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Research Article

On Presented Dimensions of Modules and Rings

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We define the presented dimensions for modules and rings to measure how far away a module is from having an infinite finite presentation and develop ways to compute the projective dimension of a module with a finite presented dimension and the right global dimension of a ring. We also make a comparison of the right global dimension, the weak global dimension, and the presented dimension and divide rings into four classes according to these dimensions.

1. Introduction

Let R be a ring and n a nonnegative integer. Following [1, 2], a right R-module M is called *n-presented* in case it has a finite n-presentation, that is, there is an exact sequence of right R-modules

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{1.1}$$

where each F_i is a finitely generated free, equivalently projective, right R-module. A module is 0-presented (resp., 1-presented) if and only if it is finitely generated (resp., finitely presented), and each m-presented module is n-presented for $m \geq n$. A ring R is called *right n-coherent* in case every n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent (resp., 1-coherent) if and only if it is right Noetherian (resp., coherent), and every n-coherent ring is m-coherent for $m \geq n$.

As in [1, 3], we set $\lambda_R(M) = \sup\{n \mid M$ has a finite n-presentation$\}$ and note that $\lambda_R(M) \geq n$ is a way to express how far away a module M is from having an infinite finite presentation. Clearly every finitely generated projective module M has an infinite finite presentation, that is, $\lambda_R(M) = \infty$. The *lambda dimension* of a ring R is the infimum of the set of integers n such that every R-module having a finite n-presentation has an infinite finite presentation. It was studied extensively by Vasconcelos in [3], where it was denoted by λ-dim(R). Note that R is right n-coherent if and only if $\lambda\text{-dim}(R) \le n$ and if and only if every n-presented module has an infinite finite presentation.

Ng [4] defined the finitely presented dimension of a module M as $\text{f.p.}\dim(M) = \inf\{n \mid$ there exists an exact sequence $P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0$ of R-modules, where each P_i is projective, and P_{n+1}, P_n are finitely generated$\}$, which measures how far away a module is from being finitely presented. Motivated by this, we define a dimension, called presented dimension, for modules and rings in this paper. It measures how far away a module is from having an infinite finite presentation and how far away a ring is from being Noetherian. In Section 2, we give the definitions and show the properties of presented dimensions. In Section 3, using strongly presented modules, we give the structure of modules with presented dimensions ≤ 1 and develop ways to compute the projective dimension of a module with a finite presented dimension and the right global dimension of a ring. In Section 4, we define the presented dimension of a ring, make a comparison of the right global dimension, the weak global dimension, and the presented dimension, and divide rings into four classes according to these dimensions. In Section 5, we provide the properties of presented dimensions of modules and rings under an almost excellent extension of rings.

Throughout rings are associative with identity, modules are unitary right R-modules, and homomorphisms are module homomorphisms. The notations $\text{pd}(M), \text{id}(M)$, and $\text{fd}(M)$denote the projective, injective, flat dimension of M, and $\text{rgD}(R)$, $\text{wD}(R)$ denote the right global dimension, weak global dimension, respectively. For other definitions and notations in this paper we refer to [5, 6].

2. Presented Dimensions of Modules

Definition 2.1. Let M be a right R-module, define the presented dimension of M as follows:

$$\begin{aligned}\text{FPd}(M) = \inf\{m \mid &\text{ there exists a projective resolution} \\ &\cdots \longrightarrow P_{m+j} \longrightarrow \cdots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \\ &\text{such that } P_{m+i}, i = 0, 1, 2, \dots \text{ are finitely generated}\}.\end{aligned} \tag{2.1}$$

If there is no such resolution, then define $\text{FPd}(M) = \infty$.

In particular, if $\text{FPd}(M) = 0$, then M has an infinite finite presentation. In this case, we call M a *strongly presented* module. Consequently, we may regard the presented dimension as a measure of how far away a module is from having infinite finite presentation.

Clearly, R is right n-coherent if and only if $\text{FPd}(M) = 0$ for each $(n-1)$-presented right R-module M, if and only if every $(n-1)$-presented module has infinite finite presentation.

Proposition 2.2. *Let M be a right R-module, then* $\text{FPd}(M) \le \text{pd}(M) + 1$.

Proof. Directly by Definition 2.1. □

We remark that $\text{FPd}(M)$ can be much smaller than $\text{pd}(M)$. Take $R = \mathbb{Z}_4$. The ideal $2\mathbb{Z}_4$ has projective dimension ∞ while $\text{FPd}(2\mathbb{Z}_4) = 0$ for R is Noetherian.

Proposition 2.3. *No finitely generated right R-module has presented dimension 1.*

Proof. Suppose that M is a finitely generated right R-module with $\mathrm{FPd}(M) = 1$. There is a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0, \tag{2.2}$$

where $P_1, P_2, \ldots$ are finitely generated; it follows that $\ker d_0$ is finitely presented and hence finitely generated. Note that $0 \to \ker d_0 \to P_0 \to M \to 0$ is exact and M is finitely generated, thus P_0 is finitely generated, so $\mathrm{FPd}(M) = 0$, a contradiction. □

It is known that every finitely presented flat right R-module is projective, that is, if $\mathrm{f.p.\,dim}(M) = 0$, then

$$\mathrm{fd}(M) < \mathrm{f.p.\,dim}(M) + 1 \Longrightarrow \mathrm{pd}(M) < \mathrm{f.p.\,dim}(M) + 1 \ . \tag{2.3}$$

For the presented dimensions of modules, we give a general result as follows.

