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Research Article

Eigenfunctions and Fundamental Solutions of the Fractional Two-Parameter Laplacian

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We deal with the following fractional generalization of the Laplace equation for rectangular domains $(x, y) \in (x_0, X_0) \times (y_0, Y_0) \subset \mathbb{R}_+ \times \mathbb{R}_+$, which is associated with the Riemann-Liouville fractional derivatives $\Delta^{\alpha,\beta} u(x, y) = \lambda u(x, y)$, $\Delta^{\alpha,\beta} := D^{1+\alpha}_{x_0+} + D^{1+\beta}_{y_0+}$, where $\lambda \in \mathbb{C}$, $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Reducing the left-hand side of this equation to the sum of fractional integrals by x and y, we then use the operational technique for the conventional right-sided Laplace transformation and its extension to generalized functions to describe a complete family of eigenfunctions and fundamental solutions of the operator $\Delta^{\alpha,\beta}$ in classes of functions represented by the left-sided fractional integral of a summable function or just admitting a summable fractional derivative. A symbolic operational form of the solutions in terms of the Mittag-Leffler functions is exhibited. The case of the separation of variables is also considered. An analog of the fractional logarithmic solution is presented. Classical particular cases of solutions are demonstrated.

1. Introduction

Let $D^{\gamma}_{a+} f$ and $I^{\gamma}_{a+} f$ be the Riemann-Liouville fractional derivative and integral of order $\gamma > 0$ defined by [1, 2]

$$\left(D^{\gamma}_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^{n} \frac{1}{\Gamma(n-\gamma)} \int_a^x \frac{f(t)}{(x-t)^{\gamma-n+1}} dt, \quad a, x > 0, \; n = [\gamma] + 1, \tag{1.1}$$

$$\left(I^{\gamma}_{a+} f\right)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(t)}{(x-t)^{1-\gamma}} dt, \quad a, x > 0, \tag{1.2}$$

where [ ] means the integer part of γ. Consider a class of the linear nonhomogeneous differential equations:

$$\left(D^{1+\alpha}_{x_0+} u\right)(x, y) + \left(D^{1+\beta}_{y_0+} u\right)(x, y) - \lambda u(x, y) = f(x, y), \tag{1.3}$$

where $\lambda \in \mathbb{C}$, $(\alpha, \beta) \in [0,1] \times [0,1]$, $f(x,y)$, $0 \le x_0 \le x \le X_0 < \infty$, $0 \le y_0 \le y \le Y_0 < \infty$ is a prescribed function, and $u(x,y)$ is to be determined. Denoting by

$$\Delta^{\alpha,\beta} := D_{x_0+}^{1+\alpha} + D_{y_0+}^{1+\beta}, \tag{1.4}$$

equation (1.3) can be written in the form $(\Delta^{\alpha,\beta} - \lambda E)u = f$, where E is the identity operator. When $\alpha = \beta = 1$, we come out with the classical Poisson equation. Therefore we call fractional partial differential equation (1.3) *the fractional two-parameter Poisson equation (FPE)*. Its homogeneous analog is naturally called *the fractional Laplace equation (FLE)* or fractional two-parameter Laplacian.

In this paper we present a general operational approach [3] to describe eigenfunctions and fundamental solutions of the fractional two-parameter Laplacian based on the conventional right-sided Laplace transform [4]

$$F(s) = \int_{T_f}^{\infty} f(t) e^{-st} dt, \quad T_f \ge 0, \ \operatorname{Re} s > a_0 \tag{1.5}$$

of absolutely integrable functions $f \in L_1((T_f, \infty); e^{-a_0 t} dt)$ with respect to the measure $e^{-a_0 t} dt$ and its distributional analog

$$F(s) = \langle f(t), e^{-st} \rangle \tag{1.6}$$

in Zemanian's space $\mathcal{L}'(a_0)$ defined below. Operational solutions will be written in terms of the generalized Mittag-Leffler function $E_{\mu,\nu}(z)$, [1, 2, 5] which is defined in terms of the power series:

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \quad \mu > 0, \ \nu \in \mathbb{R}, \ z \in \mathbb{C}. \tag{1.7}$$

In particular, the function $E_{\mu,\nu}(z)$ is entire of the order $\rho = 1/\mu$ and type $\sigma = 1$. The exponential function and trigonometric and hyperbolic functions are expressed through (1.7) as follows:

$$\begin{gathered} E_{1,1}(z) = e^z, \qquad E_{2,1}\left(-z^2\right) = \cos z, \qquad E_{2,1}(z) = \cosh z, \\ z E_{2,2}\left(-z^2\right) = \sin z, \qquad z E_{2,2}\left(z^2\right) = \sinh z. \end{gathered} \tag{1.8}$$

We will consider in the sequel the existence and uniqueness of general solutions of the fractional Laplacian and its particular cases. Possible applications and an investigation of the fractional two-parameter Poisson equation (1.3) are still out of the framework of this paper and will be done in forthcoming articles of the author.

2. Eigenfunctions and Fundamental Solutions of the Fractional Laplace Equation

We begin with the following.

Definition 2.1 (see [1]). *By $AC^n([a,b])$, $n \in \mathbb{N}$, one denotes the class of functions $f(x)$, which are continuously differentiable on the segment $[a,b]$ up to the order $n-1$ and $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$.*

It is known [1] that the class $AC^n([a,b])$ contains only functions represented in the form

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{2.1}$$

where $\varphi(t) \in L_1(a,b)$ and c_k are arbitrary constants. It is not difficult to find that $\varphi(t) = f^{(n)}(t)$, $c_k = f^{(k)}(a)/k!$, Moreover, if $f(x) \in AC^n([a,b])$, then fractional derivative (1.1) exists almost everywhere and can be represented by the formula

$$\left(D^{\gamma}_{a+}f\right)(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} + \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt, \quad n = [\gamma]+1. \tag{2.2}$$

Definition 2.2 (see [1]). *By $I^{\gamma}_{a+}(L_1)$ denotes the class of functions f represented by the left-sided fractional integral* (1.2) *of a summable function, that is, $f = I^{\gamma}_{a+}\varphi$, $\varphi \in L_1(a,b)$.*

A description of this class is given by the following.

Theorem 2.3 (see [1]). *A function $f(x) \in I^{\gamma}_{a+}(L_1)$, $\gamma > 0$ if and only if $(I^{n-\gamma}_{a+}f)(x) \in AC^n([a,b])$, $n = [\gamma]+1$ and $(I^{n-\gamma}_{a+}f)^{(k)}(a) = 0$, $k = 0,1,\ldots,n-1$.*

Definition 2.4 (see [1]). *One will say that a function $f \in L_1(a,b)$ has a summable fractional derivative $(D^{\gamma}_{a+}f)(x)$ if $(I^{n-\gamma}_{a+}f)(x) \in AC^n([a,b])$, $n = [\gamma]+1$.*

If $(D^{\gamma}_{a+}f)(x) = (d/dx)^n(I^{n-\gamma}_{a+}f)(x)$ exists in the ordinary sense, that is, $(I^{n-\gamma}_{a+}f)(x)$ is differentiable in each point up to the order n, then $f(x)$ evidently admits the derivative $(D^{\gamma}_{a+}f)(x)$ in the sense of Definition 2.4.

