

Research Article

Certain Subclasses of Starlike Functions of Complex Order Involving Generalized Hypergeometric Functions

G. Murugusundaramoorthy¹ and N. Magesh²

¹ School of Advanced Sciences, VIT University, Vellore 632014, India

² Department of Mathematics, Government Arts College (Men), Krishnagiri 635001, India

Correspondence should be addressed to G. Murugusundaramoorthy, gmsmoorthy@yahoo.com

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Making use of the generalized hypergeometric functions, we define a new subclass of uniformly convex functions and a corresponding subclass of starlike functions with negative coefficients and obtain coefficient estimates, extreme points, the radii of close-to-convexity, starlikeness and convexity, and neighborhood results for the class $TS_m^l(\alpha, \beta, \gamma)$. In particular, we obtain integral means inequalities for the function f that belongs to the class $TS_m^l(\alpha, \beta, \gamma)$ in the unit disc.

1. Introduction

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also denote by T the subclass of \mathcal{A} consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad (1.2)$$

introduced and studied by Silverman [1]. For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, it is known [2] the definition of the Hadamard product

(or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \quad (1.3)$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and β_1, \dots, β_m ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$) the generalized hypergeometric function ${}_1F_m(z)$ is defined by

$${}_1F_m(z) \equiv {}_1F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \quad (1.4)$$

$$(l \leq m + 1; l, m \in N_0 := N \cup \{0\}; z \in U),$$

where N denotes the set of all positive integers and $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), & n \in N. \end{cases} \quad (1.5)$$

The notation ${}_1F_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial, and others; for example, see [3].

Let $H(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \mathcal{A} \rightarrow \mathcal{A}$ be a linear operator defined by

$$\begin{aligned} H(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) &:= z {}_1F_m(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n, \end{aligned} \quad (1.6)$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{1}{(n-1)!}, \quad (1.7)$$

unless otherwise stated. For notational simplicity, we can use a shorter notation $H_m^l[\alpha_1]$ for $H(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$ in the sequel. The linear operator $H_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [3]), include (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [4], Owa [5], Ruscheweyh [2], and Srivastava and Owa [6]. Motivated by Goodman [7, 8], Rønning [9, 10] introduced and studied the following subclasses of \mathcal{A} . A function $f \in \mathcal{A}$ is said to be in the class $S_p(\alpha, \beta)$, uniformly β -starlike functions if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (-1 < \alpha \leq 1; \beta \geq 0) \quad z \in U \quad (1.8)$$

and is said to be in the class $UCV(\alpha, \beta)$, uniformly β -convex functions if it satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \quad (-1 < \alpha \leq 1; \beta \geq 0) z \in U. \quad (1.9)$$

Indeed it follows from (1.8) and (1.9) that

$$f \in UCV(\alpha, \beta) \iff zf' \in S_p(\alpha, \beta). \quad (1.10)$$

The interesting geometric properties of these function classes were extensively studied by Kanas et al., in [11–14]. Motivated by Altıntaş et al. [15], Murugusundaramoorthy and Srivastava [16], and Murugusundaramoorthy and Magesh [17], now, we define a new subclass uniformly starlike functions of complex order.

For $-1 \leq \alpha < 1$, $\beta \geq 0$, and $\gamma \in \mathbb{C} \setminus \{0\}$, we let $TS_m^l(\alpha, \beta, \gamma)$ be the class of functions f satisfying (1.2) with the analytic criterion

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - \alpha \right) \right\} > \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) \right|, \quad z \in U \quad (1.11)$$

where $H_m^l[\alpha_1]f(z)$ is given by (1.6).

By suitably specializing the values of l , m , $\alpha_1, \alpha_2, \dots, \alpha_l$, $\beta_1, \beta_2, \dots, \beta_m$, α , γ , and β the class $TS_m^l(\alpha, \beta, \gamma)$, leads to various new subclasses of starlike functions of complex order. As for illustrations, we present some examples for the cases.

