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Research Article

Schatten Class Toeplitz Operators on the Bergman Space

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We have shown that if the Toeplitz operator T_ϕ on the Bergman space $L_a^2(\mathbb{D})$ belongs to the Schatten class $S_p, 1 \leq p < \infty$, then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$, where $\widetilde{\phi}$ is the Berezin transform of $\phi, d\lambda(z) = dA(z)/(1-|z|^2)^2$, and $dA(z)$ is the normalized area measure on the open unit disk $\mathbb{D}$. Further, if $\phi \in L^p(\mathbb{D}, d\lambda)$ then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ and $T_\phi \in S_p$. For certain subclasses of $L^\infty(\mathbb{D})$, necessary and sufficient conditions characterizing Schatten class Toeplitz operators are also obtained.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. Let $dA(z) = (1/\pi)dx\,dy = (1/\pi)r dr d\theta$ and $L^2(\mathbb{D}, dA)$ be the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions f on $\mathbb{D}$ with the inner product

$$\langle f, g \rangle = \int f(z)\overline{g(z)}dA(z). \tag{1.1}$$

Let $L^\infty(\mathbb{D}, dA)$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{D}\} < \infty, \tag{1.2}$$

and let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $L_a^2(\mathbb{D})$ be the subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions. The space $L_a^2(\mathbb{D})$ is called the Bergman space. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L_a^2(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L_a^2(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}\,dA(w) \tag{1.3}$$

for all f in $L_a^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z,w) = \overline{K_z(w)}. \tag{1.4}$$

The function $K(z,w)$ is thus the reproducing kernel for the Bergman space $L_a^2(\mathbb{D})$ and is called the Bergman kernel. It can be shown that the sequence of functions $\{e_n(z)\} = \{\sqrt{n+1}z^n\}_{n\geq 0}$ forms the standard orthonormal basis for $L_a^2(\mathbb{D})$ and $K(z,w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}$. The Bergman kernel is independent of the choice of orthonormal basis and $K(z,w) = 1/(1-z\overline{w})^2$.

Let $k_a(z) = K(z,\overline{a})/\sqrt{K(\overline{a},\overline{a})} = (1-|a|^2)/(1-\overline{a}z)^2$. These functions k_a are called the normalized reproducing kernels of L_a^2; it is clear that they are unit vectors in L_a^2. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = (a-z)/(1-\overline{a}z)$, $z \in \mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}z|^4}. \tag{1.5}$$

When $|z| \to 1$, $k_z \to 0$ weakly and the normalized reproducing kernels k_z, $z \in \mathbb{D}$ span $L_a^2(\mathbb{D})$ [1]. Since $L_a^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D},dA)$ (see [1]), there exists an orthogonal projection P from $L^2(\mathbb{D},dA)$ onto $L_a^2(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator T_ϕ on $L_a^2(\mathbb{D})$ by $T_\phi f = P(\phi f)$, $f \in L_a^2(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, let $\widetilde{\phi}(z) = \langle T_\phi k_z, k_z\rangle$. The function $\widetilde{\phi}(z)$ is called the Berezin transform of ϕ. Let H_ϕ be the Hankel operator from L_a^2 into $(L_a^2)^\perp$ defined by $H_\phi f = (I-P)(\phi f)$. It is easy to check that $H_{\overline{\phi}}^* H_\phi = T_{|\phi|^2} - T_{\overline{\phi}}T_\phi$ (see [1]). For $\varphi \in L^\infty(\mathbb{D})$, define $S_\varphi : L_a^2 \to L_a^2$ as $S_\varphi f = PJ(\varphi f)$, where $J : L^2 \to L^2$ is defined as $Jf(z) = f(\overline{z})$. The operator S_φ is called the little Hankel operator on $L_a^2(\mathbb{D})$. Let $d\lambda(z) = K(z,z)dA(z) = dA(z)/(1-|z|^2)^2$, the Mobius invariant measure on $\mathbb{D}$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself, and let $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$.

Often it is not easy to verify that a linear operator is bounded, and it is even more difficult to determine its norm. No conditions on the matrix entries a_{ij} have been found which are necessary and sufficient for A to be bounded, nor has $\|A\|$ been determined in the general case. For the more general problem we also need analogues of the notions of operator norm. For more details see [2, 3]. The family of norms that has received much attention during the last decade is the Schatten norm.

A proper two-sided ideal $\mathfrak{T}$ in $\mathcal{L}(H)$ is said to be a norm ideal if there is a norm on $\mathfrak{T}$ satisfying the following properties:

(i) $(\mathfrak{T}, \|\cdot\|_{\mathfrak{T}})$ is a Banach space;

(ii) $\|AXB\|_{\mathfrak{T}} \leq \|A\|\|X\|_{\mathfrak{T}}\|B\|$ for all $A, B \in \mathcal{L}(H)$ and for all $X \in \mathfrak{T}$;

(iii) $\|X\|_{\mathfrak{T}} = \|X\|$ for X a rank one operator.

If $(\mathcal{T}, \|\cdot\|_{\mathcal{T}})$ is a norm ideal, then the norm $\|\cdot\|_{\mathcal{T}}$ is unitarily invariant, in the sense that $\|UAV\|_{\mathcal{T}} = \|A\|_{\mathcal{T}}$ for all A in $\mathcal{T}$ and unitary U, V in $\mathcal{L}(H)$. Each proper ideal of $\mathcal{L}(H)$ is contained in the ideal of compact operators. Two special families of unitarily invariant norms satisfying conditions (i), (ii), and (iii) are the Schatten p-norms defined on the set of compact operators and the Ky Fan norms.

For any nonnegative integer n, the nth singular value of $T \in \mathcal{LC}(H)$ is defined by

$$s_n(T) = \inf\{\|T - K\| : K \in \mathcal{L}(H), \text{ rank } K \le n\}. \tag{1.6}$$

Here $\|\cdot\|$ is the operator norm. Clearly, $s_0(T) = \|T\|$ and

$$s_0(T) \ge s_1(T) \ge s_2(T) \ge \cdots \ge 0. \tag{1.7}$$

Thus $s_n(T)$ is the distance, with respect to the operator norm, of T from the set of operators of rank at most n in $\mathcal{L}(H)$. The spectral theorem shows that the singular values of the compact operator T are the square roots of the eigenvalues of T^*T as long as H is separable and infinite dimensional. Notice that $s_n(T)$ can be defined for any $T \in \mathcal{L}(H)$ but clearly, $s_n(T) \to 0$ if and only if T is compact.