So, if $f(x) \in I^{\gamma}_{a+}(L_1)$, then $(I^{\gamma}_{a+}D^{\gamma}_{a+}f)(x) = f(x)$. Otherwise if f just admits a summable fractional derivative, then the composition of fractional operators (1.1) and (1.2) can be written in the form (see [1])

$$\left(I^{\gamma}_{a+}D^{\gamma}_{a+}f\right)(x) = f(x) - \sum_{k=0}^{n-1}\frac{(x-a)^{\gamma-k-1}}{\Gamma(\gamma-k)}\left(I^{n-\gamma}_{a+}f\right)^{(n-k-1)}(a), \quad n = [\gamma]+1. \tag{2.3}$$

Nevertheless we note that $(D^{\gamma}_{a+}I^{\gamma}_{a+}f)(x) = f(x)$ for any summable function f.

Consider now the eigenfunction problem for the fractional Laplace equation in the rectangular domain $(x, y) \in (x_0, X_0) \times (y_0, Y_0)$

$$\left(D_{x_0+}^{1+\alpha}u\right)(x,y) + \left(D_{y_0+}^{1+\beta}u\right)(x,y) = \lambda u(x,y), \tag{2.4}$$

where $\lambda \in \mathbb{C}$, $(\alpha, \beta) \in (0,1] \times (0,1]$ in the following three cases:

(i) $u(x, y)$ belongs to classes $I_{x_0+}^{1+\alpha}(L_1)$, $I_{y_0+}^{1+\beta}(L_1)$ by $x \in [x_0, X_0]$ and $y \in [y_0, Y_0]$, respectively;

(ii) $u(x, y)$ admits a summable fractional derivative $(D_{x_0+}^{1+\alpha}u)(x, y)$ by $x \in [x_0, X_0]$ and belongs to $I_{y_0+}^{1+\beta}(L_1)$ by $y \in [y_0, Y_0]$ or vice versa;

(iii) $u(x, y)$ admits summable fractional derivative $(D_{x_0+}^{1+\alpha}u)(x, y)$, $(D_{y_0+}^{1+\beta}u)(x, y)$ by $x \in [x_0, X_0]$ and $y \in [y_0, Y_0]$, respectively.

Theorem 2.5. *In case (i) trivial solution of* (2.4) *is the only solution.*

Proof. Indeed, taking the operator $I_{x_0+}^{1+\alpha}$ from both sides of (2.4) and using the identity $(I_{x_0+}^{1+\alpha}D_{x_0+}^{1+\alpha}u)(x, y) = u(x, y)$ it becomes

$$u(x,y) + \left(I_{x_0+}^{1+\alpha}D_{y_0+}^{1+\beta}u\right)(x,y) - \lambda\left(I_{x_0+}^{1+\alpha}u\right)(x,y) = 0. \tag{2.5}$$

Hence, applying the operator $I_{y_0+}^{1+\beta}$ to both sides of (2.5), we use the fact that due Fubini's theorem this operator commutes with $I_{x_0+}^{1+\alpha}$. Then we obtain

$$\left(I_{x_0+}^{1+\alpha}u\right)(x,y) + \left(I_{y_0+}^{1+\beta}u\right)(x,y) - \lambda\left(I_{x_0+}^{1+\alpha}I_{y_0+}^{1+\beta}u\right)(x,y) = 0. \tag{2.6}$$

Hence from conditions of the theorem we observe that fractional integrals of the equation (2.6) are Laplace-transformable functions. Therefore we may act on (2.6) by the conventional right-sided Laplace transform (1.5), let say, by y with $T_f = y_0$. Taking into account its convolution and operational properties [3] after straightforward calculations we arrive at the following second kind homogeneous integral equation of the Volterra type:

$$U(x,s) + \frac{(s^{1+\beta} - \lambda)}{\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{\alpha}U(t,s)dt = 0, \tag{2.7}$$

where $\lambda \in \mathbb{C}$, $s \in \mathbb{C}$, $\operatorname{Re} s \geq a_0 > 0$, $s^{1+\beta} = |s|^{1+\beta}e^{i\theta}$, $\theta = \arg s \in (-\pi/2, \pi/2)$ and

$$U(x,s) = \int_{y_0}^{Y_0} e^{-st}u(x,t)dt. \tag{2.8}$$

Appealing to [5, Chapter 3] we find that (2.7) has the only trivial solution in the space of summable functions and $U(x, s) \in L_1(x_0, X_0)$ because $U(x, s) \in I_{x_0+}^{1+\alpha}(L_1)$ for each s.

Cancelling the Laplace transform and using its uniqueness property for summable functions we get $u(x,y)=0$. Theorem 2.5 is proved.

In case (ii), (2.6) should be substituted by the following equalities (see (2.3)):

$$\begin{aligned}&\left(I_{x_0+}^{1+\alpha}u\right)(x,y)+\left(I_{y_0+}^{1+\beta}u\right)(x,y)-\lambda\left(I_{x_0+}^{1+\alpha}I_{y_0+}^{1+\beta}u\right)(x,y)\\&\quad=\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\left(I_{y_0+}^{1+\beta}f_0\right)(y)+\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\left(I_{y_0+}^{1+\beta}f_1\right)(y),\end{aligned}\tag{2.9}$$

or

$$\begin{aligned}&\left(I_{x_0+}^{1+\alpha}u\right)(x,y)+\left(I_{y_0+}^{1+\beta}u\right)(x,y)-\lambda\left(I_{x_0+}^{1+\alpha}I_{y_0+}^{1+\beta}u\right)(x,y)\\&\quad=\frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)}\left(I_{x_0+}^{1+\alpha}h_0\right)(x)+\frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}\left(I_{x_0+}^{1+\alpha}h_1\right)(x),\end{aligned}\tag{2.10}$$

where we denoted by

$$f_0(y)=\left(I_{x_0+}^{1-\alpha}u\right)(x_0,y),\tag{2.11}$$

$$f_1(y)=\left(D_{x_0+}^{\alpha}u\right)(x_0,y),\tag{2.12}$$

$$h_0(x)=\left(I_{y_0+}^{1-\beta}u\right)(x,y_0),\tag{2.13}$$

$$h_1(x)=\left(D_{y_0+}^{\beta}u\right)(x,y_0)\tag{2.14}$$

Cauchy's fractional initial conditions. Treating, for instance, (2.9) we take the Laplace transform from its both sides and arrive at the following integral equation:

$$U(x,s)+\frac{\left(s^{1+\beta}-\lambda\right)}{\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{\alpha}U(t,s)dt=F(x,s),\quad x\in(x_0,X_0),\ \lambda\in\mathbb{C},\tag{2.15}$$

where

$$F(x,s)=\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}F_0(s)+\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}F_1(s),\tag{2.16}$$

$$F_i(s)=\int_{y_0}^{Y_0}e^{-st}f_i(t)dt,\quad i=0,1.\tag{2.17}$$

It is known [5] that a unique solution of (2.15) in the class of summable functions is

$$U(x,s)=F(x,s)-\left(s^{1+\beta}-\lambda\right)\int_{x_0}^{x}(x-t)^{\alpha}E_{1+\alpha,1+\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-t)^{1+\alpha}\right)F(t,s)dt,\tag{2.18}$$

which involves as the kernel the generalized Mittag-Leffler function (1.7). Next, substituting (2.16) into (2.15), using (1.7), (2.17), index laws for fractional operators [1], and the estimates