Example 1.1. If $l = 2, m = 1$ with $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$, and f of the form of (1.2), then

$$TS(\alpha, \beta, \gamma) := \left\{ \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right\} > \beta \left| 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right|, z \in U \right\}. \quad (1.12)$$

Example 1.2. If $l = 2, m = 1$ with $\alpha_1 = \delta + 1$ ($\delta > -1$), $\alpha_2 = 1, \beta_1 = 1$, and f of the form of (1.2), then

$$TR_\delta(\alpha, \beta, \gamma) := \left\{ \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(D^\delta f(z))'}{D^\delta f(z)} - \alpha \right) \right\} > \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(D^\delta f(z))'}{D^\delta f(z)} - 1 \right) \right|, z \in U \right\}, \quad (1.13)$$

where D^δ is called Ruscheweyh derivative of order δ ($\delta > -1$) defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv H_1^2(\delta+1, 1; 1)f(z). \quad (1.14)$$

Example 1.3. If $l = 2, m = 1$ with $\alpha_1 = \mu + 1$ ($\mu > -1$), $\alpha_2 = 1, \beta_1 = \mu + 2$, and f of the form of (1.2), then

$$TB_\mu(\alpha, \beta, \gamma) = \left\{ \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(J_\mu f(z))'}{J_\mu f(z)} - \alpha \right) \right\} > \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(J_\mu f(z))'}{J_\mu f(z)} - 1 \right) \right|, z \in U \right\}, \quad (1.15)$$

where J_μ is a Bernardi operator [18] defined by

$$J_\mu f(z) := \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \equiv H_1^2(\mu + 1, 1; \mu + 2) f(z). \quad (1.16)$$

Example 1.4. If $l = 2, m = 1$ with $\alpha_1 = a$ ($a > 0$), $\alpha_2 = 1, \beta_1 = c$ ($c > 0$), and f of the form of (1.2), then

$$TL_c^a(\alpha, \beta, \gamma) := \left\{ \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - \alpha \right) \right\} > \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(L(a, c)f(z))'}{L(a, c)f(z)} - 1 \right) \right|, z \in U \right\}, \quad (1.17)$$

where $L(a, c)$ is a well-known Carlson-Shaffer linear operator [4] defined by

$$L(a, c)f(z) := \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) * f(z) \equiv H_1^2(a, 1; c) f(z). \quad (1.18)$$

Also $TL_c^a(\alpha, \beta, \gamma)$ was studied by Murugusundaramoorthy and Magesh [19].

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bound, extreme points, radii of close to convexity, starlikeness, and convexity for the class $TS_m^l(\alpha, \beta, \gamma)$. Further, we obtain neighborhood results and integral means inequalities for aforementioned class.

2. Basic Properties

First we obtain the necessary and sufficient condition for functions f in the class $TS_m^l(\alpha, \beta, \gamma)$.

Theorem 2.1. *The necessary and sufficient condition for f of the form of (1.2) to be in the class $TS_m^l(\alpha, \beta, \gamma)$ is*

$$\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Gamma_n |a_n| \leq (1 - \alpha) + |\gamma|(1 - \beta), \quad (2.1)$$

where $-1 \leq \alpha < 1, \beta \geq 0$, and $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Assume that $f \in TS_m^l(\alpha, \beta, \gamma)$, then

$$\begin{aligned} \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - \alpha \right) \right\} &> \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) \right|, \\ \operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha)\Gamma_n|a_n|z^n}{z - \sum_{n=2}^{\infty} \Gamma_n|a_n|z^n} \right) \right\} &> \beta \left| 1 - \frac{1}{\gamma} \left(\frac{\sum_{n=2}^{\infty} (n-1)\Gamma_n|a_n|z^n}{z - \sum_{n=2}^{\infty} \Gamma_n|a_n|z^n} \right) \right|. \end{aligned} \tag{2.2}$$

Letting $z \rightarrow 1^-$ along the real axis, we have

$$\left\{ 1 + \frac{1}{|\gamma|} \left(\frac{(1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha)\Gamma_n|a_n|}{1 - \sum_{n=2}^{\infty} \Gamma_n|a_n|} \right) \right\} > \beta \left[1 - \frac{1}{|\gamma|} \left(\frac{\sum_{n=2}^{\infty} (n-1)\Gamma_n|a_n|}{1 - \sum_{n=2}^{\infty} \Gamma_n|a_n|} \right) \right]. \tag{2.3}$$