Theorem 2.4. *Assume that* $\mathrm{FPd}(M) = m < \infty$ *and* $t \geq 0$ *is an integer, then*

$$\mathrm{fd}(M) < \mathrm{FPd}(M) + t \quad \text{iff } \mathrm{pd}(M) < \mathrm{FPd}(M) + t. \tag{2.4}$$

Proof. When $m = 0$, this is trivial. Now suppose that $0 < m < \infty$, then there is a projective resolution

$$\cdots \longrightarrow P_{m+n} \xrightarrow{d_{m+n}} \cdots \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0, \tag{2.5}$$

where $P_m, \ldots, P_{m+n}, \ldots$ are finitely generated.

We only need to prove the necessity. Let $K_i = \ker d_i$, $i = 0, 1, \ldots, n+m, \ldots$, and $K_{-1} = M$. For each integer $t \geq 0$, there is an exact sequence

$$0 \longrightarrow K_{m+t-2} \longrightarrow P_{m+t-2} \longrightarrow \cdots \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow K_{m+t-1} \longrightarrow P_{m+t-1} \longrightarrow K_{m+t-2} \longrightarrow 0. \tag{2.7}$$

Since $\mathrm{fd}(M) < m + t$, we have that K_{m+t-2} is flat from (2.6). Note that P_{m+t} is finitely generated and projective, thus $K_{m+t-1} = \mathrm{Im}\, d_{m+t}$ is finitely generated. From (2.7) and [7], it follows that K_{m+t-2} is projective. Thus $\mathrm{pd}(M) \leq m + t - 1$ from (2.6), so $\mathrm{pd}(M) < m + t$. □

In particular, we have the following corollary.

Corollary 2.5. *M is a projective right R-module if and only if* $\mathrm{FPd}(M) \leq 1$ *and M are flat.*

Proof. ($\Rightarrow$). Immediately from Proposition 2.2.

($\Leftarrow$). If $\mathrm{FPd}(M) = 0$, then M is finitely presented, Thus M is projective.

If $\mathrm{FPd}(M) = 1$, then $\mathrm{fd}(M) < \mathrm{FPd}(M)$. From Theorem 2.4, we have $\mathrm{pd}(M) < \mathrm{FPd}(M)$. Thus $\mathrm{pd}(M) = 0$, so M is projective. □

We recall the mapping cone construction. Suppose that $F : C' \to C$ is a morphism of complexes. Then $MC(F)$ is a complex with $MC(F)_n = C_n \oplus C'_{n-1}$, and the sequence of complexes $0 \to C \to MC(F) \to C'(-1) \to 0$ is exact (see [8]).

Assume that

$$\begin{tikzcd} C' \arrow[r, "F"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "f"] & A \end{tikzcd} \tag{2.8}$$

is a commutative diagram in which the vertical maps are projective resolutions. If f is a monomorphism, $MC(F)$ is a projective resolution of coker$f$. If f is an epimorphism, then

$$\cdots \longrightarrow MC(F)_n \longrightarrow MC(F)_{n-1} \longrightarrow \cdots \longrightarrow MC(F)_2 \longrightarrow Z_1(MC(F)) \longrightarrow \ker f \longrightarrow 0 \tag{2.9}$$

is a projective resolution of $\ker f$.

Assume that $\mathrm{FPd}(M) = m$, there is an exact sequence

$$\cdots \longrightarrow P_{m+n} \longrightarrow \cdots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{2.10}$$

where $P_i (i = 0, \ldots, m+n, \ldots)$ are projective, and $P_m, \ldots, P_{m+n}, \ldots$ are finitely generated; we call such an infinite exact sequence *a representing sequence* of M.

Theorem 2.6. *Assume that $0 \to A' \to A \to A'' \to 0$ is an exact sequence of right R-modules, $\mathrm{FPd}(A') = d', \mathrm{FPd}(A) = d, \mathrm{FPd}(A'') = d''$. If two of these are finite, then so is the third. Furthermore,*

$$d \le \max\{d', d''\}, \qquad d'' \le \max\{d, d'+1\}, \qquad d' \le \max\{d, d''-1\}. \tag{2.11}$$

Proof. Suppose that d', d'' are finite. Let P', P'' represent sequences of A', A'', respectively. There exists a projective resolution P of A such that $0 \to P' \to P \to P'' \to 0$ is an exact sequence of complexes. Thus P_m is finitely generated when $m \ge \max\{d', d''\}$. So $\mathrm{FPd}(A) \le \max\{d', d''\}$.

Suppose that d', d are finite. Let P', P represent sequences of A', A, respectively, and let $F : P' \to P$ cover $f : A' \to A$, thus $MC(F)$ is a projective resolution of A''. By the definition of $MC(F)$, we have that $MC(F)_m$ is finitely generated for each $m \ge d$ and $m \ge d'+1$. So $\mathrm{FPd}(A'') \le \max\{d, d'+1\}$.

