following proposition shows that Definition 4.1 is a generalization of Definition 2.5.

Proposition 4.2. *Let $\mathcal{T} : L^X \to \{\perp, \top\}$ be an L-topology on X and $G \in L^X$. Then $\mathrm{Con}(G) = \top$ if and only if G is connected in $(X, \mathcal{T})$.*

Theorem 4.3. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then*

$$\mathrm{Con}(G) = \bigwedge \left\{ (\mathcal{T}(A'))' \vee (\mathcal{T}(B'))' : G \wedge A \neq \underline{\perp},\ G \wedge B \neq \underline{\perp},\ G \wedge A \wedge B = \underline{\perp},\ G \leq A \vee B \right\}. \tag{4.2}$$

Proof. On one hand, we have the following inequality:

$$
\begin{aligned}
&\operatorname{Con}(G)\\
&= \bigwedge\left\{ \bigvee_{x_\lambda \le A} \operatorname{Cl}(B)(x_\lambda) \vee \bigvee_{y_\mu \le B} \operatorname{Cl}(A)(y_\mu) : A, B \in L^X \setminus \{\perp\},\ G = A \vee B \right\}\\
&= \bigwedge\left\{ \bigvee_{x_\lambda \le A} \bigwedge_{x_\lambda \not\le D \ge B} (\tau(D'))' \vee \bigvee_{y_\mu \le B} \bigwedge_{y_\mu \not\le E \ge A} (\tau(E'))' : A, B \in L^X \setminus \{\perp\},\ G = A \vee B \right\}\\
&= \bigwedge\left\{ \bigvee_{x_\lambda \le G\wedge A} \bigwedge_{x_\lambda \not\le D \ge G\wedge B} (\tau(D'))' \vee \bigvee_{y_\mu \le G\wedge B} \bigwedge_{y_\mu \not\le E \ge G\wedge A} (\tau(E'))' : G \wedge A \ne \perp, G \wedge B \ne \perp, G \le A \vee B \right\}\\
&\le \bigwedge\left\{ \bigvee_{x_\lambda \le G\wedge A} (\tau(B'))' \vee \bigvee_{y_\mu \le G\wedge B} (\tau(A'))' : G \wedge A \ne \perp,\ G \wedge B \ne \perp,\ G \wedge A \wedge B = \perp,\ G \le A \vee B \right\}\\
&= \bigwedge\left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \ne \perp,\ G \wedge B \ne \perp,\ G \wedge A \wedge B = \perp,\ G \le A \vee B \right\};
\end{aligned}
\tag{4.3}
$$

on the other hand, in order to prove the inverse, we suppose that $\operatorname{Con}(G) \not\ge a$ $(a \in J(L))$. Then there exist $A, B \in L^X \setminus \{\perp\}$ such that $G = A \vee B$ and $(\operatorname{Sep}(A,B))' \not\ge a$. By Lemma 3.5 we know that there exists $D, E \in L^X$ such that

$$
D \ge A, \quad E \ge B, \quad D \wedge B = E \wedge A = \perp, \quad (\tau(D'))' \vee (\tau(E')) \not\ge a. \tag{4.4}
$$

Obviously $G \wedge D \ne \perp$, $G \wedge E \ne \perp$, $G \wedge D \wedge E = \perp$, $G \le D \vee E$. Hence we have

$$
\bigwedge\left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \ne \perp,\ G \wedge B \ne \perp,\ G \wedge A \wedge B = \perp,\ G \le A \vee B \right\} \not\ge a. \tag{4.5}
$$

Therefore,

$$
\operatorname{Con}(G) \ge \bigwedge\left\{ (\tau(B'))' \vee (\tau(A'))' : G \wedge A \ne \perp,\ G \wedge B \ne \perp,\ G \wedge A \wedge B = \perp,\ G \le A \vee B \right\}. \tag{4.6}
$$