The Schatten Von Neumann class $S_p = S_p(H)$, $0 < p < \infty$, consists of all operators $T \in \mathcal{LC}(H)$ such that

$$\|T\|_{S_p} = \left(\sum_{n=0}^{\infty} (s_n(T))^p\right)^{1/p} < \infty. \tag{1.8}$$

If $1 \le p < \infty$, then $\|\cdot\|_{S_p}$ is a norm, which makes S_p a Banach space. For $p < 1$, $\|\cdot\|_{S_p}$ does not satisfy the triangle inequality, it is a quasinorm (i.e., $\|T_1 + T_2\|_{S_p} \le C(\|T_1\|_{S_p} + \|T_2\|_{S_p})$ for $T_1, T_2 \in S_p$ and C, a constant), which makes S_p a quasiBanach space. We will be mainly concerned with the range $1 \le p < \infty$. The space S_1 is also called the trace class and S_2 is called the Hilbert-Schmidt class. The linear functional trace is defined on S_1 by

$$\operatorname{trace} T = \sum_{n=0}^{\infty} \langle T\varepsilon_n, \varepsilon_n\rangle, \quad T \in S_1, \tag{1.9}$$

where $\{\varepsilon_n\}_{n\ge 0}$ is an orthonormal basis in H. Moreover, the right-hand side does not depend on the choice of the basis. If $1 < p < \infty$, the dual space S_p^* can be identified with S_q with respect to the pairing $\langle T, R\rangle = \operatorname{trace} TR^*$, $T \in S_p$, $R \in S_q$. Here $q = p/(p-1)$ is the dual exponent. With respect to the same pairing one can identify S_1^* with $\mathcal{L}(H)$ and $(\mathcal{LC}(H))^*$ with S_1. We refer the reader to [2, 3] for basic facts about Schatten p-classes. The Schatten p-classes should be seen as gradations of compactness for an operator. Each Schatten p-class is dense in the space of compact operators in the operator norm. For this reason, it is of interest, given a certain class of operators, to ask whether or not there are compact operators not in any Schatten p-class. For instance, this was proved by Arazy et al. [4] and Zhu [1] for Hankel operators on Bergman space.

Hankel operators are closely related to Toeplitz operators. Many problems about Toeplitz operators can also be formulated in terms Hankel operators and vice versa. The singular values of Hankel operators on the Hardy space play a crucial role in rational approximation. The celebrated results of Adamjan et al. [5] which give the achievable error in approximating a Hankel operator Γ by another one of smaller rank in terms of the singular values of the Hankel operator is an illustration of this. It may be noted here that the Adamjan, Arov, and Krein theorem has had a considerable influence on the treatment of the problem that arises in engineering applications in the context of model reduction, that is, the problem of finding a simple model to replace a relatively complicated one without too great a loss of accuracy. In view of this it would be nice to have a satisfactory characterization for Schatten class Toeplitz operators on the Bergman space.

In this paper we find necessary and sufficient conditions on ϕ that will ensure that the Toeplitz operator T_ϕ belong to S_p, $1 \le p < +\infty$. This will provide some quantitative estimates (size estimates of these operators) in terms of norms. We will also use S_∞ to denote the full algebra of bounded linear operators from the Bergman space $L_a^2(\mathbb{D})$ into itself.

For z and w in $\mathbb{D}$, let $\phi_z(w) = (z - w)/(1 - \overline{z}w)$. These are involutive Mobius transformations on D. In fact

(1) $\phi_z \circ \phi_z(w) \equiv w$;

(2) $\phi_z(0) = z$, $\phi_z(z) = 0$;

(3) ϕ_z has a unique fixed point in $\mathbb{D}$.

Given $z \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_z f(w) = k_z(w) f(\phi_z(w))$. Since $|k_z|^2$ is the real Jacobian determinant of the mapping ϕ_z (see [1]), U_z is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA)$ and $L_a^2(\mathbb{D})$. It is also easy to check that $U_z^* = U_z$, thus U_z is a self-adjoint unitary operator. If $\phi \in L^\infty(\mathbb{D}, dA)$ and $z \in \mathbb{D}$ then $U_z T_\phi = T_{\phi\circ\phi_z} U_z$. This is because $PU_z = U_z P$ and for $f \in L_a^2$, $T_{\phi\circ\phi_z} U_z f = T_{\phi\circ\phi_z}((f \circ \phi_z)k_z) = P((\phi \circ \phi_z)(f \circ \phi_z)k_z) = P(U_z(\phi f)) = U_z P(\phi f) = U_z T_\phi f$. Let $\mathrm{Aut}(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$, and G_0 the isotropy subgroup at 0; that is, $G_0 = \{\Psi \in \mathrm{Aut}(\mathbb{D}) : \Psi(0) = 0\}$.

2. Compact Operators Whose Real and Imaginary Parts Are Positive

Zhu [1] had shown that if ϕ is a nonnegative function on $\mathbb{D}$, $1 \le p \le \infty$, then T_ϕ is in the Schatten class S_p if and only if $\widetilde{\phi}(z)$ is in $L^p(\mathbb{D}, d\lambda)$. The following is an easy consequence of it.

Proposition 2.1. *Suppose that $T_\phi \in \mathcal{L}(L_a^2(\mathbb{D}))$ is such that $T_\phi = T_{\phi_1} + iT_{\phi_2}$, where $\phi_1 \ge 0$ and $\phi_2 \ge 0$. The Toeplitz operator $T_\phi \in S_p$, $1 \le p \le \infty$ if and only if $\widetilde{\phi}(z) \in L^p(\mathbb{D}, d\lambda)$. In this case, $\|T_\phi\|_p^2 \le \|T_{\phi_1}\|_p^2 + \|T_{\phi_2}\|_p^2$ if $2 \le p < \infty$ and $\|T_\phi\|_p^2 \ge \|T_{\phi_1}\|_p^2 + \|T_{\phi_2}\|_p^2$ if $1 \le p \le 2$.*

Proof. Suppose $2 \le p \le \infty$ and $T_\phi \in S_p$. Notice that $T_{\phi_1} = (T_\phi + T_\phi^*)/2$ and $T_{\phi_2} = (T_\phi - T_\phi^*)/2i$ and since S_p is a Banach space and is closed under adjoints, hence T_{ϕ_1} and T_{ϕ_2} belong to S_p. From [1], it follows that $\widetilde{\phi_1}, \widetilde{\phi_2} \in L^p(\mathbb{D}, d\lambda)$. Hence $\widetilde{\phi} = \widetilde{\phi_1} + i\widetilde{\phi_2} \in L^p(\mathbb{D}, d\lambda)$. Now suppose $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$, $2 \le p \le \infty$. This implies $\widetilde{\phi_1}, \widetilde{\phi_2} \in L^p(\mathbb{D}, d\lambda)$. From [1], it follows that $T_{\phi_1}, T_{\phi_2} \in S_p$. Hence $T_\phi \in S_p$ as S_p is a vector space and $\|T_\phi\|_p^2 \le \|T_{\phi_1}\|_p^2 + \|T_{\phi_2}\|_p^2$ (see [6]). Now let $1 \le p < 2$ and assume $T_\phi \in S_p$. This implies $T_{\phi_1}, T_{\phi_2} \in S_p$ and therefore by [1], $\widetilde{\phi_1}, \widetilde{\phi_2} \in L^p(\mathbb{D}, d\lambda)$. Hence $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ and by [6], $\|T_\phi\|_p^2 \ge \|T_{\phi_1}\|_p^2 + \|T_{\phi_2}\|_p^2$. Now suppose $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$, $1 \le p < 2$. Then $\widetilde{\phi_1}, \widetilde{\phi_2} \in L^p(\mathbb{D}, d\lambda)$ and hence $T_{\phi_1}, T_{\phi_2} \in S_p$. Since S_p is a vector space, $T_\phi \in S_p$. $\square$

Example 2.2. Let $\phi(z) = (1 - |z|)^r$, $r > 0$. Then $T_\phi \in S_2$ only if $r > 1/2$, and $T_\phi \in S_1$ only if $r > 1$. This can be seen as follows. The matrix of T_ϕ with respect to the standard orthonormal basis $\{e_n(z)\}_{n\geq 0} = \{\sqrt{n+1}z^n\}_{n\geq 0}$ of $L_a^2(\mathbb{D})$ is diagonal, $t_{kk} = \langle T_\phi e_k, e_k\rangle = (2k+2)!/((r+1)(r+2)\cdots(r+2k+2))$ and $\sum_{k=0}^{\infty} |t_{kk}|^2 = \infty$ for $r \leq 1/2$. Similarly $\sum_{k=0}^{\infty} t_{kk} < \infty$ if $r > 1$. If $r \leq 1$, $\sum_{k=0}^{\infty} t_{kk} = \infty$.