Suppose that d, d'' are finite. Let P, P'' represent sequences of A, A'', respectively, and let $G : P \to P''$ cover $g : A \to A''$. Then P' is a projective resolution of A', where $P'_m = MC(G)_{m+1} (m \ge 1)$, $P'_0 = Z_1(MC(G))$. Thus $MC(G)_m$ are finitely generated, whenever $m \ge d''$ and $m \ge d+1$. So P'_m is finitely generated for $m \ge d''-1$ and $m \ge d$. Note that P'_0 is finitely generated if $d'' \le 1$ and $d = 0$ by the split exact sequence

$$0 \longrightarrow Z_1(MC(F)) \longrightarrow MC(F)_1 \longrightarrow MC(F)_0 \longrightarrow 0. \tag{2.12}$$

So $\mathrm{FPd}(A') \le \max\{d, d''-1\}$. □

Corollary 2.7. *If* $\mathrm{FPd}(A_1),\ldots,\mathrm{FPd}(A_m)$ *are finite, then*

$$\mathrm{FPd}(A_1\oplus\cdots\oplus A_m)=\max\{\mathrm{FPd}(A_i)\mid i=1,\ldots,m\}. \tag{2.13}$$

Proof. Clearly it suffices to consider the case $m=2$. Then there exist exact sequences

$$\begin{aligned}&0\longrightarrow A_1\longrightarrow A_1\oplus A_2\longrightarrow A_2\longrightarrow 0,\\&0\longrightarrow A_2\longrightarrow A_1\oplus A_2\longrightarrow A_1\longrightarrow 0.\end{aligned} \tag{2.14}$$

By Theorem 2.6, we have

$$\begin{aligned}\mathrm{FPd}(A_1\oplus A_2)&\le\max\{\mathrm{FPd}(A_1),\mathrm{FPd}(A_2)\},\\ \mathrm{FPd}(A_1)&\le\max\{\mathrm{FPd}(A_1\oplus A_2),\mathrm{FPd}(A_2)-1\},\\ \mathrm{FPd}(A_2)&\le\max\{\mathrm{FPd}(A_1\oplus A_2),\mathrm{FPd}(A_1)-1\}.\end{aligned} \tag{2.15}$$

Suppose that $\mathrm{FPd}(A_1\oplus A_2)<\mathrm{FPd}(A_1)$. Then $\mathrm{FPd}(A_2)\le\mathrm{FPd}(A_1)-1$, thus

$$\begin{aligned}\mathrm{FPd}(A_1)&\le\max\{\mathrm{FPd}(A_1\oplus A_2),\mathrm{FPd}(A_2)-1\}\\&\le\max\{\mathrm{FPd}(A_1\oplus A_2),\mathrm{FPd}(A_1)-2\}\\&=\mathrm{FPd}(A_1\oplus A_2),\end{aligned} \tag{2.16}$$

which contradicts the hypothesis. So $\mathrm{FPd}(A_1\oplus A_2)\ge\mathrm{FPd}(A_1)$. Similarly, $\mathrm{FPd}(A_1\oplus A_2)\ge\mathrm{FPd}(A_2)$. Therefore $\mathrm{FPd}(A_1\oplus A_2)=\max\{\mathrm{FPd}(A_1),\mathrm{FPd}(A_2)\}$. □

3. Strongly Presented Modules

Theorem 3.1. $\mathrm{FPd}(M)\le 1$ *if and only if there are a projective module* P_0*, a free module* F_0*, and a strongly presented module* M_0 *such that* $M\oplus P_0=M_0\oplus F_0$.

Proof. $(\Rightarrow)$. Suppose that $\mathrm{FPd}(M)\le 1$. There is an exact sequence $0\to K\xrightarrow{i}P\xrightarrow{\pi}M\to 0$, where P is projective and K is strongly presented. Choose a projective module P_0 such that $P\oplus P_0$ is free, and let $F=P\oplus P_0$. Thus we have an exact sequence

$$0\longrightarrow K\xrightarrow{i}F\xrightarrow{\sigma=\pi\oplus I_{P_0}}M\oplus P_0\longrightarrow 0. \tag{3.1}$$

Suppose that K is generated by the set $\{g_1,g_2,\ldots,g_n\}$. Choose a basis $\{f_1,f_2,\ldots,f_j,\ldots\}$ of F such that $i(g_1),i(g_2),\ldots,i(g_n)$ can be generated by $f_1,f_2,\ldots,f_m$. Let F_1 be generated by $f_1,f_2,\ldots,f_m$, and F_2 generated by $f_{m+1},f_{m+2},\ldots,f_j,\ldots$. Then $F=F_1\oplus F_2$, and $i(K)\subseteq F_1$. Let $M_0=\sigma(F_1)$, $F_0=\sigma(F_2)$. Then M_0 is strongly presented, $F_0\cong F_2$, and $M\oplus P_0=M_0\oplus F_0$.