$$\begin{aligned}
&\left|\left(s^{1+\beta}-\lambda\right)\int_{x_0}^{x}(x-t)^{\alpha}E_{1+\alpha,1+\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-t)^{1+\alpha}\right)F(t,s)dt\right| \\
&\quad\le |F_0(s)|\sum_{n=0}^{\infty}\frac{\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}(t-x_0)^{\alpha-1}dt \\
&\qquad+|F_1(s)|\sum_{n=0}^{\infty}\frac{\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}(t-x_0)^{\alpha}dt \\
&\quad\le\left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{2\alpha} \\
&\qquad\times\left[E_{1+\alpha,2\alpha+1}\left(\left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{1+\alpha}\right)\int_{y_0}^{Y_0}e^{-a_0t}\left|f_0(t)\right|dt\right. \\
&\qquad\left.+(X_0-x_0)E_{1+\alpha,2(\alpha+1)}\left(\left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{1+\alpha}\right)\int_{y_0}^{Y_0}e^{-a_0t}\left|f_1(t)\right|dt\right],
\end{aligned} \tag{2.19}$$

we write solution (2.18) of the Volterra type equation (2.15) in terms of the Mittag-Leffler functions:

$$\begin{aligned}
U(x,s)&=(x-x_0)^{\alpha-1}E_{1+\alpha,\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-x_0)^{1+\alpha}\right)F_0(s) \\
&\quad+(x-x_0)^{\alpha}E_{1+\alpha,1+\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-x_0)^{1+\alpha}\right)F_1(s).
\end{aligned} \tag{2.20}$$

In order to cancel the Laplace transformation by s in (2.20) we will appeal to its distributional form (1.6) in Zemanian's space $\mathcal{L}'(0)$ (see [4]), which is dual of the countable-union space of test functions $\mathcal{L}(0)$ defined by

$$\mathcal{L}(0)=\bigcup_{\nu=1}^{\infty}\mathcal{L}_{a_\nu}, \tag{2.21}$$

where $\{a_\nu\}_{\nu=1}^{\infty}$ is a sequence of real numbers $a_\nu>0$, which converges monotonically to $0+$ as $\nu\to\infty$ and each $\mathcal{L}_{a_\nu}$ is a testing-function space of smooth functions $\varphi(y), y\in(y_0,\infty)$ and for each nonnegative integer k it satisfies

$$\rho_{k,\nu}(\varphi)=\sup_{y\in(y_0,\infty)}e^{a_\nu y}\left|\varphi^{(k)}(y)\right|<\infty,\quad k\in\mathbb{N}_0. \tag{2.22}$$

According to [4, Chapter III] we assign $\mathcal{L}_{a_\nu}$ a topology generated by the multinorm (2.22). Consequently, $\mathcal{L}_{a_\nu}$ is a countably multinormed space and the kernel of the Laplace transform e^{-st} is a member of $\mathcal{L}_{a_\nu}$ if and only if $\operatorname{Re}s\ge a_\nu$. Taking the space $\mathcal{L}(0)$ we have an advantage that the space of smooth functions with compact support $\mathfrak{D}$ is dense in $\mathcal{L}(0)$ and the members of the dual $\mathcal{L}'(0)$ are distributions. Moreover, any $f \in \mathcal{L}'(0)$ is a right-sided Laplace-transformable generalized function via the formula (1.6) with the right half-plane $\operatorname{Re} s > 0$ as a region of definition. Meanwhile, any analytic function $F(s)$ on the half-plane $\operatorname{Re} s \geq a > 0$, which satisfies the estimate

$$|F(s)| \leq e^{-T \operatorname{Re} s} P(|s|), \quad \operatorname{Re} s \geq a, \tag{2.23}$$

where $P(z)$ is a polynomial, may be identified as the Laplace transform (1.6) of a right-sided Laplace-transformable generalized function which is concentrated on $T \leq t < \infty$. Finally, the uniqueness and inversion properties are true and the inversion formula has the form

$$f(t) = \lim_{r \to \infty} \int_{\sigma - ir}^{\sigma + ir} F(s) e^{st} ds, \tag{2.24}$$

in the sense of convergence in $\mathfrak{D}'$ for any $\sigma > 0$.

So in order to find eigenfunctions and general fundamental solutions of the fractional Laplace equation (2.4) we will invert the Laplace transform in (2.20) by using formula (2.24). Of course, we understand that the conventional right-sided Laplace transform (1.5) is a particular case of (1.6) being applied to a regular generalized function $f \in L_1((T_f, \infty); e^{-a_0 t} dt)$, $a_0 > 0$.

Further, we have

$$\begin{aligned} \lim_{r \to \infty} \int_{\sigma - ir}^{\sigma + ir} s^{n(1+\beta)} F_i(s) e^{sy} ds &= \lim_{r \to \infty} \left(\frac{d}{dy} \right)^{[n(1+\beta)]+1} \int_{\sigma - ir}^{\sigma + ir} s^{\{n(1+\beta)\}-1} F_i(s) e^{sy} ds \\ &= \left(D_{y_0}^{n(1+\beta)} f_i \right)(y), \quad i = 0, 1, \; n \in \mathbb{N}_0, \end{aligned} \tag{2.25}$$

where $\{ \}$ is a fractional part of the number, the convergence is in $\mathfrak{D}'$, and we assume that $(I_{y_0}^{1-\{n(1+\beta)\}} f_i)(y) \in AC^{[n(1+\beta)]+1}[y_0, Y_0]$ for any $n \in \mathbb{N}_0$. Therefore, canceling the Laplace transformation in (2.20) and taking into account (2.16) after straightforward calculations we get the expression for a family of eigenfunctions of (2.4):

$$\begin{aligned} u_\lambda(x, y) = &\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x - x_0)^{n(1+\alpha)+\alpha-1} \\ &\times {}_1\Psi_1 \left((1, n+1); (1+\alpha, n(1+\alpha)+\alpha); \lambda (x - x_0)^{1+\alpha} \right) \left(D_{y_0}^{n(1+\beta)} f_0 \right)(y) \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x - x_0)^{n(1+\alpha)+\alpha} {}_1\Psi_1 \left((1, n+1); (1+\alpha, (n+1)(1+\alpha)); \lambda (x - x_0)^{1+\alpha} \right) \\ &\times \left(D_{y_0}^{n(1+\beta)} f_1 \right)(y), \end{aligned} \tag{2.26}$$

where ${}_1\Psi_1((a_1, \mathrm{b}_1); (c_1, d_1); z)$ is the generalized Wright function [2]:

$$ {}_1\Psi_1((a_1, b_1); (c_1, d_1); z) = \sum_{m=0}^{\infty} = \frac{\Gamma(a_1 m + b_1)}{\Gamma(c_1 m + d_1)} \frac{z^m}{m!}, \tag{2.27}$$

and the convergence of series in (2.26) is in $\mathfrak{D}'$. Letting in (2.26) $\lambda = 0$ we immediately come out with a classical fundamental solution of (2.4):

$$\begin{aligned} u_f(x, y) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x - x_0)^{n(1+\alpha)+\alpha-1}}{\Gamma(n(1+\alpha)+\alpha)} \left(D_{y_0}^{n(1+\beta)} f_0\right)(y) \\ &\quad + \sum_{n=0}^{\infty} (-1)^n \frac{(x - x_0)^{n(1+\alpha)+\alpha}}{\Gamma((n+1)(1+\alpha))} \left(D_{y_0}^{n(1+\beta)} f_1\right)(y). \end{aligned} \tag{2.28}$$