Hence, by maximum modulus theorem, the simple computational leads the desired inequality

$$\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]\Gamma_n|a_n| \leq (1 - \alpha) + |\gamma|(1 - \beta). \tag{2.4}$$

Conversely, suppose that (2.1) is true for $z \in U$. Then

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - \alpha \right) \right\} - \beta \left| 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) \right| > 0 \tag{2.5}$$

if

$$1 + \frac{1}{|\gamma|} \left(\frac{(1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha)\Gamma_n|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} \Gamma_n|a_n||z|^{n-1}} \right) - \beta \left[1 - \frac{1}{|\gamma|} \left(\frac{\sum_{n=2}^{\infty} (n-1)\Gamma_n|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} \Gamma_n|a_n||z|^{n-1}} \right) \right] > 0. \tag{2.6}$$

That is if

$$\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]\Gamma_n|a_n| \leq (1 - \alpha) + |\gamma|(1 - \beta), \tag{2.7}$$

which completes the proof. □

Corollary 2.2. Let the function f defined by (1.2) belong to $TS_m^l(\alpha, \beta, \gamma)$. Then

$$|a_n| \leq \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]\Gamma_n}, \tag{2.8}$$

$n \geq 2$, $-1 \leq \alpha < 1$, $\beta \geq 0$ and, $\gamma \in \mathbb{C} \setminus \{0\}$, with equality for

$$f(z) = z - \frac{[(1-\alpha) + |\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_n} z^n. \quad (2.9)$$

Next we state the following theorem on extreme points for the class $TS_m^l(\alpha, \beta, \gamma)$ without proof.

Theorem 2.3 (Extreme Points). *Let*

$$f_1(z) = z, \quad f_n(z) = z - \frac{[(1-\alpha) + |\gamma|(1-\beta)]}{[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_n} z^n \quad \text{for } n = 2, 3, 4, \dots \quad (2.10)$$

Then $f \in TS_m^l(\alpha, \beta, \gamma)$ if and only if f can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

3. Close-to-Convexity, Starlikeness, and Convexity

We determine the radii of close-to-convexity, starlikeness, and convexity results for functions in the class $TS_m^l(\alpha, \beta, \gamma)$ in the following theorems.

Theorem 3.1. *Let $f \in TS_m^l(\alpha, \beta, \gamma)$. Then f is close-to-convex of order δ ($0 \leq \delta < 1$) in the disc $|z| < r_1$, where*

$$r_1 = \inf_{n \geq 2} \left[\frac{(1-\delta)}{n} \frac{[(n+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_n}{[(1-\alpha) + |\gamma|(1-\beta)]} \right]^{1/(n-1)}. \quad (3.1)$$

Proof. Let f belong to T . It is known [20] that f is close-to-convex of order δ , if it satisfies the condition

$$|f'(z) - 1| < 1 - \delta. \quad (3.2)$$

For the left-hand side of (3.2) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}. \quad (3.3)$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} |a_n| |z|^{n-1} < 1. \quad (3.4)$$

Using the fact that $f \in TS_m^l(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(1 - \alpha) + |\gamma|(1 - \beta)} \Gamma_n |a_n| < 1. \quad (3.5)$$

we can say that (3.2) is true if

$$\frac{n}{1 - \delta} |z|^{n-1} \leq \frac{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(1 - \alpha) + |\gamma|(1 - \beta)} \Gamma_n. \quad (3.6)$$

Or, equivalently,

$$|z| \leq \left[\frac{(1 - \delta) [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{n [(1 - \alpha) + |\gamma|(1 - \beta)]} \Gamma_n \right]^{1/(n-1)}, \quad (3.7)$$

which completes the proof. \square

Theorem 3.2. Let $f \in TS_m^l(\alpha, \beta, \gamma)$. Then the following are given.