($\Leftarrow$). Suppose that $M \oplus P_0 = M_0 \oplus F_0$, where P_0 is a projective module, F_0 is a free module, and M_0 is a strongly presented module. There is a finitely generated free module F such that the following sequence:

$$0 \longrightarrow K \xrightarrow{i} F_0 \oplus F \xrightarrow{f} F_0 \oplus M_0 = M \oplus P_0 \longrightarrow 0 \tag{3.2}$$

is exact and K is strongly presented. Let $\pi : M \oplus P_0 \to P_0$ be the canonical projection. Then we have an exact sequence $0 \to K' \to F_0 \oplus F \xrightarrow{\pi f} P_0 \to 0$, where $K' = \ker \pi f$. It is clear that $i(K) \subseteq K'$. Thus $0 \to K \xrightarrow{i} K' \xrightarrow{f|_{K'}} M \longrightarrow 0$ is exact, hence

$$\ker f\big|_{K'} = \ker f \cap K' = i(K) \cap K' = i(K), \tag{3.3}$$

and $f|_{K'}$ is epimorphic. Note that K is strongly presented and K' is projective, thus $\mathrm{FPd}(M) \leq 1$. □

Corollary 3.2. *Assume that R is a ring such that every projective module is free (e.g., R is local). Then $\mathrm{FPd}(M) \leq 1$ if and only if there are a strongly presented module M_0 and a free module F such that $M = M_0 \oplus F$.*

Next, we aim to obtain a test for projectivity of modules with finite presented dimensions. In [1, Theorem 1.7], it was proved that $\mathrm{pd}(M) \leq d$ for every n-presented module M if and only if $\mathrm{Ext}_R^{d+1}(M, N) = 0$ for every n-presented module M and $(n-(d+1))$-presented module N. We generalize it as follows.

Proposition 3.3. *Assume that M is a strongly presented module and $n \geq 0$ is an integer. Then $\mathrm{pd}(M) \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for every strongly presented module N.*

Proof. The necessity is clear. Conversely, we proceed by induction on n. If $n = 0$ and $\mathrm{E}xt_R^1(M, N) = 0$ for every strongly presented module N, there is, an exact sequence

$$0 \longrightarrow M_1 \longrightarrow F \longrightarrow M \longrightarrow 0, \tag{3.4}$$

where F is finitely generated and free and M_1 is strongly presented. Thus $\mathrm{Ext}_R^1(M, M_1) = 0$, whence $\mathrm{Hom}(F, M_1) \to \mathrm{Hom}(M_1, M_1) \to 0$; is exact, so $0 \to M_1 \to F \to M \to 0$ is split, and M is a direct summand of F, hence projective, that is, $\mathrm{pd}(M) \leq 0$.

Now suppose that $n \geq 1$ and $\mathrm{E}xt_R^{n+1}(M, N) = 0$ for every strongly presented module N. Since

$$0 = \mathrm{Ext}_R^{n+1}(M, N) = \mathrm{Ext}_R^{n}(M_1, N), \tag{3.5}$$

and M_1 is strongly presented, by hypothesis $\mathrm{pd}(M_1) \leq n-1$, so $\mathrm{pd}(M) \leq n$. □

Corollary 3.4. *Assume that M is strongly presented and $n \geq 0$ is an integer. If $pd(M) = n$, then $\mathrm{Ext}_R^n(M, R) \neq 0$.*

Proof. Since $\mathrm{pd}(M) = n$, by Proposition 3.3 there is a strongly presented module N such that $\mathrm{Ext}_R^n(M,N) \neq 0$; thus there is an exact sequence $0 \to N_1 \to P \to N \to 0$, where P is finitely generated and projective, and N_1 is finitely generated. So we have an exact sequence $\mathrm{Ext}_R^n(M,P) \to \mathrm{Ext}_R^n(M,N) \to 0$ for $\mathrm{pd}(M) = n$.

Suppose that $\mathrm{Ext}_R^n(M,R) = 0$. Then $\mathrm{Ext}_R^n(M,P) = 0$ for each finitely generated projective module P, so $\mathrm{Ext}_R^n(M,N) = 0$, a contradiction. Therefore $\mathrm{Ext}_R^n(M,R) \neq 0$. □

Lemma 3.5. *Assume that* $\mathrm{FPd}(M) \leq 1$. *Then* $pd(M) \leq n$ *if and only if* $\mathrm{Ext}_R^{n+1}(M,N) = 0$ *for every strongly presented module* N.

Proof. By Theorem 3.1, $M \oplus P_0 = M_0 \oplus F$, where P_0 is projective, F is free, and M_0 is strongly presented. Thus $\mathrm{pd}(M) \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(M,B) = 0$ for every module B, if and only if $\mathrm{Ext}_R^{n+1}(M_0,B) = 0$ for every module B, if and only if $\mathrm{pd}(M_0) \leq n$, if and only if $\mathrm{Ext}_R^{n+1}(M_0,N) = 0$ for every strongly presented module N by Proposition 3.3, if and only if $\mathrm{Ext}_R^{n+1}(M,N) = 0$ for every strongly presented module N. □

Theorem 3.6. *Assume that* $\mathrm{FPd}(M) < \infty$ *and* n *is an integer. Then* $pd(M) \leq n$ *if and only if* $\mathrm{Ext}_R^{n+1}(M,N) = 0$ *for every strongly presented module* N.

Proof. Suppose that $\mathrm{FPd}(M) = m$. Then $m \leq \mathrm{pd}(M) + 1$ by Proposition 2.2, and there is a projective resolution of M

$$\cdots \longrightarrow P_{m+l} \xrightarrow{d_{m+l}} \cdots \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0, \tag{3.6}$$

where $P_m, \ldots, P_{m+l}, \ldots$ are finitely generated. Thus $\ker d_{m-1}$ is strongly presented.