Taking into account definition (1.7) of the Mittag-Leffler function, solution (2.28) may be written in the operational form

$$\begin{aligned} u_f(x, y) &= (x - x_0)^{\alpha-1} E_{1+\alpha,\alpha}\left(-(x - x_0)^{1+\alpha} D_{y_0}^{1+\beta}\right) f_0(y) \\ &\quad + (x - x_0)^{\alpha} E_{1+\alpha,1+\alpha}\left(-(x - x_0)^{1+\alpha} D_{y_0}^{1+\beta}\right) f_1(y). \end{aligned} \tag{2.29}$$

Analogously, in the case of (2.10) we show that

$$\begin{aligned} u_\lambda(x, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (y - y_0)^{n(1+\beta)+\beta-1} \\ &\quad \times {}_1\Psi_1\left((1, n+1); (1+\beta, n(1+\beta)+\beta); \lambda(y - y_0)^{1+\beta}\right) \left(D_{x_0}^{n(1+\alpha)} h_0\right)(x) \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (y - y_0)^{n(1+\beta)+\beta} {}_1\Psi_1\left((1, n+1); (1+\beta, (n+1)(1+\beta)); \lambda(y - y_0)^{1+\beta}\right) \\ &\quad \times \left(D_{h_0}^{n(1+\alpha)} h_1\right)(x), \end{aligned} \tag{2.30}$$

$$\begin{aligned} u_h(x, y) &= (y - y_0)^{\beta-1} E_{1+\beta,\beta}\left(-(y - y_0)^{1+\beta} D_{x_0}^{1+\alpha}\right) h_0(x) \\ &\quad + (y - y_0)^{\beta} E_{1+\beta,1+\beta}\left(-(y - y_0)^{1+\beta} D_{x_0}^{1+\alpha}\right) h_1(x) \end{aligned} \tag{2.31}$$

are also correspondingly eigenfunctions and a fundamental solution of (2.4).

On the other hand we may write solutions (2.29) and (2.31) in the form of the generalized Neumann series. Namely, we find

$$u_f(x,y) = \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}(-1)^n I_{x_0}^{n(1+\alpha)}(x-x_0)^{\alpha-1}\left(D_{y_0}^{n(1+\beta)} f_0\right)(y) + \frac{1}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}(-1)^n I_{x_0}^{n(1+\alpha)}(x-x_0)^{\alpha}\left(D_{y_0}^{n(1+\beta)} f_1\right)(y), \tag{2.32}$$

taking in mind the analyticity of series in (2.32) by x in the interval $0 < \varepsilon \le |x - x_0| < R$ and by $y \in (y_0, Y_0)$. In the same manner, we represent (2.31) by the expression

$$u_h(x,y) = \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}(-1)^n I_{y_0}^{n(1+\beta)}(y-y_0)^{\beta-1}\left(D_{x_0}^{n(1+\alpha)} h_0\right)(x) + \frac{1}{\Gamma(1+\beta)}\sum_{n=0}^{\infty}(-1)^n I_{y_0}^{n(1+\beta)}(y-y_0)^{\beta}\left(D_{x_0}^{n(1+\alpha)} h_1\right)(x) \tag{2.33}$$

with arbitrary h_0, h_1 assuming analyticity of the corresponding series by y in the interval $0 < \varepsilon \le |y - y_0| < R$ and by $x \in (x_0, X_0)$. Now taking into account zero values $D_{x_0+}^{1+\alpha}(x-x_0)^{\alpha-1} = 0$, $D_{x_0+}^{1+\alpha}(x-x_0)^{\alpha} = 0$, $D_{y_0+}^{1+\beta}(y-y_0)^{\beta-1} = 0$, $D_{y_0+}^{1+\beta}(y-y_0)^{\alpha} = 0$ it is not difficult to verify that (2.32) and (2.33) are classical fundamental solutions of the fractional Laplacian $\Delta^{\alpha,\beta}u = 0$ subject to conditions (2.11), (2.12), (2.13), and (2.14) respectively. Thus we have proved □

Theorem 2.6. *In case (ii) functions* (2.26) *and* (2.30) *represent eigenfunctions of the fractional Laplacian* (2.4) *and expressions* (2.28) *and* (2.31) *are unique classical fundamental solutions subject to conditions* (2.11), (2.12), (2.13), *and* (2.14) *respectively. These solutions can be written in the corresponding form of generalized Neumann series* (2.32) *and* (2.33) *under additional conditions of analyticity.*

Finally, in case (iii) an analog of (2.9) and (2.10) is

$$\left(I_{x_0+}^{1+\alpha}u\right)(x,y) + \left(I_{y_0+}^{1+\beta}u\right)(x,y) - \lambda\left(I_{x_0+}^{1+\alpha}I_{y_0+}^{1+\beta}u\right)(x,y) = \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\left(I_{y_0+}^{1+\beta}f_0\right)(y) + \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\left(I_{y_0+}^{1+\beta}f_1\right)(y) + \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)}\left(I_{x_0+}^{1+\alpha}h_0\right)(x) + \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}\left(I_{x_0+}^{1+\alpha}h_1\right)(x). \tag{2.34}$$

Consequently, in the right-hand side of Volterra's equation (2.15) we get an additional term

$$U(x,s) + \frac{\left(s^{1+\beta}-\lambda\right)}{\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{\alpha}U(t,s)dt = F(x,s) + e^{-y_0 s}\left(I_{x_0+}^{1+\alpha}(sh_0+h_1)\right)(x), \tag{2.35}$$

which will give a source for generalized eigenfunctions and fundamental solutions of the fractional Laplacian (2.4). In fact, owing to the estimates

$$\begin{aligned}
&\left|\left(s^{1+\beta}-\lambda\right)\int_{x_0}^{x}(x-t)^{\alpha}E_{1+\alpha,1+\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-t)^{1+\alpha}\right)F(t,s)dt\right| \\
&\quad\le |F_0(s)|\sum_{n=0}^{\infty}\frac{\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}(t-x_0)^{\alpha-1}dt \\
&\quad\quad+|F_1(s)|\sum_{n=0}^{\infty}\frac{\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}(t-x_0)^{\alpha}dt \\
&\quad\quad+\sum_{n=0}^{\infty}\frac{|s|\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}\int_{x_0}^{t}(t-v)^{\alpha}|h_0(v)|dv\,dt \\
&\quad\quad+\sum_{n=0}^{\infty}\frac{\left(|s|^{1+\beta}+|\lambda|\right)^{n+1}}{\Gamma((n+1)(1+\alpha))\Gamma(1+\alpha)}\int_{x_0}^{x}(x-t)^{n(1+\alpha)+\alpha}\int_{x_0}^{t}(t-v)^{\alpha}|h_1(v)|dv\,dt \\
&\quad\le \left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{2\alpha}\Bigg[E_{1+\alpha,2\alpha+1}\left(\left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{1+\alpha}\right)\int_{y_0}^{Y_0}e^{-a_0t}\left|f_0(t)\right|dt \\
&\quad\quad+(X_0-x_0)E_{1+\alpha,2(\alpha+1)}\left(\left(|s|^{1+\beta}+|\lambda|\right)(X_0-x_0)^{1+\alpha}\right) \\
&\quad\quad\times\left[\int_{y_0}^{Y_0}e^{-a_0t}\left|f_1(t)\right|dt+|s|\int_{x_0}^{X_0}|h_0(t)|dt+\int_{x_0}^{X_0}|h_1(t)|dt\right]\Bigg]<\infty,
\end{aligned} \tag{2.36}$$