(1) f is starlike of order δ ($0 \leq \delta < 1$) in the disc $|z| < r_2$, where

$$r_2 = \inf_{n \geq 2} \left\{ \frac{(1 - \delta) [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(n - \delta) [(1 - \alpha) + |\gamma|(1 - \beta)]} \Gamma_n \right\}^{1/(n-1)}. \quad (3.8)$$

(2) f is convex of order δ ($0 \leq \delta < 1$) in the unit disc $|z| < r_3$, where

$$r_3 = \inf_{n \geq 2} \left\{ \frac{(1 - \delta) [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{n(n - \delta) [(1 - \alpha) + |\gamma|(1 - \beta)]} \Gamma_n \right\}^{1/(n-1)}. \quad (3.9)$$

Each of these results is sharp for the extremal function f given by (2.10).

Proof. Let $f \in T$. It is known [1] that f is starlike of order δ , if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta. \quad (3.10)$$

For the left-hand side of (3.10) we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n| |z|^{n-1}}. \quad (3.11)$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n - \delta}{1 - \delta} |a_n| |z|^{n-1} < 1. \quad (3.12)$$

Using the fact that $f \in TS_m^l(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(1 - \alpha) + |\gamma|(1 - \beta)} \Gamma_n |a_n| < 1. \quad (3.13)$$

we can say that (3.10) is true if

$$\frac{n - \delta}{1 - \delta} |z|^{n-1} < \frac{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(1 - \alpha) + |\gamma|(1 - \beta)} \Gamma_n. \quad (3.14)$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1 - \delta)[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]}{(n - \delta)[(1 - \alpha) + |\gamma|(1 - \beta)]} \Gamma_n, \quad (3.15)$$

which yields the starlikeness of the family.

Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar to those the proof of (1). \square

4. Integral Means

In order to find the integral means inequality and to verify the Silverman Conjecture [21] for $f \in TS_m^l(\alpha, \beta, \gamma)$ we need the following subordination result due to Littlewood [22].

Lemma 4.1 (see [22]). *If the functions f and g are analytic in U with $g < f$, then*

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta, \quad \eta > 0, \quad z = re^{i\theta}, \quad 0 < r < 1. \quad (4.1)$$

Applying Theorem 2.1 with the extremal function and Lemma 4.1, we prove the following theorem.

Theorem 4.2. *Let $\eta > 0$. If $f \in TS_m^l(\alpha, \beta, \gamma)$ and $\{\Phi(\alpha, \beta, \gamma, n)\}_{n=2}^{\infty}$ is nondecreasing sequence, then, for $z = re^{i\theta}$ and $0 < r < 1$, one has*

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta, \quad (4.2)$$

where $f_2(z) = z - ((1 - \alpha) + |\gamma|(1 - \beta))z^2 / \Phi(\alpha, \beta, \gamma, 2)$, and $\Phi(\alpha, \beta, \gamma, n) = [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)]\Gamma_n$.

Proof. Let f of the form of (1.2) and $f_2(z) = z - ((1 - \alpha) + |\gamma|(1 - \beta))z^2 / \Phi(\alpha, \beta, \gamma, 2)$, then we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^n d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Phi(\alpha, \beta, \gamma, 2)} z \right|^n d\theta. \quad (4.3)$$

By Lemma 4.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} < 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Phi(\alpha, \beta, \gamma, 2)} z. \quad (4.4)$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Phi(\alpha, \beta, \gamma, 2)} \omega(z), \quad (4.5)$$

from (4.5) and (2.1) we obtain

$$\begin{aligned} |\omega(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Phi(\alpha, \beta, \gamma, n)}{(1 - \alpha) + |\gamma|(1 - \beta)} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\alpha, \beta, \gamma, n)}{(1 - \alpha) + |\gamma|(1 - \beta)} |a_n| \\ &\leq |z| < 1. \end{aligned} \quad (4.6)$$