Suppose that $n = m - 1$. Then $\mathrm{FPd}(\ker d_{m-2}) \leq 1$, hence $\mathrm{pd}(\ker d_{m-2}) \leq n - m + 1$ if and only if $\mathrm{Ext}_R^{n-m+2}(\ker d_{m-2}, N) = 0$ for every strongly presented module N by Lemma 3.5. Suppose that $n \geq m$; by Proposition 3.3 $\mathrm{pd}(\ker d_{m-1}) \leq n - m$ if and only if $\mathrm{Ext}_R^{n-m+1}(\ker d_{m-1}, N) = 0$ for every strongly presented module N.

Therefore $\mathrm{pd}(M) \leq n$ if and only if $Ext_R^{n+1}(M,N) = 0$ for every strongly presented module N. □

Now we obtain a way to compute the right global dimension of a ring.

Corollary 3.7. *Assume that* $\mathrm{rgD}(R) < \infty$. *Then*

$$\mathrm{rgD}(R) = \sup\{\mathrm{id}(N) \mid N \textit{ is strongly presented}\}. \tag{3.7}$$

Proof. By Proposition 2.2. $\mathrm{FPd}(R/I) < \infty$ for each right ideal I of R, thus $\mathrm{pd}(R/I) \leq n$ for each right ideal I of R if and only if $\mathrm{Ext}_R^{n+1}(R/I,N) = 0$ for each strongly presented module N and each right ideal I of R by Theorem 3.6, if and only if $\mathrm{id}(N) \leq n$ for each strongly presented module N by the Baer Criterion for injectivity. Therefore the result holds. □

4. Presented Dimensions of Rings

Definition 4.1. Define the presented dimension of R as follows:

$$\mathrm{FPD}(R) = \sup\{\mathrm{FPd}(M) \mid M \text{ is a finitely generated right } R\text{-module}\}. \tag{4.1}$$

It is easy to see that $\mathrm{FPD}(R) = 0$ if and only if every finitely generated module has an infinite finite presentation, if and only if every finitely generated module is finitely presented, if and only if R is right Noetherian. Thus we may regard the presented dimension of a ring as a measure of how far it is from being right Noetherian.

Proposition 4.2. $\mathrm{FPD}(R) \leq \mathrm{rgD}(R) + 1$.

Proof. By Proposition 2.2, $\mathrm{FPD}(M) \leq \mathrm{pd}(M) + 1$, thus the result follows immediately. □

Note that $\mathrm{FPD}(R)$ can be much smaller than $\mathrm{rgD}(R)$. Take $R = \mathbb{Z}_4$. Then $\mathrm{rgD}(R) = \infty$ while $\mathrm{FPD}(R) = 0$ for R is Noetherian.

Following Proposition 2.3, we have the following corollary.

Corollary 4.3. *No ring can have presented dimension 1.*

In the following, we investigate the relations of the right global, weak global, and presented dimensions of rings.

Theorem 4.4. *Let R be a ring.*

(1) *If* $\mathrm{FPD}(R) \leq \mathrm{wD}(R)$, *then* $\mathrm{rgD}(R) = \mathrm{wD}(R)$.

(2) *If* $\mathrm{FPD}(R) > \mathrm{wD}(R)$, *then* $\mathrm{rgD}(R) = \mathrm{FPD}(R)$ *or* $\mathrm{FPD}(R) - 1$.

(3) *If* $\mathrm{rgD}(R) > \mathrm{wD}(R)$, *then* $\mathrm{FPD}(R) = \mathrm{rgD}(R) + 1$.

Proof. (1) It suffices to prove that $\mathrm{wD}(R) \geq \mathrm{rgD}(R)$ and suppose that $\mathrm{wD}(R) = s < \infty$. Let M be finitely generated. Since $\mathrm{FPD}(R) \leq \mathrm{wD}(R) = s$, we have $\mathrm{FPd}(M) = m \leq s$, thus there is a projective resolution

$$\cdots \longrightarrow P_{m+n} \xrightarrow{d_{m+n}} \cdots \longrightarrow P_m \xrightarrow{d_m} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{4.2}$$

where $P_m, \ldots, P_{m+n}, \ldots$ are finitely generated. Since $\mathrm{wD}(R) = s$, it follows that $\ker d_{s-1}$ is flat. Note that $s \geq m$, hence $\ker d_{s-1}$ is finitely presented, whence projective, that is, $\mathrm{pd}(M) \leq s$. So $\mathrm{wD}(R) \geq \mathrm{rgD}(R)$.

(2) If $\mathrm{FPD}(R) = \infty$, the result follows immediately from Proposition 4.2. Now suppose that $\mathrm{FPD}(R) = m < \infty$. Since $\mathrm{FPD}(R) > \mathrm{wD}(R) \geq 0$, by Corollary 4.3 we have $m \geq 2$. Let M be finitely generated and $\mathrm{fd}(M) = k$, thus $\ker d_t$ is finitely presented for each $t \geq m - 1$.

If $\mathrm{fd}(M) \leq \mathrm{FPd}(M)$, then $\ker d_{m-1}$ is flat, hence projective, so $\mathrm{pd}(M) \leq \mathrm{FPd}(M)$.