we write solution of the Volterra type equation (2.35) in terms of the Mittag-Leffler functions and generalized Neumann series:

$$\begin{aligned}
U(x,s)&=(x-x_0)^{\alpha-1}E_{1+\alpha,\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-x_0)^{1+\alpha}\right)F_0(s) \\
&\quad+(x-x_0)^{\alpha}E_{1+\alpha,1+\alpha}\left(-\left(s^{1+\beta}-\lambda\right)(x-x_0)^{1+\alpha}\right)F_1(s) \\
&\quad+e^{-y_0s}\sum_{n=0}^{\infty}(-1)^n\left(s^{1+\beta}-\lambda\right)^n\left(I_{x_0}^{(n+1)(1+\alpha)}(sh_0+h_1)\right)(x).
\end{aligned} \tag{2.37}$$

Cancelling the Laplace transformation we take in mind the relations

$$\begin{aligned}
&\lim_{r\to\infty}\int_{\sigma-ir}^{\sigma+ir}s^{n(1+\beta)}e^{s(y-y_0)}ds \\
&\quad=\lim_{r\to\infty}\left(\frac{d}{dy}\right)^{[n(1+\beta)]+1}\int_{\sigma-ir}^{\sigma+ir}\left\langle\frac{t_+^{-\{n(1+\beta)\}}}{\Gamma(1-\{n(1+\beta)\})}*\delta(t-y_0),e^{-st}\right\rangle e^{sy}ds \\
&\quad=\left(\frac{d}{dy}\right)^{[n(1+\beta)]+1}\left(\frac{t_+^{-\{n(1+\beta)\}}}{\Gamma(1-\{n(1+\beta)\})}*\delta(t-y_0)\right)(y)=\left(D_{y_0}^{n(1+\beta)}\delta\right)(y),
\end{aligned} \tag{2.38}$$

where δ is Dirac's delta-function and $*$ denotes the convolution product. Therefore, after straightforward calculations we get the expression for a family of eigenfunctions of (2.4):

$$\begin{aligned}
u_\lambda(x,y) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-x_0)^{n(1+\alpha)+\alpha-1} \\
&\quad\times {}_1\Psi_1\Big((1,n+1);(1+\alpha,n(1+\alpha)+\alpha);\lambda(x-x_0)^{1+\alpha}\Big)\Big(D_{y_0}^{n(1+\beta)}f_0\Big)(y) \\
&\quad+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-x_0)^{n(1+\alpha)+\alpha}\,{}_1\Psi_1\Big((1,n+1);(1+\alpha,(n+1)(1+\alpha));\lambda(x-x_0)^{1+\alpha}\Big) \\
&\quad\times\Big(D_{y_0}^{n(1+\beta)}f_1\Big)(y)+\int_{x_0}^{x}h_1(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \\
&\quad\times(x-t)^{n(1+\alpha)+\alpha}\,{}_1\Psi_1\Big((1,n+1);(1+\alpha,(n+1)(1+\alpha));\lambda(x-t)^{1+\alpha}\Big) \\
&\quad\times\Big(D_{y_0}^{n(1+\beta)}\delta\Big)(y)\,dt+\int_{x_0}^{x}h_0(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-t)^{n(1+\alpha)+\alpha} \\
&\quad\times {}_1\Psi_1\Big((1,n+1);(1+\alpha,(n+1)(1+\alpha));\lambda(x-t)^{1+\alpha}\Big) \\
&\quad\times\Big(D_{y_0}^{n(1+\beta)}\delta'\Big)(y)\,dt,
\end{aligned} \tag{2.39}$$

where the convergence of series in (2.39) is in $\mathfrak{D}'$. Letting in (2.39) $\lambda = 0$ we derive a generalized fundamental solution of (2.4):

$$\begin{aligned}
u(x,y) &= \sum_{n=0}^{\infty}(-1)^n\frac{(x-x_0)^{n(1+\alpha)+\alpha-1}}{\Gamma(n(1+\alpha)+\alpha)}\Big(D_{y_0}^{n(1+\beta)}f_0\Big)(y) \\
&\quad+\sum_{n=0}^{\infty}(-1)^n\frac{(x-x_0)^{n(1+\alpha)+\alpha}}{\Gamma((n+1)(1+\alpha))}\Big(D_{y_0}^{n(1+\beta)}f_1\Big)(y) \\
&\quad+\int_{x_0}^{x}h_1(t)\sum_{n=0}^{\infty}(-1)^n\frac{(x-t)^{(n+1)(1+\alpha)-1}}{\Gamma((n+1)(1+\alpha))}\Big(D_{y_0}^{n(1+\beta)}\delta\Big)(y)\,dt \\
&\quad+\int_{x_0}^{x}h_0(t)\sum_{n=0}^{\infty}(-1)^n\frac{(x-t)^{(n+1)(1+\alpha)-1}}{\Gamma((n+1)(1+\alpha))}\Big(D_{y_0}^{n(1+\beta)}\delta'\Big)(y)\,dt,
\end{aligned} \tag{2.40}$$

which may be written in the operational form

$$\begin{aligned}
u(x,y) &= (x-x_0)^{\alpha-1}E_{1+\alpha,\alpha}\Big(-(x-x_0)^{1+\alpha}D_{y_0}^{1+\beta}\Big)f_0(y) \\
&\quad+(x-x_0)^{\alpha}E_{1+\alpha,1+\alpha}\Big(-(x-x_0)^{1+\alpha}D_{y_0}^{1+\beta}\Big)f_1(y) \\
&\quad+\int_{x_0}^{x}(x-t)^{\alpha}E_{1+\alpha,1+\alpha}\Big(-(x-t)^{1+\alpha}D_{y_0}^{1+\beta}\Big)\delta(y)h_1(t)\,dt \\
&\quad+\int_{x_0}^{x}(x-t)^{\alpha}E_{1+\alpha,1+\alpha}\Big(-(x-t)^{1+\alpha}D_{y_0}^{1+\beta}\Big)\delta'(y)h_0(t)\,dt.
\end{aligned} \tag{2.41}$$