The proof is completed. □

Example 4.4. Let $X = \{x, y\}$ and $L = [0,1]$. Define $B \in [0,1]^X$ by $B(x) = 0.5$ and $B(y) = 0$, and define $C \in [0,1]^X$ by $C(y) = 0.5$ and $C(x) = 0$, respectively. Let $\mathcal{T} : [0,1]^X \to [0,1]$ be defined as follows:

$$\mathcal{T}(A) = \begin{cases} 1, & A \in \{\underline{\top}, \underline{\bot}, \underline{0.5}\}, \\ 0.5, & A \in \{B', C'\}, \\ 0, & \text{others}. \end{cases} \tag{4.7}$$

Then $\mathcal{T}$ is an L-fuzzy topology on X. It is easy to verify that $\operatorname{Con}(\underline{a}) = 0.5$ for any $a \in (0, 0.5]$ and $\operatorname{Con}(\underline{b}) = 1$ for any $b \in (0.5, 1]$.

Corollary 4.5. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Then*

$$\begin{aligned} \operatorname{Con}(\underline{\top}) &= \bigwedge\left\{(\mathcal{T}(A'))' \vee (\mathcal{T}(B'))' : A \neq \underline{\bot},\ B \neq \underline{\bot},\ A \wedge B = \underline{\bot},\ \underline{\top} = A \vee B\right\} \\ &= \bigwedge\{(\mathcal{T}(A))' \vee (\mathcal{T}(B))' : A \neq \underline{\bot},\ B \neq \underline{\bot},\ A \wedge B = \underline{\bot},\ \underline{\top} = A \vee B\}. \end{aligned} \tag{4.8}$$

Remark 4.6. Yue and Fang [22] introduced a definition of connectivity in a $[0,1]$-fuzzy topological space. It is easy to see that Yue and Fang's definition is a special case of our definition from Corollary 4.5.

Theorem 4.7. *For any $e \in J(L^X)$, it follows that* $\operatorname{Con}(e) = \top$.

Proof. From Theorem 4.3 we have

$$\begin{aligned} \operatorname{Con}(e) &= \bigwedge\left\{(\mathcal{T}(A'))' \vee (\mathcal{T}(B'))' : e \wedge A \neq \underline{\bot},\ e \wedge B \neq \underline{\bot},\ e \wedge A \wedge B = \underline{\bot},\ e \leq A \vee B\right\} \\ &= \bigwedge \emptyset = \top. \end{aligned} \tag{4.9}$$

□

Theorem 4.8. *For any $G \in L^X$, one has*

$$\bigvee_{r \in J(L)} \operatorname{Con}\left(\bigvee (\operatorname{Cl}(G))_{[r]}\right) \geq \operatorname{Con}(G). \tag{4.10}$$

Proof. Let $a \in J(L)$ and $a \leq \operatorname{Con}(G)$. Now we prove $\bigvee_{r \in J(L)} \operatorname{Con}(\bigvee(\operatorname{Cl}(G))_{[r]}) \geq a$. Suppose that $\bigvee_{r \in J(L)} \operatorname{Con}(\bigvee(\operatorname{Cl}(G))_{[r]}) \not\geq a$. Then $\operatorname{Con}(\bigvee(\operatorname{Cl}(G))_{[a]}) \not\geq a$. By Theorem 4.3 we know that there exists $A, B \in L^X$ such that

$$\begin{gathered} \left(\bigvee (\operatorname{Cl}(G))_{[a]}\right) \wedge A \neq \underline{\bot}, \quad \left(\bigvee (\operatorname{Cl}(G))_{[a]}\right) \wedge B \neq \underline{\bot}, \quad \left(\bigvee (\operatorname{Cl}(G))_{[a]}\right) \wedge A \wedge B = \underline{\bot}, \\ \bigvee (\operatorname{Cl}(G))_{[a]} \leq A \vee B, \quad (\mathcal{T}(B'))' \vee (\mathcal{T}(A'))' \not\geq a. \end{gathered} \tag{4.11}$$

By $(\bigvee(\mathrm{Cl}(G))_{[a]}) \wedge A \neq \perp$ we know that there exists $x_\lambda \leq A$ such that $\mathrm{Cl}(G)(x_\lambda) \geq a$. Furthermore by $(\bigvee(\mathrm{Cl}(G))_{[a]}) \wedge A \wedge B = \perp$ we obtain $x_\lambda \nleq B$.