If $\mathrm{fd}(M) > \mathrm{FPd}(M)$, then $\ker d_{k-1}$ is flat, hence projective, so

$$\mathrm{pd}(M) \leq \mathrm{fd}(M) \leq \mathrm{wD}(R). \tag{4.3}$$

Therefore $\mathrm{rgD}(R) \leq \mathrm{FPD}(R)$.

On the other hand, by Proposition 4.2, $\mathrm{FPD}(R) \leq \mathrm{rgD}(R) + 1$. So $\mathrm{rgD}(R) = \mathrm{FPD}(R)$ or $\mathrm{FPD}(R) - 1$.

(3) From (1) and (2), we have $\mathrm{FPD(R)} = \mathrm{rgD}(R) + 1$ or $\mathrm{FPD}(R) = \mathrm{rgD}(R)$. Thus we need only to consider $\mathrm{rgD}(R) = m < \infty$ and prove $\mathrm{FPD}(R) \neq \mathrm{rgD}(R)$. Suppose that

$\mathrm{FPD}(R) = \mathrm{rgD}(R)$. Let M be a finitely generated right R-module with $\mathrm{FPd}(M) = m$, then there is an exact sequence

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \xrightarrow{d_{m-1}} P_{m-2} \xrightarrow{d_{m-2}} \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{4.4}$$

where P_i is projective and P_m is strongly presented. Let $K_{m-2} = \ker d_{m-2}$. Note that $m \neq 1$ and $m > \mathrm{wD}(R) \geq 0$. We consider the exact sequence

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow K_{m-2} \longrightarrow 0. \tag{4.5}$$

Since $\mathrm{wD}(R) < \mathrm{rgD}(R) = m$, K_{m-2} is flat. Suppose that Q such that $P_{m-1} \oplus Q = F$ is free. Then

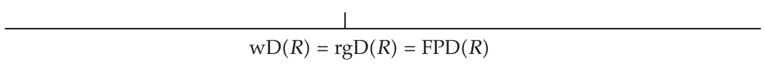
$$0 \longrightarrow P_m \longrightarrow F \longrightarrow K_{m-2} \oplus Q \longrightarrow 0 \tag{4.6}$$

is exact, and $K_{m-2} \oplus Q$ is flat. Let $p_1, \ldots, p_m$ generate P_m. Using the flatness of $K_{m-2} \oplus Q$, there exists a homomorphism $F \longrightarrow P_m$ such that $p_i \mapsto p_i (i = 1, 2, \ldots, t)$. Thus the above short sequence splits, and so $F \cong P_m \oplus K_{m-2} \oplus Q$. Thus K_{m-2} is projective, therefore $\mathrm{pd}(M) \leq m-1$, and so $\mathrm{rgD}(R) \leq m-1$, a contradiction. Hence $\mathrm{FPD}(R) \neq \mathrm{rgD}(R)$, so $\mathrm{FPD}(R) = \mathrm{rgD}(R) + 1$. □

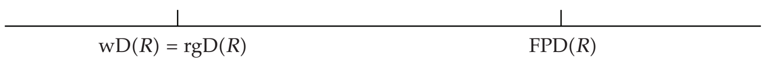
Corollary 4.5. $\mathrm{rgD}(R) = \max\{\mathrm{wD}(R), \mathrm{FPD}(R) - 1\}$.

From the foregoing discussion, we can classify rings by the right global dimensions, weak global dimensions, and presented dimensions of rings.

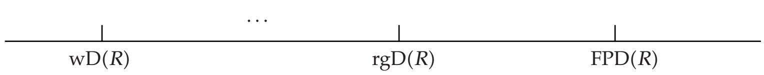
Case 1:



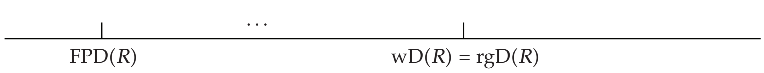
Case 2:



Case 3:



Case 4:



In the diagrams, represents two consecutive numbers while means that the numbers may not be consecutive.

5. On Ring Extensions

In this section, assume that $S \geq R$ is a unitary ring extension. We aim to investigate properties of the presented dimensions of modules and rings. We first recall some concepts.

(1) The ring S is called *right R-projective* [9] in case, for any right S-module M_S with an S-module N_S, $N_R \mid M_R$ implies $N_S \mid M_S$, where $N \mid M$ means that N is a direct summand of M. For example, every $n \times n$ matrix ring R_n is right R-projective [9].

(2) The ring extension $S \geq R$ is called *a finite normalizing extension* [10] in case there is a finite subset $\{s_1, \ldots, s_n\} \subseteq S$ such that $S = \sum_{i=1}^{n} s_i R$ and $s_i R = R s_i$ for $i = 1, \ldots, n$.

(3) A finite normalizing extension $S \geq R$ is called *an almost excellent extension* [11] in case ${}_RS$ is flat, S_R is projective, and the ring S is right R-projective.

(4) An almost excellent extension $S \geq R$ is *an excellent extension* [9] in case both ${}_RS$ and S_R are free modules with a common basis $\{s_1, \ldots, s_n\}$.