Analogously, functions

$$
\begin{aligned}
u_\lambda(x,y) = {} & \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(y-y_0)^{n(1+\beta)+\beta-1} \\
& \times {}_1\Psi_1\Big((1,n+1);(1+\beta,n(1+\beta)+\beta);\lambda(y-y_0)^{1+\beta}\Big)\Big(D_{x_0}^{n(1+\alpha)}h_0\Big)(x) \\
& + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(y-y_0)^{n(1+\beta)+\beta}\,{}_1\Psi_1\Big((1,n+1);(1+\beta,(n+1)(1+\beta));\lambda(y-y_0)^{1+\beta}\Big) \\
& \times \Big(D_{h_0}^{n(1+\alpha)}h_1\Big)(x) + \int_{y_0}^{y} h_1(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \\
& \times (y-t)^{n(1+\beta)+\beta}\,{}_1\Psi_1\Big((1,n+1);(1+\beta,(n+1)(1+\beta));\lambda(y-t)^{1+\beta}\Big) \\
& \times \Big(D_{x_0}^{n(1+\alpha)}\delta\Big)(x)dt + \int_{y_0}^{y} h_0(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(y-t)^{n(1+\beta)+\beta} \\
& \times {}_1\Psi_1\Big((1,n+1);(1+\beta,(n+1)(1+\beta));\lambda(y-t)^{1+\beta}\Big) \\
& \times \Big(D_{x_0}^{n(1+\alpha)}\delta'\Big)(x)dt,
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
u(x,y) = {} & (y-y_0)^{\beta-1}E_{1+\beta,\beta}\Big(-(y-y_0)^{1+\beta}D_{x_0}^{1+\alpha}\Big)h_0(x) \\
& + (y-y_0)^{\beta}E_{1+\beta,1+\beta}\Big(-(y-y_0)^{1+\beta}D_{x_0}^{1+\alpha}\Big)h_1(x) \\
& + \int_{y_0}^{y}(y-t)^{\beta}E_{1+\beta,1+\beta}\Big(-(y-t)^{1+\beta}D_{x_0}^{1+\alpha}\Big)\delta(x)f_1(t)dt \\
& + \int_{y_0}^{y}(y-t)^{\beta}E_{1+\beta,1+\beta}\Big(-(y-t)^{1+\beta}D_{x_0}^{1+\alpha}\Big)\delta'(x)f_0(t)dt
\end{aligned}
\tag{2.43}
$$

are also correspondingly eigenfunctions and generalized fundamental solutions of (2.4).

Theorem 2.7. *In case (iii) functions* (2.39) *and* (2.42) *represent eigenfunctions and expressions* (2.40), (2.41), *and* (2.43) *are generalized fundamental solutions of the fractional Laplacian* (2.4).

Example 2.8. As a particular case, it is not difficult to obtain from (2.28), (2.31) the classical fundamental solution $u(x,y) = (1/2)\log((x-x_0)^2 + (y-y_0)^2)$ of the Laplace equation $\Delta u = 0$ ($\alpha = \beta = 1$). Indeed, putting $f_1(y) = h_1(x) = 0$, $f_0(y) = \log(y-y_0)$, $h_0(x) = \log(x-x_0)$ we assume, correspondingly, $(x-x_0)/(y-y_0) < 1$, $(x-x_0)/(y-y_0) > 1$ in (2.28), (2.31), and for instance, solution $u_f(x,y)$ becomes

$$\begin{aligned} u_f(x,y) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-x_0)^{2n}}{(2n)!} \left(\log(y-y_0)\right)^{(2n)} \\ &= \log(y-y_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \left(\frac{x-x_0}{y-y_0}\right)^{2n} \\ &= \log(y-y_0) + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+1)} \left(\frac{x-x_0}{y-y_0}\right)^{2(n+1)} \\ &= \log(y-y_0) + \frac{1}{2}\log\left(1+\left(\frac{x-x_0}{y-y_0}\right)^2\right) \\ &= \frac{1}{2}\log\left((x-x_0)^2+(y-y_0)^2\right). \end{aligned} \tag{2.44}$$

Analogously we treat solution (2.31).

3. Separation of Variables: Analytic Solutions

The method of separation of variables allows us to simplify eigenfunctions and fundamental solutions of the fractional Laplacian. Indeed, putting $u(x,y) = u_1(x)u_2(y)$, substituting in (2.34), and taking into account initial conditions (2.11), (2.12), (2.13), and (2.14) it becomes

$$\begin{aligned} u_2(y)\left(I_{x_0+}^{1+\alpha}u_1\right)(x) + u_1(x)\left(I_{y_0+}^{1+\beta}u_2\right)(y) - \lambda\left(I_{x_0+}^{1+\alpha}u_1\right)(x)\left(I_{y_0+}^{1+\beta}u_2\right)(y) \\ = a_1\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\left(I_{y_0+}^{1+\beta}u_2\right)(y) + a_2\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\left(I_{y_0+}^{1+\beta}u_2\right)(y) \\ + b_1\frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)}\left(I_{x_0+}^{1+\alpha}u_1\right)(x) + b_2\frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}\left(I_{x_0+}^{1+\alpha}u_1\right)(x), \end{aligned} \tag{3.1}$$

where $a_i, b_i \in \mathbb{C}$, $i = 1,2$ are arbitrary constants. If $(I_{x_0+}^{1+\alpha}u_1)(x)(I_{y_0+}^{1+\beta}u_2)(y) \neq 0$, $(x,y) \in (x_0, X_0) \times (y_0, Y_0)$ we divide (3.1) by this product and separate variables, getting two Abel's type second kind integral equations to define u_i, $i = 1,2$, namely,

$$\begin{aligned} u_1(x) + \mu\left(I_{x_0+}^{1+\alpha}u_1\right)(x) &= a_1\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} + a_2\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}, \\ u_2(y) - (\lambda+\mu)\left(I_{y_0+}^{1+\beta}u_2\right)(y) &= b_1\frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} + b_2\frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}, \end{aligned} \tag{3.2}$$

where $\lambda, \mu \in \mathbb{C}$ are constants. We note that the equality $(I_{x_0+}^{1+\alpha}u_1)(x)(I_{y_0+}^{1+\beta}u_2)(y) = 0$ for at least one point (ξ, η) agrees with (3.1) and (3.2). So we solve the latter equations similarly to (2.18), arriving at the following family of eigenfunctions $u_{\lambda,\mu}(x,y) = u_1(x)u_2(y)$, where

$$\begin{aligned}
u_1(x) &= a_1(x-x_0)^{\alpha-1}E_{1+\alpha,\alpha}\left(\mu(x-x_0)^{1+\alpha}\right)\\
&\quad + a_2(x-x_0)^{\alpha}E_{1+\alpha,1+\alpha}\left(\mu(x-x_0)^{1+\alpha}\right),\\
u_2(y) &= b_1(y-y_0)^{\beta-1}E_{1+\beta,\beta}\left((\lambda-\mu)(y-y_0)^{1+\beta}\right)\\
&\quad + b_2(y-y_0)^{\beta}E_{1+\beta,1+\beta}\left((\lambda-\mu)(y-y_0)^{1+\beta}\right).
\end{aligned} \tag{3.3}$$