Proposition 2.3. *Let $\phi \in L^\infty(\mathbb{D})$ and suppose that ϕ is not the zero function. If T_ϕ is compact then* Range T_ϕ *is not closed.*

Proof. Since T_ϕ is compact, hence Range T_ϕ contains no closed infinite-dimensional subspace of $L_a^2(\mathbb{D})$. If now Range T_ϕ is closed then Range T_ϕ is finite dimensional. That is, T_ϕ is of finite rank. This implies by [7] that $\phi \equiv 0$. This is a contradiction as ϕ is not the zero function. □

Recall the following.

Suppose that A is a positive operator on a Hilbert space H and x is unit vector in H, then (i) $\langle A^p x, x\rangle \geq \langle Ax, x\rangle^p$ for all $p \geq 1$; (ii) $\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p$ for all $0 < p \leq 1$. For proof see [1].

Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. Define the composition operator C_φ from $L_a^2(\mathbb{D})$ into itself as $C_\varphi f = f \circ \varphi$. It is shown in [1] that C_φ is a bounded linear operator on $L_a^2(\mathbb{D})$ and $\|C_\varphi\| \leq (1 + |\varphi(0)|)/(1 - |\varphi(0)|)$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f$ by $C_a f(z) = f(\phi_a(z))$, where $\phi_a \in \mathrm{Aut}(\mathbb{D})$. The map C_a is a composition operator on $L_a^2(\mathbb{D})$.

Proposition 2.4. *If $\phi \in L^\infty(\mathbb{D})$ then T_ϕ is compact if and only if $T_{\phi\circ\phi_z}$ is compact.*

Proof. This follows from the fact that $T_{\phi\circ\phi_z} = U_z T_\phi U_z$ and as $U_z^2 = I$. □

Proposition 2.5. *For $\phi \in L^\infty(\mathbb{D})$, $\mathrm{MO}(\phi)^2(z) = \widetilde{|\phi|^2}(z) - |\tilde{\phi}(z)|^2 \leq \|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2$.*

Proof. Observe that

$$\begin{aligned}
\|H_\phi k_z\| &= \|(I-P)(\phi k_z)\| \\
&= \|(I-P)U_z(\phi\circ\phi_z)\| \\
&= \|U_z(I-P)(\phi\circ\phi_z)\| \\
&= \|(I-P)(\phi\circ\phi_z)\| \\
&= \|\phi\circ\phi_z - P(\phi\circ\phi_z)\|.
\end{aligned} \tag{2.1}$$

Similarly, we have $\|H_{\overline{\phi}} k_z\| = \|\overline{\phi}\circ\phi_z - P(\overline{\phi}\circ\phi_z)\| = \|\phi\circ\phi_z - \overline{P(\overline{\phi}\circ\phi_z)}\|$. Since $\tilde{\phi}(z) = P(\phi\circ\phi_z)(0)$ and $P\overline{g}(z) = \overline{g}(0)$ for any $g \in L_a^2$ and all $z \in \mathbb{D}$, we have

$$\begin{aligned}
\mathrm{MO}(\phi)^2(z) &= \widetilde{|\phi|^2}(z) - \left|\tilde{\phi}(z)\right|^2 \\
&= \left\|\phi\circ\phi_z - P(\phi\circ\phi_z)(0)\right\|^2 \\
&= \left\|\phi\circ\phi_z - P(\phi\circ\phi_z)\right\|^2 + \left\|P(\phi\circ\phi_z) - P(\phi\circ\phi_z)(0)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|H_\phi k_z\|^2 + \left\| P(\phi \circ \phi_z) - \overline{P\left(\overline{\phi} \circ \phi_z\right)(0)} \right\|^2 \\
&= \|H_\phi k_z\|^2 + \left\| P\left(\phi \circ \phi_z - \overline{P\left(\overline{\phi} \circ \phi_z\right)}\right) \right\|^2 \\
&\leq \|H_\phi k_z\|^2 + \left\| \phi \circ \phi_z - \overline{P\left(\overline{\phi} \circ \phi_z\right)} \right\|^2 \\
&= \|H_\phi k_z\|^2 + \left\| H_{\overline{\phi}} k_z \right\|^2.
\end{aligned}
\tag{2.2}
$$

□

Let $h > 1$. The generalized Kantorvich constant $K(p)$ is defined by

$$K(p) = \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p - 1}{h^p - h}\right)^p \tag{2.3}$$

for any real number p and it is known that $K(p) \in (0,1]$ for $p \in [0,1]$. We state below the known results on the generalized Kantorvich constant $K(p)$. Let A be strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = M/m > 1$. Then the following [8] inequalities (2.4) and (2.5) hold for every unit vector x and are equivalent:

$$K(p)\langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq \langle Ax, x\rangle^p \quad \text{for any } p > 1 \text{ or any } p < 0; \tag{2.4}$$

$$\langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq K(p)\langle Ax, x\rangle^p \quad \text{for any } p \in (0,1]. \tag{2.5}$$

The Kantorvich constant $K(p)$ is symmetric with respect to $p = 1/2$ and $K(p)$ is an increasing function of p for $p \geq 1/2$, $K(p)$ is a decreasing function of p for $p \leq 1/2$, and $K(0) = K(1) = 1$. Further, $K(p) \geq 1$ for $p \geq 1$ or $p \leq 0$, and $1 \geq K(p) \geq 2h^{1/4}/(h^{1/2} + 1)$ for $p \in [0,1]$.

Proposition 2.6. *Let T_ϕ be strictly positive satisfying $MI \geq T_\phi \geq mI > 0$, where $M > m > 0$. The following hold.*

(i) *If $0 < p < \infty$ and $T_\phi \in S_p$ then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$.*

(ii) *If $0 < p \leq 1$, $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ then $T_\phi \in S_p$.*

(iii) *Let $p \in [1, \infty)$ be such that $K(p) < \infty$. If $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ then $T_\phi \in S_p$.*

Proof. Suppose $p > 1$ and $T_\phi \in S_p$. Then

$$\int_{\mathbb{D}} \left\langle T_\phi^p k_z, k_z \right\rangle d\lambda(z) = \int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z\rangle d\lambda(z) < \infty. \tag{2.6}$$

Hence by (2.4), $\int_{\mathbb{D}} \langle T_\phi k_z, k_z\rangle^p d\lambda(z) < \infty$. That is, $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$. Suppose $0 < p \leq 1$ and $T_\phi \in S_p$. Then $\int_{\mathbb{D}} \langle T_\phi^p k_z, k_z\rangle d\lambda(z) = \int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z\rangle d\lambda(z) < \infty$. Hence from (2.5), it follows that $K(p)\int_{\mathbb{D}} \langle T_\phi k_z, k_z\rangle^p d\lambda(z) < \infty$. Since $K(p) \in (0,1]$ for $p \in [0,1]$, hence $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$.