Excellent extensions were introduced by Passman [9] and named by Bonami [12]. Examples include the $n \times n$ matrix rings and the crossed products $R * G$ where G is a finite group with $|G|^{-1} \in R$. Almost excellent extensions were introduced and studied by Xue [11] as a nontrivial generalization of excellent extensions and recently studied in [2, 13–15].

Proposition 5.1. *Assume that $S \geq R$ is a finite normalizing extension and ${}_RS$ is flat. Then for each right R-module M_R, we have*

$$\mathrm{FPd}(M \otimes_R S)_S \leq \mathrm{FPd}(M_R). \tag{5.1}$$

Proof. If $\mathrm{FPd}(M_R) = \infty$, it is clear. Suppose $\mathrm{FPd}(M_R) = m < \infty$. There is a projective resolution of M

$$\cdots \longrightarrow P_{m+n} \longrightarrow \cdots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{5.2}$$

where $P_m, \ldots, P_{m+n}$ are finitely generated. Since ${}_RS$ is flat, there is an exact sequence of right S-modules

$$\cdots \longrightarrow P_{m+n} \otimes_R S \longrightarrow \cdots \longrightarrow P_m \otimes_R S \longrightarrow \cdots \longrightarrow P_0 \otimes_R S \longrightarrow M \otimes_R S \longrightarrow 0, \tag{5.3}$$

where $P_i \otimes_R S$ is a projective right S-module, and $P_m \otimes_R S, \ldots, P_{m+n} \otimes_R S, \ldots$ are finitely generated. So $\mathrm{FPd}(M \otimes_R S) \leq m$, therefore $\mathrm{FPd}(M \otimes_R S)_S \leq \mathrm{FPd}(M_R)$. □

Proposition 5.2. *Assume that $S \geq R$ is a finite normalizing extension, ${}_RS$ is flat, and S is right R-projective. Then for each right S-module M_S, one has*

$$\mathrm{FPd}(M_S) \leq \mathrm{FPd}(M \otimes_R S). \tag{5.4}$$

Proof. By [11, Lemma 1.1], M_S is isomorphic to a direct summand of $M \otimes_R S$. By Corollary 2.7, $\mathrm{FPd}(M \otimes_R S) \geq \mathrm{FPd}(M_S)$. □

Proposition 5.3. *Assume that $S \geq R$ is an almost excellent extension. Then for each right S-module M_S, one has* $\mathrm{FPd}(M_R) \leq \mathrm{FPd}(M_S)$.

Proof. If $\mathrm{FPd}(M_S) = \infty$, then it clear. Suppose that $\mathrm{FPd}(M_S) = m < \infty$. Then there is a projective resolution

$$\cdots \longrightarrow P_{m+n} \longrightarrow \cdots \longrightarrow P_m \cdots \longrightarrow P_0 \longrightarrow M_S \longrightarrow 0, \tag{5.5}$$

where P_i $(i = 0, 1, \ldots)$ are right S-modules and $P_m, \ldots, P_{m+1}, \ldots$ are finitely generated. Since $S \geq R$ is an almost excellent extension, it follows that P_i $(i = 0, 1, \ldots)$ are projective right R-modules, and $P_m, P_{m+1}, \ldots$ are finitely generated right R-modules. Thus

$$\cdots \longrightarrow P_{m+n} \longrightarrow \cdots \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \tag{5.6}$$

is a projective resolution of M_R. So $\mathrm{FPd}(M_S) \geq \mathrm{FPd}(M_R)$. □

Corollary 5.4. *Assume that $S \geq R$ is an almost excellent extension. Then for each right S-module M_S, one has* $\mathrm{FPd}(M_R) = \mathrm{FPd}(M_S) = \mathrm{FPd}(M \otimes_R S)$.

Theorem 5.5. *Assume that $S \geq R$ is a finite normalizing extension and $_RS$ is flat.*

(1) *If S is right R-projective and* $\mathrm{FPD}(S) < \infty$, *then* $\mathrm{FPD}(S) \leq \mathrm{FPD}(R)$;

(2) *If* $\mathrm{FPD}(R) < \infty$, *then*

$$\mathrm{FPD}(R) \leq \mathrm{FPD}(S) + \max\{l, s\}, \tag{5.7}$$

where $l = pd(S_R)$ and $s = \sup\{\mathrm{FPd}(A_R) \mid A \in \text{Mod-}S$ *and* $\mathrm{FPd}(A_S) = 0\}$.

Proof. (1) Suppose that $\mathrm{FPD}(S) = m$, M is a finitely generated right S-module, and $\mathrm{FPd}(M_S) = m$. Since S is right R-projective, there is an exact sequence of S-modules

$$0 \longrightarrow M_S \longrightarrow M \otimes_R S \longrightarrow C \longrightarrow 0, \tag{5.8}$$

where $\mathrm{FPd}(C) \leq m$ for $\mathrm{FPD}(S) = m$. By Theorem 2.6 we have

$$\begin{gathered} \mathrm{FPd}(M \otimes_R S)_S \leq \max\{\mathrm{FPd}(M_S), \mathrm{FPd}(C_S)\} = m, \\ m = \mathrm{FPd}(M_S) \leq \max\{\mathrm{FPd}(M \otimes_R S)_S, \mathrm{FPd}(C_S) - 1\} \leq \mathrm{FPd}(S) = m, \end{gathered} \tag{5.9}$$

thus $\mathrm{FPd}(M \otimes_R S_S) = m$. It follows from Proposition 5.1 that $\mathrm{FPd}(M \otimes_R S_S) \leq \mathrm{FPd}(M_R)$. So $\mathrm{FPD}(S) \leq \mathrm{FPD}(R)$.