On the other hand, we may write these solutions in terms of the generalized Neumann series. Precisely, denoting by

$$\begin{aligned}
v_\alpha(x) &= a_1\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} + a_2\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)},\\
v_\beta(y) &= b_1\frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} + b_2\frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)},
\end{aligned} \tag{3.4}$$

and recalling index properties for the fractional integral (1.2) we get representations of (3.3) in the respective resolvent form for fractional integral operators $I_{x_0}^{1+\alpha} : L_1(x_0, X_0) \to L_1(x_0, X_0)$, $I_{y_0}^{1+\beta} : L_1(y_0, Y_0) \to L_1(y_0, Y_0)$:

$$\begin{aligned}
u_1(x) &= \sum_{n=0}^{\infty} I_{x_0}^{n(1+\alpha)} v_\alpha \mu^n = \left(E - \mu I_{x_0}^{1+\alpha}\right)^{-1} v_\alpha = \mathcal{R}\left(\mu; I_{x_0}^{1+\alpha}\right) v_\alpha,\\
u_2(y) &= \sum_{n=0}^{\infty} I_{y_0}^{n(1+\beta)} v_\beta (\lambda-\mu)^n = \left(E - (\lambda-\mu) I_{y_0}^{1+\beta}\right)^{-1} v_\beta\\
&= \mathcal{R}\left(\lambda-\mu; I_{y_0}^{1+\beta}\right) v_\beta,
\end{aligned} \tag{3.5}$$

where E as usual is the identity operator. It is easily seen that series (3.5) are analytic with respect to $x \in (x_0, X_0)$ and $y \in (y_0, Y_0)$. Further, since (see (1.2))

$$\left\| I_{x_0}^{1+\alpha} f \right\|_{L_1(x_0,X_0)} \le \frac{(X_0-x_0)^{\alpha+1}}{\Gamma(1+\alpha)} \|f\|_{L_1(x_0,X_0)}, \tag{3.6}$$

we have

$$\left\| \mathcal{R}\left(z; I_{x_0}^{1+\alpha}\right) \right\| \le \sum_{n=0}^{\infty} \left\| I_{x_0}^{1+\alpha} \right\|^n |z|^n \le \sum_{n=0}^{\infty} \left(\frac{(X_0-x_0)^{\alpha+1}}{\Gamma(1+\alpha)} \right)^n |z|^n. \tag{3.7}$$

Therefore the resolvent functions $\mathcal{R}(z; I_{x_0}^{1+\alpha})$, $\mathcal{R}(z; I_{y_0}^{1+\beta})$ are analytic in the open discs:

$$|z| < \frac{\Gamma(1+\alpha)}{(X_0 - x_0)^{\alpha+1}}, \qquad |z| < \frac{\Gamma(1+\beta)}{(Y_0 - y_0)^{\beta+1}}, \tag{3.8}$$

respectively. Thus we write a family of eigenfunctions for (2.4) in the resolvent form

$$u_\lambda(x,y) = \mathcal{R}\left(\mu; I_{x_0}^{1+\alpha}\right) v_\alpha \; \mathcal{R}\left(\lambda - \mu; I_{y_0}^{1+\beta}\right) v_\beta. \tag{3.9}$$

Indeed, substituting (3.9) into (2.4) taking into account the values $D_{x_0+}^{1+\alpha} v_\alpha = D_{y_0+}^{1+\beta} v_\beta = 0$ after a simple change of the summation index into the series we easily satisfy (2.4). But we will extend our family of eigenfunctions considering

$$u_\lambda(x,y) = \mathcal{R}\left(\mu; I_{x_0}^{1+\alpha}\right) f \; \mathcal{R}\left(\lambda - \mu; I_{y_0}^{1+\beta}\right) g, \quad \mu \in \mathbb{C}, \tag{3.10}$$

with arbitrary $f(x), g(y)$ such that $I_{x_0}^{n(1+\alpha)+1-\alpha} f \in AC^2([x_0, X_0])$, $I_{y_0}^{n(1+\beta)+1-\beta} g \in AC^2([y_0, Y_0])$, and the corresponding resolvent function (3.5) are analytic by x and y. So substituting (3.10) into (2.4) and ignoring trivial cases $D_{x_0+}^{1+\alpha} f = D_{y_0+}^{1+\beta} g = 0$ which drive immediately to (3.10) with $f = v_\alpha$, $g = v_\beta$ (see (2.3)), after separation of variables we obtain fractional differential equations to define f, g:

$$\begin{aligned} D_{x_0+}^{1+\alpha} f &= c\mathcal{R}\left(\mu; I_{x_0}^{1+\alpha}\right) f, \\ D_{y_0+}^{1+\beta} g &= -c\mathcal{R}\left(\lambda - \mu; I_{y_0}^{1+\beta}\right) g, \end{aligned} \tag{3.11}$$

where c is an arbitrary constant. Hence acting by inverse operators $E - \mu I_{x_0}^{1+\alpha}$ and $E - (\lambda - \mu) I_{y_0}^{1+\beta}$ on (3.11) with the use of (2.3) we get, correspondingly,

$$\begin{aligned} D_{x_0+}^{1+\alpha} f &= (\mu + c) f(x) + \mu v_\alpha, \\ D_{y_0+}^{1+\beta} g &= (\lambda - \mu - c) g(y) + (\lambda - \mu) v_\beta. \end{aligned} \tag{3.12}$$

The latter equations are solved, for instance, in [1, 2] and we obtain the following solutions ($c_i, d_i, i = 1, 2$ are constants):

$$\begin{aligned} f(x) &= c_1(x-x_0)^{\alpha-1}E_{1+\alpha,\alpha}\Big((\mu+c)(x-x_0)^{1+\alpha}\Big) \\ &\quad + c_2(x-x_0)^{\alpha}E_{1+\alpha,1+\alpha}\Big((\mu+c)(x-x_0)^{1+\alpha}\Big) \\ &\quad + \mu a_1(x-x_0)^{2\alpha}E_{1+\alpha,2\alpha+1}\Big((\mu+c)(x-x_0)^{1+\alpha}\Big) \\ &\quad + \mu a_2(x-x_0)^{2\alpha+1}E_{1+\alpha,2(\alpha+1)}\Big((\mu+c)(x-x_0)^{1+\alpha}\Big), \\ g(y) &= d_1(y-y_0)^{\beta-1}E_{1+\beta,\beta}\Big((\lambda-\mu-c)(y-y_0)^{1+\beta}\Big) \\ &\quad + d_2(y-y_0)^{\beta}E_{1+\beta,1+\beta}\Big((\lambda-\mu-c)(y-y_0)^{1+\beta}\Big) \\ &\quad + (\lambda-\mu)b_1(y-y_0)^{2\beta}E_{1+\beta,2\beta+1}\Big((\lambda-\mu-c)(y-y_0)^{1+\beta}\Big) \\ &\quad + (\lambda-\mu)b_2(y-y_0)^{2\beta+1}E_{1+\beta,2(\beta+1)}\Big((\lambda-\mu-c)(y-y_0)^{1+\beta}\Big). \end{aligned} \tag{3.13}$$

Consequently, (2.4) has families of eigenfunctions (3.9) and (3.10) with f and g given by (3.13). The case $\lambda = 0$ naturally gives classical fundamental solutions $u_{0,\mu}(x,y) = u_1(x)u_2(y)$ with u_1, u_2, for instance, in the form (3.3).

Remark 3.1. Letting $\alpha = \beta = 1$ in (3.3) and using (1.8) we obtain familiar trigonometric eigenfunctions of the Laplace equation $\Delta u = \lambda u$.

