Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 989865, 8 pages
doi:10.1155/2009/989865

Research Article

On $(N, p, q)(E, 1)$ Summability of Fourier Series

H. K. Nigam and Ajay Sharma

Department of Mathematics, Faculty of Engineering & Technology, Mody Institute of Technology and Science (Deemed University), Laxmangarh, Sikar (Rajasthan) 332311, India

Correspondence should be addressed to Ajay Sharma, ajaymathematicsanand@gmail.com

Received 7 November 2008; Revised 17 March 2009; Accepted 30 March 2009

Recommended by Hüseyin Bor

A new theorem on $(N, p, q)(E, 1)$ summability of Fourier series has been established.

1. Introduction

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$\begin{aligned} P_n &= p_0 + p_1 + p_2 + \cdots + p_n = \sum_{\nu=0}^{n} p_\nu \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ Q_n &= q_0 + q_1 + q_2 + \cdots + q_n = \sum_{\nu=0}^{n} q_\nu \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \\ R_n &= p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 = \sum_{\nu=0}^{n} p_\nu q_{n-\nu} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{1.1}$$

Given two sequences $\{p_n\}$ and $\{q_n\}$ convolution $(p * q)$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k. \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its nth partial sums $\{s_n\}$.

We write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k s_k. \tag{1.3}$$

If $R_n \neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to S$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to S by generalized Nörlund method (Borwein [1]) and is denoted by

$$S_n \longrightarrow \mathrm{S}(N,p,q). \tag{1.4}$$

The necessary and sufficient conditions for (N,p,q) method to be regular are

$$\sum_{k=0}^{n} \left|p_{n-k} q_k\right| = O(|R_n|), \tag{1.5}$$

and $p_{n-k} = o(|R_n|)$, as $n \to \infty$ for every fixed $k \geq 0$, for which $q_k \neq 0$.

Now

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k} s_k. \tag{1.6}$$

If $E_n^1 \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be $(E,1)$ summable to s (Hardy [2]):

$$\begin{aligned} t_n^{p,q,E} &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k E_k^1 \\ &= \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \cdot \frac{1}{2^k}\sum_{\nu=0}^{k} \binom{k}{\nu} s_\nu. \end{aligned} \tag{1.7}$$

If $t_n^{p,q,E} \to \infty$, as $n \to \infty$, then we say that the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is summable to S by $(N,p,q)(E,1)$ summability method.

Particular Cases

(1) $(N,p,q)(E,1)$ mean reduces to $(N,p_n)(E,1)$ summability mean if $q_n = 1$, $\forall n$.

(2) $(N,p,q)(E,1)$ mean reduces to $(N,1/(n+1))(E,1)$ mean if $p_n = 1/(n+1)$ and $q_n = 1$, $\forall n$.

(3) $(N,p,q)(E,1)$ method reduces to $(\overline{N},q_n)(E,1)$ if $p_n = 1$, $\forall n$.

(4) $(N,p,q)(E,1)$ method reduces to $(C,\alpha)(E,1)$ if $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, and $q_n = 1$, $\forall n$.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$.

Let its Fourier series be given by

$$f(t) \sim \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} (a_n \cos\, nt + b_n \sin nt). \tag{1.8}$$

We use the following notations:

$$\begin{gathered}
\phi(t) = f(x+t) - f(x-t) - 2f(x), \\
\Phi(t) = \int_0^t |\phi(u)|\, du, \\
\tau = \left[\frac{1}{t}\right] \;=\; \text{the integral part of } \frac{1}{t}, \\
R\left(\frac{1}{t}\right) = R_\tau, \qquad R_n = R(n), \\
K_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{\cos^k (t/2) \cos(k+1)(t/2)}{\sin\,(t/2)}.
\end{gathered} \tag{1.9}$$

2. Theorem

A quite good amount of work is known for Fourier series by ordinary summability method. The purpose of this paper is to study Fourier series by $(N, p, q)(E, 1)$ summability method in the following form.