(2) Suppose that $\mathrm{FPD}(R) = m$, M is a finitely generated right R-module, and $\mathrm{FPd}(M_R) = m < \infty$. Since $_RS$ is flat, by [16, Lemma 2.3], there is an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow M \otimes_R S \longrightarrow C \longrightarrow 0. \tag{5.10}$$

Note that $_RS$ is finitely generated, which implies thats $M \otimes_R S$ and C are finitely generated, thus $\mathrm{FPd}(C) \leq m$ for $\mathrm{FPD}(R) = m$. By Theorem 2.6, we have

$$m = \mathrm{FPd}(M_R) \leq \max\{\mathrm{FPd}(M \otimes_R S_R), \mathrm{FPd}(C_R) - 1\} \leq \mathrm{FPD}(R) = m, \tag{5.11}$$

hence $\mathrm{FPd}(M\otimes_R S_R) = m$. Let $\mathrm{FPd}(M\otimes_R S)_S = t \leq \mathrm{FPD}(S)$. Then there is a projective resolution of the right S-module $M\otimes_R S$

$$\cdots \longrightarrow Q_{t+1} \longrightarrow Q_t \longrightarrow Q_{t-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M\otimes_R S \longrightarrow 0, \tag{5.12}$$

where $Q_t, Q_{t+1}, \ldots$ are finitely generated. Thus we have the following exact sequences:

$$\begin{gathered} 0 \longrightarrow K_{t-1} \longrightarrow Q_{t-1} \longrightarrow K_{t-2} \longrightarrow 0, \\ 0 \longrightarrow K_{t-2} \longrightarrow Q_{t-2} \longrightarrow K_{t-3} \longrightarrow 0, \\ \cdots \\ 0 \longrightarrow K_0 \longrightarrow Q_0 \longrightarrow M\otimes_R S \longrightarrow 0, \end{gathered} \tag{5.13}$$

where $K_i = \mathrm{Im}(Q_{i+1} \to Q_i)$ and $\mathrm{FPd}(K_{t-1}) = 0$. By Proposition 2.2,

$$\mathrm{FPd}(Q_i)_R \leq \mathrm{pd}(Q_i)_R + 1 \leq \mathrm{pd}(S_R) + 1 = l + 1, \tag{5.14}$$

and $\mathrm{FPd}(K_{t-1})_R \leq s$. Following Theorem 2.6 and Proposition 2.2, we have

$$\mathrm{FPd}_R(K_{t-2}) \leq \max\{\mathrm{FPd}_R(Q_{t-1}), \mathrm{FPd}_R(K_{t-1}) + 1\} \leq \max\{l+1, s+1\} = 1 + \max\{l, s\}. \tag{5.15}$$

Again by Theorem 2.6, we have

$$\begin{gathered} \mathrm{FPd}_R(K_{t-3}) \leq 2 + \max\{l, s\}, \\ \cdots \\ \mathrm{FPd}_R(K_0) \leq t - 1 + \max\{l, s\}, \\ m = \mathrm{FPd}(S\otimes_R M)_R \leq t + \max\{l, s\} \leq \mathrm{FPD}(S) + \max\{l, s\}. \end{gathered} \tag{5.16}$$

Therefore $\mathrm{FPD}(R) \leq \mathrm{FPD}(S) + \max\{l, s\}$. □

Note that if $S \geq R$ is an almost excellent extension, then $\mathrm{pd}(S_R) = 0$, and

$$s = \sup\{\mathrm{FPd}(A_R) \mid A \in \text{Mod-}S \text{ and } \mathrm{FPd}(A_S) = 0\} = 0. \tag{5.17}$$

thus

Corollary 5.6. *Assume that $S \geq R$ is an almost excellent extension. Then* $\mathrm{FPD}(R) = \mathrm{FPD}(S)$.

Proof. Suppose that $S \geq R$ is an almost excellent extension. Then S_R is a finitely generated projective R-module and S is right R-projective. By Theorem 5.5 and Proposition 5.2, we have $\mathrm{FPD}(R) = \mathrm{FPD}(S)$. □

To close this section, we give an example of an excellent extension $S \geq R$, which is provided by Xue in [16]. Let R be a ring graded by a finite group G. The smash product $R\#G$ is a free right and left R-module with a basis $\{p_g \mid g \in G\}$ and the multiplication determined by $(rp_g)(r'p_h) = rr'_{gh^{-1}}ph$, where $g, h \in G, r, r' \in R$, and $r'_{gh^{-1}}$ is the gh^{-1}-component of r'.

Example 5.7. Let R be a ring graded by a finite group G with $|G|^{-1} \in R$. Then

$$\mathrm{FPD}(R) = \mathrm{FPD}(R\#G). \tag{5.18}$$

Proof. By [17, Theorem 1.3], we know that G acts as automorphisms on $R\#G$ and the skew group ring $(R\#G) * G \cong R_n$ where $n = |G|$. Since skew group rings and finite matrix rings are excellent extensions, the result follows. □

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