Now we prove $G \wedge A \neq \perp$. In fact, if $G \wedge A = \perp$, then by $G \leq \bigvee(\mathrm{Cl}(G))_{[a]} \leq A \vee B$ we have $G \leq B$, hence it follows that

$$a \leq \mathrm{Cl(G)}(x_\lambda) = \bigwedge_{x_\lambda \nleq E \geq G} (\tau(E'))' \leq (\tau(B'))', \tag{4.12}$$

contradicting $a \nleq (\tau(B'))'$. Analogously, we can prove $G \wedge B \neq \perp$. Thus by

$$\begin{gathered} G \wedge A \neq \perp, \qquad G \wedge B \neq \perp, \qquad G \wedge A \wedge B = \perp, \\ G \leq A \vee B, \qquad (\tau(B'))' \vee (\tau(A'))' \ngeq a, \end{gathered} \tag{4.13}$$

and Theorem 4.3, we know that $\mathrm{Con}(G) \ngeq a$, contradicting $\mathrm{Con}(G) \geq a$. It is proved that $\bigvee_{r \in J(L)} \mathrm{Con}(\bigvee(\mathrm{Cl}(G))_{[r]}) \geq \mathrm{Con}(G)$. □

Theorem 4.9. *For any $G, H \in L^X$, one has*

$$\mathrm{Con}(G \vee H) \geq (\mathrm{Sep}(G, H))' \wedge \mathrm{Con}(G) \wedge \mathrm{Con}(H). \tag{4.14}$$

Proof. Let $a \in J(L)$ and $a \leq (\mathrm{Sep}(G,H))' \wedge \mathrm{Con}(G) \wedge \mathrm{Con}(H)$. Now we prove $\mathrm{Con}(G \vee H) \geq a$. Suppose that $\mathrm{Con}(G \vee H) \ngeq a$. Then by Theorem 4.3 we know that there exist $A, B \in L^X$ such that

$$\begin{gathered} (G \vee H) \wedge A \neq \perp, \qquad (G \vee H) \wedge B \neq \perp, \qquad (G \vee H) \wedge A \wedge B = \perp, \\ G \vee H \leq A \vee B, \qquad (\tau(B'))' \vee (\tau(A'))' \ngeq a. \end{gathered} \tag{4.15}$$

By $(G \vee H) \wedge A \neq \perp$ we know that one of $G \wedge A \neq \perp$ and $H \wedge A \neq \perp$ must be true.

Suppose that $G \wedge A \neq \perp$ (the case of $H \wedge A \neq \perp$ is analogous). Then we must have $G \wedge B = \perp$, otherwise if $G \wedge B \neq \perp$, then by

$$G \wedge A \neq \perp, \quad G \wedge B \neq \perp, \quad G \wedge A \wedge B = \perp, \quad G \leq A \vee B, \quad (\tau(B'))' \vee (\tau(A'))' \ngeq a \tag{4.16}$$

we know that $\mathrm{Con}(G) \ngeq a$, contradicting $\mathrm{Con}(G) \geq a$. In this case by $(G \vee H) \wedge B \neq \perp$ we know that $H \wedge B \neq \perp$. Analogously we can prove $H \wedge A = \perp$. Thus by $G \vee H \leq A \vee B$ we can obtain that $G \leq A$ and $H \leq B$. Hence by

$$G \leq A, \quad H \leq B, \quad G \wedge B = H \wedge A = \perp, \quad (\tau(B'))' \vee (\tau(A'))' \ngeq a \tag{4.17}$$

and Lemma 3.5 we know that $(\mathrm{Sep}(G,H))' \ngeq a$, contradicting $(\mathrm{Sep}(G,H))' \geq a$. This shows that $\mathrm{Con}(G \vee H) \geq a$. It is proved that $\mathrm{Con}(G \vee H) \geq (\mathrm{Sep}(G,H))' \wedge \mathrm{Con}(G) \wedge \mathrm{Con}(H)$. □

By Lemma 3.3 we can obtain the following result.