This completes the proof of Theorem 4.2. \square

5. Inclusion Relations Involving $N_\delta(e)$

To study about the inclusion relations involving $N_\delta(e)$ we need the following definitions due to Goodman [23] and Ruscheweyh [24]. The n, δ neighborhood of function $f \in T$ is given by

$$N_\delta(f) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n, \sum_{n=2}^{\infty} n |a_n - b_n| \leq \delta \right\}. \quad (5.1)$$

Particularly for the identity function $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n, \sum_{n=2}^{\infty} n |b_n| \leq \delta \right\}. \quad (5.2)$$

Theorem 5.1. *Let*

$$\delta = \frac{2[(1-\alpha) + |\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2}, \quad (5.3)$$

where $\Gamma_2 = (\alpha_1 \cdots \alpha_2) / (\beta_1 \cdots \beta_2)$. Then $TS_m^l(\alpha, \beta, \gamma) \subset N_\delta(e)$.

Proof. For $f \in TS_m^l(\alpha, \beta, \gamma)$, Theorem 2.1 yields

$$[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2 \sum_{n=2}^{\infty} |a_n| \leq (1-\alpha) + |\gamma|(1-\beta) \quad (5.4)$$

so that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(1-\alpha) + |\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2}. \quad (5.5)$$

On the other hand, from (2.1) and (5.5) we have

$$\begin{aligned} (1-\beta)\Gamma_2 \sum_{n=2}^{\infty} n|a_n| &\leq (1-\alpha) + |\gamma|(1-\beta) + [(\alpha-\beta) - |\gamma|(1-\beta)]\Gamma_2 \sum_{n=2}^{\infty} |a_n| \\ &\leq (1-\alpha) + |\gamma|(1-\beta) + [(\alpha-\beta) - |\gamma|(1-\beta)] \\ &\quad \times \Gamma_2 \frac{(1-\alpha) + |\gamma|(1-\beta)}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2} \\ &\leq \frac{[(1-\alpha) + |\gamma|(1-\beta)]2(1-\beta)}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]}, \\ \sum_{n=2}^{\infty} n|a_n| &\leq \frac{2[(1-\alpha) + |\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2}. \end{aligned} \quad (5.6)$$

□

Now we determine the neighborhood for each of the class $TS_m^l(\alpha, \beta, \gamma)$ which we define as follows. A function $f \in T$ is said to be in the class $TS_m^l(\alpha, \beta, \gamma)$ if there exists a function $g \in TS_m^l(\alpha, \beta, \gamma)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta, \quad (z \in \mathcal{U}, 0 \leq \eta < 1). \quad (5.7)$$

Theorem 5.2. *If $g \in TS_m^l(\alpha, \beta, \gamma)$ and*

$$\eta = 1 - \frac{\delta[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2}{2\{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2 - ((1-\alpha) + |\gamma|(1-\beta))\}}, \quad (5.8)$$

where $\Gamma_2 = (\alpha_1 \cdots \alpha_2) / (\beta_1 \cdots \beta_2)$, then $N_\delta(g) \subset TS_m^l(\alpha, \beta, \gamma, \eta)$.

Proof. Suppose that $f \in N_\delta(g)$, then we find from (5.6) that

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \delta, \quad (5.9)$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}. \quad (5.10)$$

Next, since $g \in TS_m^l(\alpha, \beta, \gamma)$, we have

$$\sum_{n=2}^{\infty} |b_n| \leq \frac{2[(1-\alpha) + |\gamma|(1-\beta)]}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2} \quad (5.11)$$

so that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} |b_n|} \\ &\leq \frac{\delta}{2} \times \frac{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2}{[(2+|\gamma|)(1-\beta) - (\alpha-\beta)]\Gamma_2 - ((1-\alpha) + |\gamma|(1-\beta))} \\ &\leq 1 - \eta, \end{aligned} \quad (5.12)$$

provided that η is given precisely by (5.8). Thus by definition, $f \in TS_m^l(\alpha, \beta, \gamma, \eta)$ for η given by (5.8), which completes the proof. \square

Concluding Remarks

By suitably specializing the various parameters involved in Theorems 2.1 to 5.2, we can state the corresponding results for the new subclasses defined in Examples 1.1 to 1.4 and also for many relatively more familiar function classes.

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