Now assume $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$. Then if $0 < p \leq 1$ then by (2.5), we have $\int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z\rangle d\lambda(z) < \infty$ and hence $T_\phi \in S_p$. If $1 \leq p < \infty$, then by (2.4) and (2.5), if $K(p) < \infty$ and $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ then $\int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z\rangle d\lambda(z) < \infty$ and $T_\phi \in S_p$. □

3. Schatten Class Operators

In this section, we will obtain conditions to describe Schatten class Toeplitz operators on Bergman space $L_a^2(\mathbb{D})$. The results of this paper hold for Bergman spaces $L_a^2(\Omega)$, where Ω is any bounded symmetric domain in $\mathbb{C}$. For simplicity, we consider only the case of the open unit disk $\mathbb{D}$ in $\mathbb{C}$.

Let $BT = \{f \in L^1 : \|f\|_{BT} = \sup_{z\in\mathbb{D}} \widetilde{|f|}(z) < \infty\}$. The space L^∞ is properly contained in BT (see [9]) and if $\phi \in BT$ then T_ϕ is bounded on L_a^2 and there is a constant C such that $\|T_\phi\| \leq C\|\phi\|_{BT}$.

Theorem 3.1. *Suppose* $1 \leq p < \infty$ *and* $d\lambda(z) = dA(z)/(1-|z|^2)^2$. *Then the following hold.* (1) *If* $T_\phi \in S_p$, *then* $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$. (2) *If* $\phi \in L^p(\mathbb{D}, d\lambda)$ *then* $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ *and* $T_\phi \in S_p$.

Proof. Suppose $T_\phi \in S_p$. Then

$$\int_{\mathbb{D}} \langle |T_\phi|^p k_w, k_w\rangle d\lambda(w) < \infty. \tag{3.1}$$

That is, $\int_{\mathbb{D}} \langle (T_\phi^* T_\phi)^{p/2} k_w, k_w\rangle d\lambda(w) < \infty$. If $2 \leq p < \infty$, then

$$\int_{\mathbb{D}} \left\langle T_\phi^* T_\phi k_w, k_w \right\rangle^{p/2} d\lambda(w) \leq \int_{\mathbb{D}} \left\langle \left(T_\phi^* T_\phi\right)^{p/2} k_w, k_w \right\rangle d\lambda(w) < \infty. \tag{3.2}$$

This implies

$$\begin{aligned} \int_{\mathbb{D}} \|P(\phi \circ \phi_w)\|^p d\lambda(w) &= \int_{\mathbb{D}} \|P(U_w(\phi k_w))\|^p d\lambda(w) \\ &= \int_{\mathbb{D}} \|U_w T_\phi k_w\|^p d\lambda(w) \\ &= \int_{\mathbb{D}} \|T_\phi k_w\|^p d\lambda(w) \\ &= \int_{\mathbb{D}} \left\langle T_\phi^* T_\phi k_w, k_w \right\rangle^{p/2} d\lambda(w) < \infty. \end{aligned} \tag{3.3}$$

Now $|P(\phi \circ \phi_w)(0)| = |\langle P(\phi \circ \phi_w), 1\rangle| = |\langle U_w(T_\phi k_w), 1\rangle| = |\langle T_\phi k_w, U_w 1\rangle| = |\langle T_\phi k_w, k_w\rangle| \leq \|T_\phi k_w\| = \|P(\phi \circ \phi_w)\|$. Thus $\int_{\mathbb{D}} |P(\phi \circ \phi_w)(0)|^p d\lambda(w) < \infty$. That is, $\int_{\mathbb{D}} |\widetilde{\phi}(w)|^p d\lambda(w) < \infty$ and $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$. Suppose $1 \leq p < 2$. Then by Heinz inequality [10], it follows that

$$\begin{aligned}
\infty &> \int_{\mathbb{D}} \langle |T_\phi|^p k_w, k_w \rangle d\lambda(w) \\
&= \int_{\mathbb{D}} \left\langle |T_\phi|^{2\cdot p/2} k_w, k_w \right\rangle d\lambda(w) \\
&\geq \int_{\mathbb{D}} \frac{|\langle T_\phi k_w, k_w \rangle|^2}{\left\langle \left|T_\phi^*\right|^{2(1-p/2)} k_w, k_w \right\rangle} d\lambda(w) \\
&= \int_{\mathbb{D}} \frac{\left|\tilde{\phi}(w)\right|^2}{\left\|P(\overline{\phi} \circ \phi_w)\right\|^{2-p}} d\lambda(w) \\
&= \int_{\mathbb{D}} \left|\tilde{\phi}(w)\right|^2 \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^{p-2} d\lambda(w) \\
&\geq \int_{\mathbb{D}} \frac{\left|\tilde{\phi}(w)\right|^2}{\left\|P(\overline{\phi} \circ \phi_w)\right\|^2} \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^p d\lambda(w) \\
&\geq \int_{\mathbb{D}} \frac{\left|\tilde{\phi}(w)\right|^2}{C^2 \|\phi\|_{BT}^2} |P(\phi \circ \phi_w)(0)|^p d\lambda(w) \\
&= \int_{\mathbb{D}} \frac{\left|\tilde{\phi}(w)\right|^2}{C^2 \|\phi\|_{BT}^2} \left|\tilde{\phi}(w)\right|^p d\lambda(w),
\end{aligned} \tag{3.4}$$

since

$$\begin{aligned}
\left\langle \left|T_\phi^*\right|^{2-p} k_w, k_w \right\rangle &= \left\langle \left|T_\phi^*\right|^{2\cdot(2-p)/2} k_w, k_w \right\rangle \\
&\leq \left\langle \left|T_\phi^*\right|^2 k_w, k_w \right\rangle^{(2-p)/2} \\
&= \left\langle T_\phi T_\phi^* k_w, k_w \right\rangle^{(2-p)/2} \\
&= \left\|T_\phi^* k_w\right\|^{2-p} \\
&= \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^{2-p}.
\end{aligned} \tag{3.5}$$

Hence

$$\int_{\mathbb{D}} \left|\tilde{\phi}(w)\right|^{p+2} d\lambda(w) < \infty, \tag{3.6}$$

and therefore $\int_{\mathbb{D}} |\tilde{\phi}(w)|^p d\lambda(w) < \infty$. Thus $\tilde{\phi} \in L^p(\mathbb{D}, d\lambda)$.