Returning again to functions $f_i(y)$, $h_i(x)$, $i = 0,1$ from Section 2 we suppose the following power-logarithmic analytic expansions in the neighbourhood of points y_0, x_0, $0 < r_1 \le |y-y_0| \le r_2 = |Y_0 - y_0|$, $0 < \rho_1 \le |x-x_0| \le \rho_2 = |X_0 - x_0|$, namely,

$$\begin{aligned} f_i(y) &= a_i \log(y-y_0) + (y-y_0)^{\mu_i}\sum_{k=0}^{\infty} a_{ik}(y-y_0)^k, \quad i = 1,2,\ \mu_i > -1, \\ h_i(x) &= b_i \log(x-x_0) + (x-x_0)^{\nu_i}\sum_{k=0}^{\infty} b_{ik}(y-y_0)^k, \quad i = 1,2,\ \nu_i > -1. \end{aligned} \tag{3.14}$$

Hence owing to [1] and straightforward calculations we get for each $n \in \mathbb{N}_0$

$$\begin{aligned} &\Big(D_{y_0}^{n(1+\beta)} f_i\Big)(y) \\ &\quad = a_i D_{y_0}^{n(1+\beta)} \log(y-y_0) + (y-y_0)^{\mu_i - n(1+\beta)} \\ &\quad\quad \times \frac{(-1)^n}{\pi}\sin(\pi(\{n\beta\}-\mu_i))\sum_{k=0}^{\infty}(-1)^k a_{ik}\Gamma(k+\mu_i+1)\Gamma(n(1+\beta)-\mu_i-k)(y-y_0)^k, \\ &\Big(D_{x_0}^{n(1+\alpha)} h_i\Big)(x) \\ &\quad = b_i D_{x_0}^{n(1+\alpha)} \log(x-x_0) + (x-x_0)^{\nu_i - n(1+\alpha)} \\ &\quad\quad \times \frac{(-1)^n}{\pi}\sin(\pi(\{n\alpha\}-\nu_i))\sum_{k=0}^{\infty}(-1)^k b_{ik}\Gamma(k+\nu_i+1)\Gamma(n(1+\alpha)-\nu_i-k)(x-x_0)^k, \end{aligned} \tag{3.15}$$

where $i = 0, 1$. Substituting the latter expressions into (2.32), (2.33) we get

$$\begin{aligned}
u_f(x,y) = {} & \sum_{n=0}^{\infty}(-1)^n \frac{(x-x_0)^{n(1+\alpha)+\alpha-1}}{\Gamma(n(1+\alpha)+\alpha)}\left[a_0 + \frac{(x-x_0)a_1}{n(1+\alpha)+\alpha}\right] D_{y_0}^{n(1+\beta)} \log(y-y_0) \\
& + (x-x_0)^{\alpha-1}(y-y_0)^{\mu_0}\frac{1}{\pi}\sum_{k,n=0}^{\infty}(-1)^k a_{0k}\sin(\pi(\{n\beta\}-\mu_0)) \\
& \times \frac{\Gamma(n(1+\beta)-\mu_0-k)\Gamma(k+\mu_0+1)}{\Gamma(n(1+\alpha)+\alpha)}\left(\frac{(x-x_0)^{1+\alpha}}{(y-y_0)^{1+\beta}}\right)^n (y-y_0)^k \\
& + (x-x_0)^{\alpha}(y-y_0)^{\mu_1}\frac{1}{\pi}\sum_{k,n=0}^{\infty}(-1)^k a_{1k}\sin(\pi(\{n\beta\}-\mu_1)) \\
& \times \frac{\Gamma(n(1+\beta)-\mu_1-k)\Gamma(k+\mu_1+1)}{\Gamma((n+1)(1+\alpha))}\left(\frac{(x-x_0)^{1+\alpha}}{(y-y_0)^{1+\beta}}\right)^n (y-y_0)^k,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
u_h(x,y) = {} & \sum_{n=0}^{\infty}(-1)^n \frac{(y-y_0)^{n(1+\beta)+\beta-1}}{\Gamma(n(1+\beta)+\beta)}\left[b_0 + \frac{(y-y_0)b_1}{n(1+\beta)+\beta}\right] D_{x_0}^{n(1+\alpha)} \log(x-x_0) \\
& + (y-y_0)^{\beta-1}(x-x_0)^{\nu_0}\frac{1}{\pi}\sum_{k,n=0}^{\infty}(-1)^k b_{0k}\sin(\pi(\{n\alpha\}-\nu_0)) \\
& \times \frac{\Gamma(n(1+\alpha)-\nu_0-k)\Gamma(k+\nu_0+1)}{\Gamma(n(1+\beta)+\beta)}\left(\frac{(y-y_0)^{1+\beta}}{(x-x_0)^{1+\alpha}}\right)^n (x-x_0)^k \\
& + (y-y_0)^{\beta}(x-x_0)^{\nu_1}\frac{1}{\pi}\sum_{k,n=0}^{\infty}(-1)^k b_{1k}\sin(\pi(\{n\alpha\}-\nu_1)) \\
& \times \frac{\Gamma(n(1+\alpha)-\nu_1-k)\Gamma(k+\nu_1+1)}{\Gamma((n+1)(1+\beta))}\left(\frac{(y-y_0)^{1+\beta}}{(x-x_0)^{1+\alpha}}\right)^n (x-x_0)^k,
\end{aligned} \tag{3.17}$$

where double series in (3.16), (3.17) are absolutely and uniformly convergent on the compact $0 < r_1 \le |y-y_0| \le r_2 = |Y_0 - y_0|$, $0 < \rho_1 \le |x-x_0| \le \rho_2 = |X_0 - x_0|$ owing to conditions

$$\begin{aligned}
& \sum_{k,n=0}^{\infty}\frac{|a_{1k}\Gamma(n(1+\beta)-\mu_1-k)|\Gamma(k+\mu_1+1)}{\Gamma((n+1)(1+\alpha))}\left(\frac{\rho_2^{1+\alpha}}{r_1^{1+\beta}}\right)^n r_2^k < \infty, \\
& \sum_{k,n=0}^{\infty}\frac{|b_{1k}\Gamma(n(1+\alpha)-\nu_1-k)|\Gamma(k+\nu_1+1)}{\Gamma((n+1)(1+\beta))}\left(\frac{r_2^{1+\beta}}{\rho_1^{1+\alpha}}\right)^n \rho_2^k < \infty.
\end{aligned} \tag{3.18}$$

We will call the ordinary series in (3.16), (3.17) fractional logarithmic solutions. Taking into account, for instance, the representation (see (1.1))

$$
\begin{aligned}
&D_{y_0}^{n(1+\beta)} \log(y-y_0) \\
&= \left(\frac{d}{dy}\right)^{[n(1+\beta)]+1}\left[(y-y_0)^{1-\{n\beta\}}\int_0^1 (1-t)^{-\{n\beta\}}\log((y-y_0)t)dt\right] \\
&= (-1)^{[n(1+\beta)]+1}(y-y_0)^{-n(1+\beta)}\left[\frac{\Gamma(n(1+\beta))}{\Gamma(\{n\beta\}-1)}\left[d_n-\frac{\log(y-y_0)}{\{n\beta\}-1}\right.\right. \\
&\qquad\left.\left.+\frac{([n(1+\beta)]+1)!}{\Gamma(\{n\beta\})}\sum_{m=1}^{[n(1+\beta)]+1}\frac{\Gamma(n(1+\beta)-m)}{m([n(1+\beta)]+1-m)!}\right]\right],
\end{aligned}
\tag{3.19}
$$

where $d_n = \int_0^1 (1-t)^{-\{n\beta\}} \log t\, dt$, we may substitute it in (3.16) to write the ordinary series in a different form and to guarantee its absolute convergence in the region $(x-x_0)^{1+\alpha}/(y-y_0)^{1+\beta} < 1$. Finally we note that in the similar manner we treat the ordinary series in (3.17).

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