Theorem 2.1. *Let* $\{p_n\}$ *and* $\{q_n\}$ *be positive monotonic, nonincreasing sequences of real numbers such that*

$$R_n = \sum_{k=0}^{n} p_k q_{n-k} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{2.1}$$

Let $\alpha(t)$ *be a positive, nondecreasing function of* t*. If*

$$\Phi(t) = \int_0^t |\phi(u)| du = o\left(\frac{t}{\alpha(1/t)}\right), \quad \text{as } t \longrightarrow +0, \tag{2.2}$$

$$\alpha(n) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \tag{2.3}$$

then a sufficient condition that the Fourier Series (1.8) *be summable* $(N, p, q)(E, 1)$ *to* $f(x)$ *at the point* $t = x$ *is*

$$\int_1^n \frac{R(u)}{u\alpha(u)}\, du = O(R_n), \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

3. Lemmas

Proof of the theorem needs some lemmas.

Lemma 3.1. *For* $0 \le t \le 1/n$,

$$|K_n(t)| = O(n). \tag{3.1}$$

Proof.

$$\begin{aligned}|K_n(t)| &= \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_{n-k}q_k \frac{\cos^k(t/2)\sin(k+1)(t/2)}{\sin(t/2)}\right| \\ &\le \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_{n-k}q_k \frac{(k+1)|\sin(t/2)|}{|\sin(t/2)|} = O(n)\frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}q_k = O(n).\end{aligned} \tag{3.2}$$

□

Lemma 3.2. *If* $\{p_n\}$ *and* $\{q_n\}$ *are nonnegative and nonincreasing, then for* $0 \le a \le b < \infty$, $0 \le t \le \pi$, *and any* n *we have*

$$\frac{1}{2\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \frac{\cos^k(t/2)\sin(k+1)(t/2)}{\sin(t/2)}\right| = O\left(\frac{R_\tau}{tR_n}\right). \tag{3.3}$$

Proof.

$$\begin{aligned}&\frac{1}{2\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \frac{\cos^k(t/2)\ \sin(k+1)(t/2)}{\sin(t/2)}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)\sin(k+1)\frac{t}{2}\right| \\ &\quad= \frac{1}{t\pi R_n}\left|\operatorname{Im}\left\{\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{i(k+1)(t/2)}\right\}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ikt/2}\right|\left|e^{it/2}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left|\sum_{k=a}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ikt/2}\right| \\ &\quad\le \frac{1}{t\pi R_n}\left\{\left|\sum_{k=a}^{\tau-1} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{\frac{ikt}{2}}\right| + \left|\sum_{k=\tau}^{b} p_{n-k}q_k \cos^k\left(\frac{t}{2}\right)e^{ik\left(\frac{t}{2}\right)}\right|\right\}.\end{aligned} \tag{3.4}$$

Now considering first term of (3.4), we have

$$\begin{aligned}\frac{1}{t\pi R_n}\left|\sum_{k=a}^{\tau-1} p_{n-k}q_k\cos^k\left(\frac{t}{2}\right)e^{ik(t/2)}\right| &\leq \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{n-k}q_k\left|e^{ik(t/2)}\right| \leq \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{n-k}q_k \\ &\leq \frac{1}{t\pi R_n}\sum_{k=a}^{\tau-1} p_{\tau-k}q_k \leq \frac{1}{t\pi R_n}(R_\tau) = O\left(\frac{R_\tau}{tR_n}\right).\end{aligned} \tag{3.5}$$

Now considering second term of (3.4) and using Abel's lemma, we have

$$\begin{aligned}\frac{1}{t\pi R_n}\left|\sum_{k=\tau}^{b} p_{n-k}q_k\cos^k\left(\frac{t}{2}\right)e^{ik(t/2)}\right| &\leq \frac{1}{t\pi R_n}\left|\sum_{k=\tau}^{b} p_{n-k}q_k e^{ik(t/2)}\right| \\ &\leq \frac{2p_{n-b}q_\tau}{t\pi R_n}\max_{\tau+1\leq k\leq b}\left|\frac{1-e^{i(k+1)(t/2)}}{1-e^{it/2}}\right| \\ &\leq \frac{4p_{n-b}q_\tau}{t\pi R_n}\left|\frac{e^{-it/4}}{e^{it/4}-e^{-it/4}}\right| \\ &\leq \frac{2q_\tau}{t\pi R_n}\left(\frac{p_{n-b}}{P_\tau}\right)P_\tau\left|\frac{1}{\sin(t/4)}\right| \quad \left(\text{where } P_\tau = \sum_{k=0}^{\tau} p_{\tau-k}\right) \\ &\leq \frac{8q_\tau}{t\pi R_n}\left(\frac{p_{n-b}}{P_\tau}\right)P_\tau\left|\frac{1}{t}\right| \\ &\leq \frac{8q_\tau P_\tau}{t\pi R_n} \\ &\leq \frac{8R_\tau}{t\pi R_n} \quad \left(\text{since } R_\tau = \sum_{k=0}^{\tau} p_{\tau-k}q_k \geq P_\tau q_\tau\right) \\ &= O\left(\frac{R_\tau}{tR_n}\right).\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we get the required result of Lemma 3.2. □