Now suppose $\phi \in L^1(\mathbb{D}, d\lambda)$. Then the change of the order of integration

$$\begin{aligned}
\int_{\mathbb{D}} \left|\tilde{\phi}(w)\right| d\lambda(w) &= \int_{\mathbb{D}} \left|\tilde{\phi}(w)\right| \frac{dA(w)}{\left(1-|w|^2\right)^2} \\
&\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |\phi(z)| \frac{\left(1-|w|^2\right)^2}{|1-\overline{\mathbf{w}}z|^4} dA(z) \right) \frac{dA(w)}{\left(1-|w|^2\right)^2} \\
&= \int_{\mathbb{D}} |\phi(z)| \int_{\mathbb{D}} \frac{dA(w)}{|1-\overline{\mathbf{w}}z|^4} dA(z) = \int_{\mathbb{D}} |\phi(z)| \langle k_z, k_z \rangle dA(z) \\
&= \int_{\mathbb{D}} |\phi(z)| \frac{dA(z)}{\left(1-|z|^2\right)^2}
\end{aligned} \tag{3.7}$$

is justified by the positivity of the integrand. Hence $\tilde{\phi} \in L^1(\mathbb{D}, d\lambda)$. Similarly if $\phi \in L^\infty(\mathbb{D})$ then $\tilde{\phi} \in L^\infty(\mathbb{D})$ as $|\tilde{\phi}(w)| = |\langle \phi k_w, k_w \rangle| \leq \|\phi k_w\|_2 \|k_w\|_2 \leq \|\phi\|_\infty \|k_w\|_2^2 = \|\phi\|_\infty$. By Marcinkiewicz interpolation theorem it follows that if $\phi \in L^p(\mathbb{D}, d\lambda)$ then $\tilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ for $1 \leq p \leq \infty$. Now suppose $\phi \in L^p(\mathbb{D}, d\lambda)$, $1 \leq p \leq \infty$. We will prove $T_\phi \in S_p$. The case $p = +\infty$ is trivial. By interpolation we need only to prove the result for $p = 1$. Suppose $\phi \in L^1(\mathbb{D}, d\lambda)$ and $\{e_n\} = \{\sqrt{n+1} z^n\}_{n=0}^\infty$ is the standard orthonormal basis for $L_a^2(\mathbb{D})$. Now $\langle T_\phi e_n, e_n \rangle = \int_{\mathbb{D}} |e_n(z)|^2 \phi(z) dA(z)$ and

$$\begin{aligned}
\sum_{n=0}^\infty \left|\langle T_\phi e_n, e_n \rangle\right| &\leq \int_{\mathbb{D}} \sum_{n=0}^\infty |e_n(z)|^2 |\phi(z)| dA(z) \\
&\leq \int_{\mathbb{D}} K(z,z) |\phi(z)| dA(z) \\
&= \int_{\mathbb{D}} |\phi(z)| d\lambda(z).
\end{aligned} \tag{3.8}$$

Thus $T_\phi \in S_1$ and $\|T_\phi\|_{S_1} \leq \int_{\mathbb{D}} |\phi(z)| d\lambda(z)$. □

It is not so difficult to verify the conditions in Theorem 3.1.

Example 3.2. Let $\Phi(z) = (1-|z|^2)\log(1-|z|^2)$. Then $\Phi \in L^2(\mathbb{D}, d\lambda)$. This can be verified as follows:

$$\begin{aligned}
&\int_{\mathbb{D}} \left(1-|z|^2\right)^2 \left|\log\left(1-|z|^2\right)\right|^2 d\lambda(z) \\
&\quad = \int_{\mathbb{D}} \left(1-|z|^2\right)^2 \left|\log\left(1-|z|^2\right)\right|^2 \frac{dA(z)}{\left|\left(1-|z|^2\right)\right|^2}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \left|\log\left(1-|z|^2\right)\right|^2 dA(z) \\
&= \int_0^1 \left|\log\left(1-r^2\right)\right|^2 2r\,dr \\
&= \int_0^1 |\log(1-t)|^2 dt \\
&= \int_0^1 |\log t|^2 dt,
\end{aligned}
\tag{3.9}
$$

and changing the variable to $y = -\log t$, this reduces to $\int_0^\infty y^2 e^{-y} dy = \Gamma(3) = 2 < \infty$. Thus $T_\Phi \in S_2$.

Example 3.3. Let $g(z) = \ln|z|^2$. Then

$$
\begin{aligned}
\int_{\mathbb{D}} |g(z)|^2 d\lambda(z) &= \int_0^1 \left(\frac{\ln t}{1-t}\right)^2 dt \\
&= \int_0^1 \sum_{n=0}^\infty \sum_{m=0}^\infty t^{m+n} \ln^2 t\,dt \\
&= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{2}{(m+n+1)^3} \\
&= \sum_{k=0}^\infty \frac{2}{(k+1)^2} \\
&= \frac{\pi^2}{3} < \infty.
\end{aligned}
\tag{3.10}
$$

Thus $g \in L^2(\mathbb{D}, d\lambda)$. Direct computation reveals that $\tilde{g}(x) = |x|^2 - 1$ and $\tilde{g} \in L^2(\mathbb{D}, d\lambda)$.

Example 3.4. Let $\phi(z) = 1 - |z|$. Then by Example 2.2, $T_\phi \in S_2$ and

$$
\begin{aligned}
\int_{\mathbb{D}} |\phi(z)|^2 d\lambda(z) &= \int_{\mathbb{D}} (1-|z|)^2 \frac{dA(z)}{\left(1-|z|^2\right)^2} \\
&= \int_{\mathbb{D}} \frac{dA(z)}{(1+|z|)^2} \\
&\le \int_{\mathbb{D}} dA(z) = 1.
\end{aligned}
\tag{3.11}
$$

Thus $\phi \in L^2(\mathbb{D}, d\lambda)$ and hence $\tilde{\phi} \in L^2(\mathbb{D}, d\lambda)$.

It may be noted that the space $L^p(\mathbb{D}, d\lambda)$, $1 \le p < \infty$, contains no nonzero harmonic functions and even no nonzero constants. To see this, for example, for $L^2(\mathbb{D}, d\lambda)$, let

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^2 dt. \tag{3.12}$$

This is a nonnegative and nondecreasing function of r and $\|f\|^2_{L^2(\mathbb{D},d\lambda)} = \int_0^1 M(r)(2r/(1-r^2)^2)dr < \infty$. So $M(r)$ must tend to 0 as $r \to 1$. Thus $M(r) \equiv 0$, and therefore $f \equiv 0$.

But although there is no nonzero harmonic functions in $L^2(\mathbb{D}, d\lambda)$, there are plenty of subharmonic functions. Consider the function $f(z) = \ln|z|^2$. We have verified in Example 3.3 that $f \in L^2(\mathbb{D}, d\lambda)$. Suppose that f is real-valued subharmonic and $f \in L^2(\mathbb{D}, d\lambda)$. The subharmonicity of f implies that $\widetilde{f}(w) = \int_{\mathbb{D}} f(\phi_w(z))dA(w) \ge f(\phi_w(0)) = f(w)$. Hence $\widetilde{f} \ge f$. Let $\Delta_h := (1-|z|^2)^2(\partial^2/\partial z\partial\overline{\mathbf{z}})$. It can be verified that $\Delta_h(f\circ\phi_a) = (\Delta_h f)\circ\phi_a$ and note that $\Delta_h\widetilde{f} = \widetilde{\Delta_h f} \ge 0$ since $\Delta_h f \ge 0$. In other words, $\widetilde{f}$ is also subharmonic. Proceeding by induction, if we define $Bg = \widetilde{g}$ on $L^2(\mathbb{D}, d\lambda)$ then we obtain $B^n f$ is subharmonic for all $n \in \mathbb{N}$ and $\{B^n f\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of functions.