Corollary 4.10. *Let* $(X, \mathcal{T})$ *be an L-fuzzy topological space and* $G, H \in L^X$*. If* $A \wedge B \neq \perp$*, then* $\mathrm{Con}(G \vee H) \geq \mathrm{Con}(G) \wedge \mathrm{Con}(H)$*.*

Theorem 4.11. *Let* $(X, \mathcal{T})$ *be an L-fuzzy topological space and* $G \in L^X$*. Then*

$$\mathrm{Con}(G) = \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \mathrm{Con}\left(D_{x_\lambda y_\mu}\right) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \tag{4.18}$$

Proof. It is obvious that

$$\mathrm{Con}(G) \leq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \mathrm{Con}\left(D_{x_\lambda y_\mu}\right) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \tag{4.19}$$

Now we prove that

$$\mathrm{Con}(G) \geq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \mathrm{Con}\left(D_{x_\lambda y_\mu}\right) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \tag{4.20}$$

Suppose that $\bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \{\mathrm{Con}(D_{x_\lambda y_\mu}) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G\} \geq a$ ($a \in J(L)$). Take a fixed $x_\lambda \leq G$. Then for any $y_\mu \leq G$, there exists a $D_{x_\lambda y_\mu} \in L^X$ such that $x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G$ and $\mathrm{Con}(D_{x_\lambda y_\mu}) \geq a$. Let $D_{x_\lambda} = \bigvee_{y_\mu \leq G} D_{x_\lambda y_\mu}$. Obviously $D_{x_\lambda} = G$ and $\bigwedge_{y_\mu \leq G} D_{x_\lambda y_\mu} \neq \perp$. By Corollary 4.10 we easily obtain $\mathrm{Con}(G) = \mathrm{Con}(D_{x_\lambda}) \geq \bigwedge_{y_\mu \leq G} \mathrm{Con}(D_{x_\lambda y_\mu}) \geq a$. This shows

$$\mathrm{Con}(G) \geq \bigwedge_{x_\lambda, y_\mu \leq G} \bigvee \left\{ \mathrm{Con}\left(D_{x_\lambda y_\mu}\right) : x_\lambda, y_\mu \leq D_{x_\lambda y_\mu} \leq G \right\}. \tag{4.21}$$

□

Theorem 4.12. *If* $f_L^{\rightarrow} : (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{T}_2)$ *is continuous, then* $\mathrm{Con}(f_L^{\rightarrow}(G)) \geq \mathrm{Con}(G)$*.*

Proof. This can be proved from Theorem 4.3 and the following inequality:

$$\begin{aligned}
\mathrm{Con}(f_L^{\rightarrow}(G)) = \bigwedge\Big\{ & (\mathcal{T}_2(A'))' \vee (\mathcal{T}_2(B'))' : f_L^{\rightarrow}(G) \wedge A \neq \perp,\ f_L^{\rightarrow}(G) \wedge B \neq \perp, \\
& f_L^{\rightarrow}(G) \wedge A \wedge B = \perp,\ f_L^{\rightarrow}(G) \leq A \vee B \} \\
\geq \bigwedge\Big\{ & (\mathcal{T}_2(A'))' \vee (\mathcal{T}_2(B'))' : G \wedge f_L^{\leftarrow}(A) \neq \perp,\ G \wedge f_L^{\leftarrow}(B) \neq \perp, \\
& G \wedge f_L^{\leftarrow}(A) \wedge f_L^{\leftarrow}(B) = \perp,\ G \leq f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B) \Big\} \\
\geq \bigwedge\Big\{ & (\mathcal{T}_1(f_L^{\leftarrow}(A')))' \vee (\mathcal{T}_1(f_L^{\leftarrow}(B')))' : G \wedge f_L^{\leftarrow}(A) \neq \perp,\ G \wedge f_L^{\leftarrow}(B) \neq \perp, \\
& G \wedge f_L^{\leftarrow}(A) \wedge f_L^{\leftarrow}(B) = \perp,\ G \leq f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B) \Big\} \\
\geq \mathrm{Con}(G). &
\end{aligned} \tag{4.22}$$

□

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