4. Proof of Theorem

Following Zygmund [3], the nth partial sum $s_n(x)$ of the series (1.8) at $t = x$ is given by

$$s_n(x) = f(x) + \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\frac{\sin(n+1/2)t}{\sin(t/2)}dt. \tag{4.1}$$

So the $(E,1)$ mean of the series (1.8) at $t = x$ is given by

$$
\begin{aligned}
E_n^1(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k(x) \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\left\{\sum_{k=0}^{n}\binom{n}{k}\ \sin\left(k+\frac{1}{2}\right)t\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}\left(1+e^{it}\right)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}(1+\cos\ t + i\sin t)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}2^n\cos^n\left(\frac{t}{2}\right)\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)^n\right\}dt \\
&= f(x) + \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin(t/2)}\operatorname{Im}\left\{e^{it/2}2^n\cos^n\left(\frac{t}{2}\right)\left\{\left(\cos\frac{nt}{2}+i\sin\frac{nt}{2}\right)\right\}\right\}dt \\
&= f(x) + \frac{1}{2\pi}\int_0^{\pi}\phi_x(t)\ \ \frac{\cos^n(t/2)\sin(n+1)(t/2)}{\sin(t/2)}\ dt.
\end{aligned}
\tag{4.2}
$$

Therefore (N,p,q) transform of $\{E_n^1(x)\}$ is given by

$$
\begin{aligned}
t_n^{p,q,E}(x) &= \frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_kE_k^1(x) = f(x) + \frac{1}{2\pi}\int_0^{\pi}\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\phi_x(t)\frac{\cos^k(t)\sin(k+1)(t/2)}{\sin(t/2)} \\
&= f(x) + \int_0^{\pi}K_n(t)\phi_x(t)dt,
\end{aligned}
$$

$$
t_n^{p,q,E}(x) - f(x) = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right]K_n(t)\phi_x(t)dt = I_1 + I_2 + I_3 \quad (\text{say}).
\tag{4.3}
$$

We have

$$
\begin{aligned}
|I_1| &\le \int_0^{1/n}|K_n(t)|\ \left|\phi_x(t)\right|dt \\
&= O(n)\int_0^{1/n}\ \left|\phi_x(t)\right|dt \quad (\text{using Lemma 3.1}) \\
&= O(n)o\left(\frac{1}{n\alpha(n)}\right) \quad (\text{by (2.2)}) \\
&= o\left(\frac{1}{\alpha(n)}\right) = o(1) \quad \text{as } n \longrightarrow \infty \quad (\text{by (2.3)}).
\end{aligned}
\tag{4.4}
$$