Corollary 3.5. *If $\phi \in h^\infty(\mathbb{D})$, the space of bounded harmonic functions on $\mathbb{D}$ and $1 \le p < \infty$, then $T_\phi \in S_p$ if and only if $\phi \equiv 0$.*

Proof. The proof of the corollary follows from the above discussion and the fact [1] that $\phi \in h^\infty(\mathbb{D})$ if and only if $\widetilde{\phi} = \phi$. □

Corollary 3.6. *If ϕ is a real-valued bounded subharmonic function on $\mathbb{D}$, $1 \le p < \infty$, then $T_\phi \in S_p$ if and only if $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$.*

Proof. By Theorem 3.1, if $T_\phi \in S_p$ then $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$. Conversely if ϕ is real-valued, subharmonic, bounded on $\mathbb{D}$ and $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ then $\widetilde{\phi}$ is also subharmonic and the subharmonicity of ϕ implies that $\widetilde{\phi}(w) = \int_{\mathbb{D}} \phi(\phi_w(z))dA(z) \ge \phi(\phi_w(0)) = \phi(w)$. Hence $\int_{\mathbb{D}}|\widetilde{\phi}(w)|^p d\lambda(w) = \int_{\mathbb{D}}|\phi(w)|^p d\lambda(w)$ as $\phi \in L^p(\mathbb{D}, d\lambda)$ implies $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ and the result follows from Theorem 3.1. □

Corollary 3.7 follows immediately from Corollary 3.5. We present a proof of Corollary 3.7 to show a different method of approach.

Corollary 3.7. *If $\phi \in H^\infty(\mathbb{D}) \cup \overline{H^\infty(\mathbb{D})}$ then the Toeplitz operator $T_\phi \in S_p, 1 \le p < \infty$ if and only if $\phi \equiv 0$.*

Proof. Notice that if $\phi \in H^\infty(\mathbb{D}) \cup \overline{H^\infty(\mathbb{D})}$ then the following two possibilities hold.

(*) For all $z \in \mathbb{D}$, either $P(\phi\circ\phi_z) \ne P(\phi\circ\phi_z)(0)$ or $P(\overline{\phi}\circ\phi_{\mathbf{z}}) \ne P(\overline{\phi}\circ\phi_{\mathbf{z}})(0)$.

(**) There exists $z \in D$ such that $P(\phi\circ\phi_z) = P(\phi\circ\phi_z)(0)$ and $P(\overline{\phi}\circ\phi_{\mathbf{z}}) = P(\overline{\phi}\circ\phi_{\mathbf{z}})(0)$.

Suppose that (**) holds and let $z \in \mathbb{D}$ be such that $P(\phi\circ\phi_z) = P(\phi\circ\phi_z)(0)$ and $P(\overline{\phi}\circ\phi_{\mathbf{z}}) = P(\overline{\phi}\circ\phi_{\mathbf{z}})(0)$. Then this implies $T_\phi k_z = c_z k_z$ and $T_{\overline{\phi}} k_z = \overline{\mathbf{c}}_{\mathbf{z}} k_z$. Hence T_ϕ has an eigenvalue. From [1], it follows that ϕ is a constant and T_ϕ is not compact unless $\phi \equiv 0$.

Now suppose that (*) holds, $1 \leq p < \infty$. Then for all $z \in \mathbb{D}$, $\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2 = \|\phi \circ \phi_z - P(\phi \circ \phi_z)\|^2 + \|\overline{\phi} \circ \phi_z - P(\overline{\phi} \circ \phi_z)\|^2 < \|\phi \circ \phi_z - P(\phi \circ \phi_z)(0)\|^2 + \|\overline{\phi} \circ \phi_z - P(\overline{\phi} \circ \phi_z)(0)\|^2 = 2(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2)$. Let $c_z = (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2)/(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2)$. Note that $1 \leq c_z < 2$ for all $z \in \mathbb{D}$. It follows from Proposition 2.5 and from the previous discussion.

Notice T_ϕ compact implies that $T_\phi^* T_\phi = T_{|\phi|^2} - H_\phi^* H_\phi$ is compact. Thus $0 \leq \langle T_\phi^* T_\phi k_z, k_z \rangle = \widetilde{|\phi|^2}(z) - \|H_\phi k_z\|^2 \to 0$ as $|z| \to 1^-$. Similarly since $T_\phi T_\phi^*$ is compact, $0 \leq \widetilde{|\phi|^2}(z) - \|H_{\overline{\phi}} k_z\|^2 \to 0$ as $|z| \to 1^-$.

Thus $0 \leq 2\widetilde{|\phi|^2}(z) - (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2) \to 0$ as $|z| \to 1^-$. Hence it follows that $0 \leq 2\widetilde{|\phi|^2}(z) - c_z(\widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2) = 2\widetilde{|\phi|^2}(z) - (\|H_\phi k_z\|^2 + \|H_{\overline{\phi}} k_z\|^2) \to 0$ as $|z| \to 1^-$. So $0 \leq (2 - c_z)\widetilde{|\phi|^2}(z) + c_z|\widetilde{\phi}(z)|^2 \to 0$ as $|z| \to 1^-$, where $1 \leq c_z < 2$. Since T_ϕ is compact, $|\widetilde{\phi}(z)| \to 0$ as $|z| \to 1^-$. Thus $\widetilde{|\phi|^2}(z) \to 0$ as $|z| \to 1^-$. Similarly since $T_{\phi \circ \phi_z}$ is compact we can show that $\widetilde{|\phi \circ \phi_z|^2}(z) \to 0$ as $|z| \to 1^-$. Thus

$$\int_{\mathbb{D}} \left|(\phi \circ \phi_z)(w)\right|^2 |k_z(w)|^2 dA(w) \longrightarrow 0 \tag{3.13}$$

as $|z| \to 1^-$. Hence $\int_{\mathbb{D}} |\phi(w)|^2 dA(w) = 0$ as $U_z k_z = k_z(\phi_z(w))k_z(w) = 1$. It follows therefore that $\phi(w) = 0$ almost everywhere and hence $\phi \equiv 0$. Therefore, $T_\phi \equiv 0$. □

Let $\widetilde{T}(z) = \langle T k_z, k_z \rangle$ for $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. It follows from Heinz inequality [10] that

$$\left|\langle T_\phi k_z, k_z \rangle\right|^2 \leq \left\langle \left|T_\phi\right|^{2\alpha} k_z, k_z \right\rangle \left\langle \left|T_\phi^*\right|^{2(1-\alpha)} k_z, k_z \right\rangle \tag{3.14}$$

for all $z \in \mathbb{D}$ and $0 \leq \alpha \leq 1$. That is, $|\widetilde{\phi}(z)|^2 \leq \widetilde{|T_\phi|^{2\alpha}}(z) \widetilde{|T_\phi^*|^{2(1-\alpha)}}(z)$ for all $z \in \mathbb{D}$. Hence if $\widetilde{|T_\phi|^{2\alpha}} \in L^p(\mathbb{D}, d\lambda)$ and $\widetilde{|T_\phi^*|^{2(1-\alpha)}} \in L^q(\mathbb{D}, d\lambda)$, $1/p + 1/q = 1$, it follows from Holders inequality that $\widetilde{\phi} \in L^2(\mathbb{D}, d\lambda)$. From Theorem 3.1, it follows that $T_{\widetilde{\phi}} \in S_2$.

We say T majorizes $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ if $\|Sf\| \leq M\|Tf\|$ for all $f \in L_a^2(\mathbb{D})$.