Now we consider

$$
\begin{aligned}
|I_2| &\leq \int_{1/n}^{\delta} |K_n(t)|\left|\phi_x(t)\right| dt \quad (\text{where } 0<\delta<1)\\
&= \int_{1/n}^{\delta} O\left(\frac{R(1/t)}{tR_n}\right)\left|\phi_x(t)\right| dt \quad (\text{using Lemma 3.2})\\
&= O\left(\frac{1}{R_n}\right)\int_{1/n}^{\delta} \left(\frac{R(1/t)}{t}\right)\left|\phi_x(t)\right| dt\\
&= O\left(\frac{1}{R_n}\right)\left[\left\{\left(\frac{R(1/t)}{t}\right)\phi_x(t)\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} d\left(\frac{R(1/t)}{t}\right)\phi_x(t)\right]\\
&= O\left(\frac{1}{R(n)}\right)\left[\left\{o\left(\frac{R(1/t)}{\alpha(1/t)}\right)\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta} \phi_x(t) d\left(\frac{R(1/t)}{t}\right)\right] \quad (\text{by } (2.1))\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)\left[\int_{1/n}^{\delta} \phi_x(t)\left\{d\left(\frac{R(1/t)\alpha(1/t)}{t\alpha(1/t)}\right)\right\}\right]\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)\\
&\quad \times \left[\int_{1/n}^{\delta} o\left(\frac{t}{\alpha(1/t)}\right)\left\{d\alpha(1/t)\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right\} + \alpha\left(\frac{1}{t}\right)d\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right]\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1)\left[\int_{1/n}^{\delta} \frac{d\alpha(1/t)}{\{\alpha(1/t)\}^2} + o\left(\frac{1}{R(n)}\right)\int_{1/n}^{\delta} t\, d\left(\frac{R(1/t)}{t\alpha(1/t)}\right)\right]\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1)\left\{\frac{1}{\alpha(1/t)}\right\}_{1/n}^{\delta}\\
&\quad + o\left(\frac{1}{R(n)}\right)\left[\left\{\frac{tR(1/t)}{t\alpha(1/t)}\right\}_{1/n}^{\delta} - \int_{1/n}^{\delta}\left(\frac{R(1/t)}{t\alpha(1/t)}\right)dt\right]\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)\int_{1/n}^{1}\left(\frac{R(1/t)}{t\alpha(1/t)}\right)dt\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)\int_{1}^{n}\left(\frac{R(u)}{u\alpha(u)}\right)du \quad \left\{\because \frac{1}{t} = u\right\}\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{R(n)}\right)O(R_n) \quad (\text{by } (2.4))\\
&= o\left(\frac{1}{R(n)}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1)\\
&= o(1), \quad \text{as } n \to \infty \ (\text{by virtue of } (2.1) \text{ and } (2.2)).
\end{aligned}
\tag{4.5}
$$

Now by Riemann-Lebesgue theorem and by regularity of the method of summability we have

$$I_3 = \int_\delta^\pi |k_n(t)| \, \left|\phi_x(t)\right| dt \\ = o(1), \quad \text{as } n \longrightarrow \infty. \tag{4.6}$$

This completes the proof of the theorem.

5. Corollaries

Following corollaries can be derived from our main theorem.

Corollary 5.1. *If*

$$\Phi(t) = o\left[\frac{t}{\log(1/t)}\right], \quad as\ t \longrightarrow +0, \tag{5.1}$$

then the Fourier series (1.8) *is* $(C,1)(E,1)$ *summable to* $f(x)$ *at the point* $t = x$*.*

Corollary 5.2. *If*

$$\Phi(t) = o(t), \quad as\ t \longrightarrow +0, \tag{5.2}$$

then the Fourier series (1.8) *is* $(N,p_n)(E,1)$ *summable to* $f(x)$ *at the point* $t = x$*, provided that* $\{p_n\}$ *be a positive, monotonic, and nonincreasing sequence of real numbers such that*

$$p_n = p_0 + p_1 + \cdots + p_n \longrightarrow \infty, \quad as\ n \longrightarrow \infty. \tag{5.3}$$

Acknowledgments

The authors are grateful to Professor Shyam Lal, Department of Mathematics, Harcourt Butler Technological Institute, Kanpur, India, for his valuable suggestions and guidence in preparation of this paper. The authors are also thankful to Professor M. P. Jain, Vice Chancellor, Mody Institute of Technology and Science (Deemed University), Lakshmangarh, Sikar, Rajasthan, India, and to Professor S. N. Puri, Former Dean, Faculty of Engineering, Technology, Mody Institute of Technology and Science (Deemed University), Laxmangarh, Sikar, Rajasthan.

References

[1] D. Borwein, "On product of sequences," *Journal of the London Mathematical Society*, vol. 33, pp. 352–357, 1958.
[2] G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, UK, 1st edition, 1949.
[3] A. Zygmund, *Trigonometric Series. Vol. I*, Cambridge University Press, Cambridge, UK, 2nd edition, 1959.