Corollary 3.8. *If $\phi \geq 0$, $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$, $1 \leq p \leq 2$, and T_ϕ majorizes T_ψ then $T_\psi \in S_p$.*

Proof. Since T_ϕ majorizes T_ψ, it follows that $\|T_\psi f\| \leq M\|T_\phi f\|$ for some $M > 0$ and for all $f \in L_a^2$. Hence $\langle T_\psi^* T_\psi f, f \rangle \leq M^2 \langle T_\phi^* T_\phi f, f \rangle$ for all $f \in L_a^2$. That is, $T_\psi^* T_\psi \leq M^2 T_\phi^* T_\phi$. Since $1 \leq p \leq 2$, we obtain from [10, 11] that $(T_\psi^* T_\psi)^{p/2} \leq M^p (T_\phi^* T_\phi)^{p/2}$. That is, $|T_\psi|^p \leq M^p |T_\phi|^p$ for $1 \leq p \leq 2$. Now if $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$, $\phi \geq 0$, then by [1], $T_\phi \in S_p$ and $\int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z \rangle d\lambda(z) < \infty$. Hence $\int_{\mathbb{D}} \langle |T_\psi|^p k_z, k_z \rangle d\lambda(z) < \infty$ and $T_\psi \in S_p$. □

Corollary 3.9. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. If* Range $T_{\overline{\psi}} \subseteq$ Range $T_{\overline{\phi}}$ *and $\phi \in L^p(\mathbb{D}, d\lambda)$, $1 \leq p \leq 2$ then $T_\psi \in S_p$ and $\widetilde{\psi} \in L^p(\mathbb{D}, d\lambda)$.*

Proof. Range $T_{\overline{\psi}} \subseteq$ Range $T_{\overline{\phi}}$ implies that T_ϕ majorizes T_ψ. Hence $\|T_\psi f\| \leq M\|T_\phi f\|$ for some $M > 0$ and for all $f \in L_a^2$. Hence $\langle T_\psi^* T_\psi f, f \rangle \leq M^2 \langle T_\phi^* T_\phi f, f \rangle$ for all $f \in L_a^2$. That is, $T_\psi^* T_\psi \leq M^2 T_\phi^* T_\phi$. Since $1 \leq p \leq 2$, we obtain from [10, 11] that $(T_\psi^* T_\psi)^{p/2} \leq M^p (T_\phi^* T_\phi)^{p/2}$. That is,

$|T_\psi|^p \leq M^p |T_\phi|^p$ for $1 \leq p \leq 2$. Now if $\phi \in L^p(\mathbb{D}, d\lambda)$ then $T_\phi \in S_p$ and $\int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z \rangle d\lambda(z) < \infty$. Hence $\int_{\mathbb{D}} \langle |T_\psi|^p k_z, k_z \rangle d\lambda(z) < \infty$. Thus $T_\psi \in S_p$ and $\widetilde{\psi} \in L^p(\mathbb{D}, d\lambda)$. □

Definition 3.10. A function $G \in L_a^2(D)$ $(G \in H^2)$ is called an inner function [12] in $L_a^2(D)$ (resp., H^2) if $|G|^2 - 1$ is orthogonal to H^∞. If for some inner function $G \in L_a^2(D)$ the following conditions hold then we call it a finite zero divisor in $L_a^2(D)$. (i) G vanishes on $\mathbf{a} = \{a_j\}_{j=1}^N$, a finite sequence of points in D. (ii) $\|G\|_{L^2} = 1$. (iii) G is equal to a constant plus a linear combination of the Bergman kernel functions $K(z, a_1), K(z, a_2), \dots, K(z, a_n)$ and certain of their derivatives. (iv) $|G|^2 - 1$ is orthogonal to L_h^1, the class of harmonic functions in L^1 of the disc.

Corollary 3.11. *Suppose $\phi \in L^\infty(\mathbb{D})$ and $\ker T_\phi \subseteq GL_a^2$, where G is a finite inner divisor in $L_a^2(\mathbb{D})$. If $\phi \in L^p(\mathbb{D}, d\lambda), 1 \leq p \leq 2$, then $S_\psi \in S_p$, where S_ψ is the little Hankel operator on $L_a^2(\mathbb{D})$ such that* $\ker S_\psi = GL_a^2$.

Proof. Let $\mathbf{b} = (b_j)_{j=1}^N$ be a finite set of points in $\mathbb{D}$ that are the zeroes of the finite inner divisor G counting multiplicities and $\mathcal{I} = I(\mathbf{b}) = \{f \in L_a^2(D) : f = 0 \text{ on } \mathbf{b}\}$. Then G is the solution of the extremal problem

$$\sup\left\{\operatorname{Re} f^{(n)}(0) : f \in \mathcal{I},\ \|f\|_{L^2} \leq 1\right\}, \tag{3.15}$$

where n is the number of times zero appears in the sequence $\mathbf{b}$ (i.e., the functions in $\mathcal{I}$ have a common zero of order n at the origin). It is shown in [12] that for

$$\overline{\psi} = \sum_{j=1}^{N} \sum_{\nu=0}^{m_j - 1} c_{j\nu} \frac{\partial^\nu}{\partial \bar{b}_j^{\,\nu}} K_{b_j}(z), \tag{3.16}$$

where $c_{j\nu} \neq 0$ for all j, ν and m_j is the number of times b_j appear in $\mathbf{b}$, $\ker S_\psi = GL_a^2(D)$. Now $\ker T_\phi \subseteq GL_a^2 = \ker S_\psi$ implies [13] the operator T_ϕ that majorizes S_ψ. Since $\phi \in L^p(\mathbb{D}, d\lambda)$, $1 \leq p \leq 2$, $T_\phi \in S_p$. By similar arguments as in Corollary 3.9 one can show that $S_\psi \in S_p$. □

For any $a \in \mathbb{D}$, let γ_a be the unique geodesic such that $\gamma_a(0) = 0$, $\gamma_a(1) = a$. Then there exists a unique $\phi_a \in \operatorname{Aut}(\mathbb{D})$ such that $\phi_a \circ \phi_a(z) \equiv z$ and $\gamma_a(1/2)$ is an isolated fixed point of ϕ_a. Further ϕ_a is the geodesic symmetry at $\gamma_a(1/2)$. We denote by m_a the geodesic midpoint $\gamma_a(1/2)$ of 0 and a. It can easily be verified that each ϕ_a has m_a as a unique fixed point and for $a \in \mathbb{D}$, $m_a = ((1 - \sqrt{1 - |a|^2})/|a|^2)a$.

Given $\lambda \in \mathbb{D}$ and a measurable function f on $\mathbb{D}$, we have $f \circ \phi_\lambda = f$ if and only if there exists an even function g on $\mathbb{D}$ such that $f = g \circ \phi_{m_\lambda}$; $f \circ \phi_\lambda = -f$ if and only if there exists an odd function g on $\mathbb{D}$ such that $f = g \circ \phi_{m_\lambda}$. For proof of this fact see [14].

Corollary 3.12. *If $\phi = \theta \circ \phi_{m_z}$ for some $z \in \mathbb{D}$, $\theta \in L^\infty(\mathbb{D})$ and where θ is an even function and $T_\phi \in S_p$, $1 \leq p < \infty$, then*

$$\left\| T_{\psi \circ \phi_z - \psi + \phi} \right\|_{S_p} \geq \left\| T_\phi \right\|_{S_p} \tag{3.17}$$

for all $T_\psi \in S_p$.

Proof. Since $\phi = \theta \circ \phi_{m_z}$ where θ is an even function, by [14], we have $\phi \circ \phi_z = \phi$. Hence $U_z T_\phi U_z = T_{\phi \circ \phi_z} = T_\phi$. As U_z is unitary, from [15], it follows that

$$\left\| U_z T_\psi U_z - T_\psi + T_\phi \right\|_{S_p} \geq \left\| T_\phi \right\|_{S_p} \tag{3.18}$$

for all $T_\psi \in S_p$. Hence

$$\left\| T_{\psi \circ \phi_z - \psi + \phi} \right\|_{S_p} \geq \left\| T_\phi \right\|_{S_p} \tag{3.19}$$

for all $T_\psi \in S_p$. □

Theorem 3.13. *Suppose there exist constants $C, M > 0$ such that $C > M$ and $\|P(\phi \circ \phi_z) - C\| \leq M$ for all $z \in \mathbb{D}$, $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ and $p \in [1, \infty)$ is such that the Kantorvich constant $K(p) \in (0, \infty)$. If T_ϕ is bounded below then $T_\phi \in S_p$.*

Proof. If there exist $C > M > 0$ such that for all $z \in \mathbb{D}$, we have $\|P(\phi \circ \phi_z) - C\| \leq M$ then $\|T_\phi k_z - C k_z\| = \|U_z P(\phi \circ \phi_z) - C U_z 1\| = \|P(\phi \circ \phi_z) - C\| \leq M$ for all $z \in \mathbb{D}$. Let $\Gamma = C + M$ and $\gamma = C - M$. Then $0 < \gamma < \Gamma$ and $C = (\Gamma + \gamma)/2$ and $M = (\Gamma - \gamma)/2$. Thus for all $z \in \mathbb{D}$, $\|T_\phi k_z - ((\Gamma + \gamma)/2) k_z\| \leq (1/2)|\Gamma - \gamma| \|k_z\|$. By [16], it follows that there exists $L > 0$ such that $|\widetilde{\phi}(z)| = |\langle T_\phi k_z, k_z \rangle| \geq L \|T_\phi k_z\|$ for all $z \in \mathbb{D}$. Notice that the reproducing kernels $\{k_z : z \in \mathbb{D}\}$ span $L^2_a(\mathbb{D})$ and T_ϕ is bounded below. Hence there exist $r > 0$, $s > 0$ such that $s \leq T^*_\phi T_\phi \leq r$. Now if $\widetilde{\phi} \in L^p(\mathbb{D}, d\lambda)$ and $p \in [1, \infty)$ is such that the Kantorvich constant $K(p) \in (0, \infty)$ then $\int_{\mathbb{D}} \|T_\phi k_z\|^p d\lambda(z) < \infty$. From inequalities (2.4) and (2.5), it follows that $\int_{\mathbb{D}} \langle |T_\phi|^p k_z, k_z \rangle d\lambda(z) < \infty$. Hence $T_\phi \in S_p$. □

Let $L^2(\mathbb{T})$ be the usual Lebesgue space of functions on the unit circle $\mathbb{T}$, and let the Hardy space H^2 be the class of all $L^2(\mathbb{T})$ functions whose negative Fourier coefficients are zero. Let $L^\infty(\mathbb{T})$ be the algebra of essentially bounded complex valued functions on the unit circle $\mathbb{T}$ and $H^\infty(\mathbb{T})$ be the subalgebra of $L^\infty(\mathbb{T})$ consisting of functions whose negative Fourier coefficients are zero. Let $\mathcal{R}_n$ denote the rational functions with at most n poles (counting multiplicities) all of which are in the interior of $\mathbb{T}$, and let $C(\mathbb{T})$ denote the algebra of continuous functions on $\mathbb{T}$. Notice that the following inclusion relations hold:

$$H^\infty(\mathbb{T}) \subset H^\infty(\mathbb{T}) + \mathcal{R}_1 \subset H^\infty(\mathbb{T}) + \mathcal{R}_2 \subset \cdots H^\infty(\mathbb{T}) + C(\mathbb{T}). \tag{3.20}$$

For $\phi \in L^\infty(\mathbb{T})$, let $B_\phi : H^2 \to H^2$ be the Toeplitz operator on H^2 with symbol ϕ, and let $\Gamma_\phi : H^2 \to H^2$ be the Hankel operator with symbol ϕ. Nehari [17], Adamjan et al. [5], and Hartman [18], respectively, proved the following results:

(i) $\|\Gamma_\phi\| = d(\phi, H^\infty(\mathbb{T}))$;

(ii) $s_k(\Gamma_\phi) = d(\phi, H^\infty(\mathbb{T}) + \mathcal{R}_n)$;

(iii) $\|\Gamma_\phi\|_e = d(\phi, H^\infty(\mathbb{T}) + C(\mathbb{T}))$, where $\|\Gamma_\phi\|_e$ denotes the essential norm of Γ_ϕ. Feintuch in [19, 20] obtained the following operator theoretic analogues of these distance formulae. Given $T \in \mathcal{L}(H^2(\mathbb{T}))$, define a sequence $\{\Gamma_n(\mathbb{T})\}$ of operators on $H^2(\mathbb{T})$ by $\Gamma_n(\mathbb{T}) = J_n T S^{n+1}$, $n = 1, 2, \ldots$, where for $z \in \mathbb{T}$,

$$J_n z^i = \begin{cases} z^{n-i}, & \text{if } 0 \le i \le n, \\ 0, & \text{if } i > n, \end{cases} \tag{3.21}$$

and S is the unilateral shift on $H^2(\mathbb{T})$. Let $\widehat{\mathcal{C}}$ denote the family of operators T for which $\{\Gamma_n(\mathbb{T})\}$ converges strongly and

$$\begin{aligned} \mathcal{L}_0 &= \left\{T \in \widehat{\mathcal{C}} : \Gamma_n(T) \text{ converges strongly to } 0\right\}; \\ \mathcal{L}_k &= \mathcal{L}_0 + \left\{B_\phi : \phi \in H^\infty(\mathbb{T}) + \mathcal{R}_k\right\}; \\ \mathcal{L}_\infty &= \mathcal{L}_0 + \left\{B_\phi : \phi \in H^\infty(\mathbb{T}) + C(\mathbb{T})\right\}. \end{aligned} \tag{3.22}$$

Then (i) $d(T, \mathcal{L}_0) = \|\Gamma(\mathbb{T})\|$; (ii) $d(T, \mathcal{L}_k) = s_k(\Gamma(\mathbb{T}))$; (iii) $d(T, \mathcal{L}_\infty) = \|\Gamma(\mathbb{T})\|_e$. In particular when $T = B_\phi$, $\phi \in L^\infty(\mathbb{T})$ we have

(i) $d(B_\phi, \mathcal{L}_0) = \|\Gamma_\phi\| = d(\phi, H^\infty(\mathbb{T}))$;

(ii) $d(B_\phi, \mathcal{L}_k) = s_k(\Gamma_\phi) = d(\phi, H^\infty(\mathbb{T}) + \mathcal{R}_k)$;

(iii) $d(B_\phi, \mathcal{L}_\infty) = \|\Gamma_\phi\|_e = d(\phi, H^\infty(\mathbb{T}) + C(\mathbb{T}))$.

Yamada [21] also obtained distance formulas involving the norm of Toeplitz operators on the Hardy space.

It is well known that [1] there are no compact Toeplitz operators on the Hardy space other than the zero operator. In the Bergman space setting, however, there are lots of nontrivial compact Toeplitz operators belonging to different Schatten classes. In view of this it is of interest to know whether such distance formulae is possible for Toeplitz and Hankel operators defined on the Bergman space and the characterization of Schatten class Toeplitz operators is also important in this context. These results play an important role in approximation theory.

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